## Lecture 11

Exam 1 average: this section 72%

all 408L 68%

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## Partial fractions (cont)

For integrals of form $\int \frac{P(x)}{Q(x)}\,dx$ P, Q polynomials

like $\frac{7x^4-3x^3-18-\frac{1}{2}}{x^3+4x}$

Strategy: factor $Q(x)$, use that factorization to split up $\frac{P(x)}{Q(x)}$.

Q $\int \frac{1}{x^2+x}\,dx = \int \frac{1}{x(x+1)}\,dx \quad \frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$

$$1 = A(x+1) + B(x)$$

$$x=0 \quad 1 = A(1) + B(0) \rightarrow A=1$$

$$x=-1 \quad 1 = A(0) + B(-1) \rightarrow B=-1$$

$$\int \frac{1}{x}\,dx + \int \frac{-1}{x+1}\,dx$$

$$\underline{\underline{\ln|x| - \ln|x+1| + C}}$$

$x(x+1)$

OR:
$$1 = A(x+1) + Bx$$
$$0x + 1 = (A+B)x + A$$
$$\longrightarrow \quad 0 = A + B \qquad \text{(compare x terms)}$$
$$1 = A \qquad \text{(compare const terms)}$$
$$\longrightarrow \quad A = 1,\ B = -1$$

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Last time: $\dfrac{x^2+2x-1}{2x^3+3x^2-2x}$ $\longleftarrow$ factors into $x(2x-1)(x+2)$

then set
$$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

$\dfrac{2x^2-x+4}{x^3+4x}$ $\longleftarrow$ factors into $x(x^2+4)$

then set
$$\frac{2x^2-x+4}{x^3+4x} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

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What if a factor appears more than once in denom?

Ex $\displaystyle\int \frac{1}{x^3+2x^2+x}\,dx$ Factor: $x^3+2x^2+x = x(x^2+2x+1)$
$= x(x+1)^2$

Write
$$\frac{1}{x^3+2x^2+x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$\left[\text{or: } \dfrac{A}{x} + \dfrac{Bx+C}{(x+1)^2}\right]$ multiply thru by $x(x+1)^2$:

$$1 = A(x+1)^2 + Bx(x+1) + Cx$$

$x=0 \longrightarrow 1 = A(1)^2 + B(0) + C(0)$
$1 = A \longrightarrow A = 1$

$x=-1 \longrightarrow 1 = A(0) + B(-1)(0) + C(-1)$
$1 = -C \longrightarrow C = -1$

$x=1 \longrightarrow 1 = A(2)^2 + B(1)(2) + C(1)$
$1 = 4A + 2B + C$
$1 = 4 + 2B - 1 \longrightarrow B = -1$
$-2 = 2B$

so $$\int \frac{1}{x^3+2x^2+x}\,dx = \int \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}\,dx$$
$$= \int \frac{1}{x} + \frac{-1}{x+1} + \frac{-1}{(x+1)^2}\,dx$$
$$= \underline{\underline{\left(\ln|x| - \ln|x+1| + \frac{1}{x+1}\right) + K}}$$

$\left[u = x+1,\ -\int \frac{1}{u^2}\,du = \frac{1}{u}\right]$

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__Ex__ if we want to do $\displaystyle\int \frac{2x-3}{(x+4)^4 x^2 (x-1)}\,dx$ :

$$\frac{2x-3}{(x+4)^4x^2(x-1)} = \frac{A}{x-1} + \frac{B}{x} + \frac{C}{x^2}$$

$$+ \frac{D}{x+4} + \frac{E}{(x+4)^2} + \frac{F}{(x+4)^3} + \frac{G}{(x+4)^4}$$

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Q $\int_0^1 \frac{x^3-4x-10}{x^2-x-6}\,dx = ?$

