Lecture 18

Sequences

A *sequence* is an ordered list of numbers

$$\{a_n\} = a_1, a_2, a_3, a_4, \ldots, a_{100}, \ldots, a_{1000}, \ldots$$

Ex $a_n = n$: $1, 2, 3, 4, 5, \ldots, \underset{a_{100}}{100}, \ldots$

$a_n = n^2$: $1, 4, 9, 16, 25, \ldots, 10000, \ldots$

$a_n = 4n+1$: $5, 9, 13, 17, 21, \ldots$

$a_n = (-1)^n$: $-1, 1, -1, 1, -1, \ldots$

$a_n = \frac{1}{n^2+1}$: $\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \frac{1}{17}, \ldots, \frac{1}{1001}, \ldots$

a_n = the closing price of 1 Bitcoin in USD on n^{th} day of the year:

$\ldots, 6371.10, 6403, \ldots$

a_n = the n^{th} term of the Fibonacci sequence: (ie $a_1 = 1$, $a_2 = 1$, $a_{n+2} = a_n + a_{n+1}$)

$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$

Q $a_n = \frac{1}{n}\cos\left(\frac{n\pi}{2}\right)$ write this sequence out

$0, \frac{-1}{2}, 0, \frac{1}{4}, 0, \frac{-1}{6}, 0, \frac{1}{8}, \ldots$

Q take the sequence $3, 8, 13, 18, \ldots$ where each term is 5 bigger than the previous one.

what is a_n?

$$\underline{\underline{a_n = 5n - 2}}$$

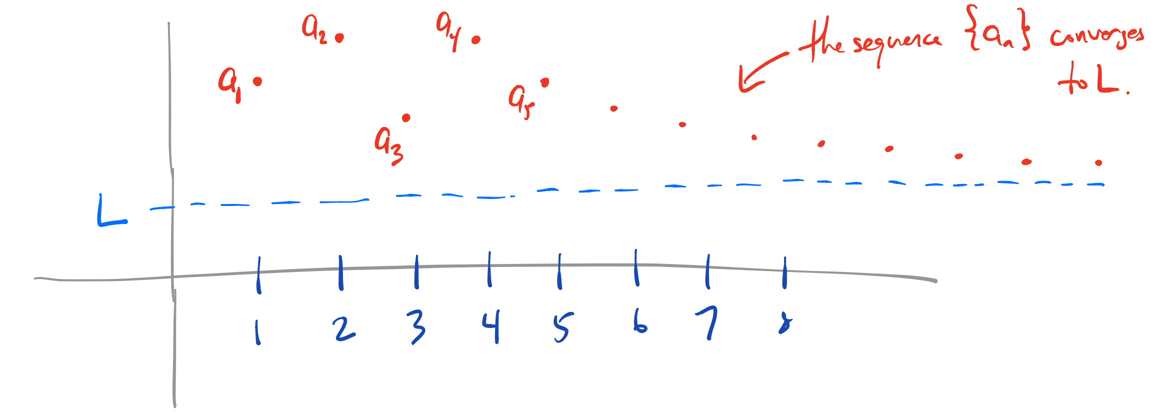
Ex $a_n = n!$

$n! = n(n-1)(n-2)\cdots 1$

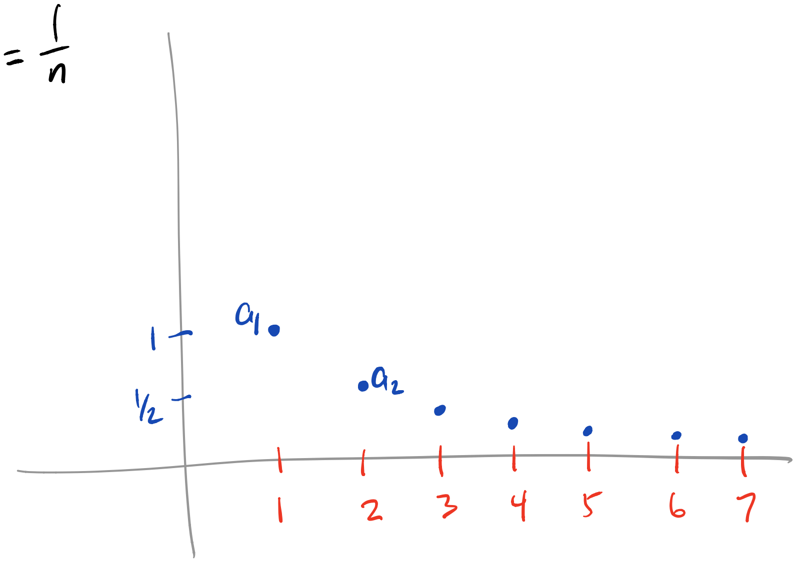
e.g. $5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = \underline{120}$

$\{a_n\} = 1, 2, \underset{3\cdot 2}{\underset{\parallel}{6}}, \underset{4\cdot 3\cdot 2}{\underset{\parallel}{24}}, 120, \ldots$

Fundamental question about a sequence $\{a_n\}$: $\underline{\text{does it converge}}$?



Ex $a_n = \frac{1}{n}$