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$$\begin{array}{r l}
 & x+1 \\
x^2-x-6 \,\big)\!\! & x^3+0x^2-4x-10 \\
 & x^3-x^2-6x \\
\hline
 & 0+x^2+2x-10 \\
 & x^2-x-6 \\
\hline
 & 3x-4
\end{array}$$

so $\displaystyle\int_0^1 \frac{x^3-4x-10}{x^2-x-6}\,dx = \int_0^1 x+1 + \frac{3x-4}{x^2-x-6}\,dx$

par frac: $x^2-x-6 = (x-3)(x+2)$

$$\frac{3x-4}{x^2-x-6} = \frac{A}{x-3} + \frac{B}{x+2}$$

$$3x-4 = A(x+2) + B(x-3)$$

$$\rightarrow A=1,\ B=2$$

$$\int_0^1 x+1+\frac{1}{x-3}+\frac{2}{x+2}\,dx$$

$$= \tfrac{1}{2}x^2+x+\ln|x-3|+2\ln|x+2| \Big|_0^1$$

$$= \tfrac{1}{2}+1+(\ln 2-\ln 3)+2\ln 3-2\ln 2$$

$$= \tfrac{1}{2}+1-\ln 2+\ln 3$$

$$= \underline{\underline{\tfrac{3}{2}+\ln\tfrac{3}{2}}}$$

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$A=1$
$B=-1$
$C=-1$
$D=+1$
$E=0$

Q $$\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx = ?$$

$$\frac{1-x+2x^2-x^3}{x(x^2+1)^2} = \frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}$$

$x(x^2+1)^2$

$$1-x+2x^2-x^3 = A(x^2+1)^2+(Bx+C)(x)(x^2+1)+(Dx+E)(x)$$
$$= \underbrace{A(x^4+2x^2+1)}+\underbrace{(Bx+C)(x^3+x)}+Dx^2+Ex$$
$$= Ax^4+2Ax^2+A+Bx^4+Cx^3+Bx^2+Cx+Dx^2+Ex$$

$$0x^4-x^3+2x^2-x+1 = (A+B)x^4+Cx^3+(2A+B+D)x^2+(C+E)x+A$$

$$\left.\begin{aligned} 0&=A+B\\ -1&=C\\ 2&=2A+B+D\\ -1&=C+E\\ 1&=A \end{aligned}\right\} \rightarrow \begin{pmatrix} A=1\\ B=-1\\ C=-1\\ D=1\\ E=0 \end{pmatrix}$$

$$\rightarrow \int \frac{1}{x} + \frac{-x-1}{x^2+1} + \frac{x}{(x^2+1)^2}\,dx$$

$\ln|x|$

$\int \frac{-x}{x^2+1}\,dx + \int \frac{-1}{x^2+1}\,dx$

$= -\frac{1}{2}\ln|x^2+1| - \tan^{-1}x$

$u = x^2+1$

$du = 2x\,dx$

$\int \frac{\frac{1}{2}\,du}{u^2} = -\frac{1}{2u}$

$= -\frac{1}{2(x^2+1)}$

So... $\int = \underline{\ln|x| - \frac{1}{2}\ln|x^2+1| - \tan^{-1}x - \frac{1}{2(x^2+1)} + K}$

Ex $\displaystyle\int_0^1 \frac{1}{1+\sqrt[3]{x}}\,dx = \int_0^1 \frac{1}{1+x^{1/3}}\,dx$

$u = x^{1/3}$

$u^3 = x$

$3u^2\,du = dx$

$= \displaystyle\int_0^1 \frac{3u^2\,du}{1+u}$

then by division...

$$\begin{array}{r} 3u - 3 \\ u+1 \,\big)\overline{3u^2 + 0u + 0} \\ \underline{3u^2 + 3u} \\ -3u + 0 \\ \underline{-3u - 3} \\ 3 \end{array}$$

$= \displaystyle\int_0^1 (3u-3) + \frac{3}{1+u}\,du$

$= \frac{3}{2}u^2 - 3u + 3\ln|1+u|\,\Big|_0^1$

$$= \frac{3}{2} - 3 + 3\ln 2$$

$$= -\frac{3}{2} + 3\ln 2$$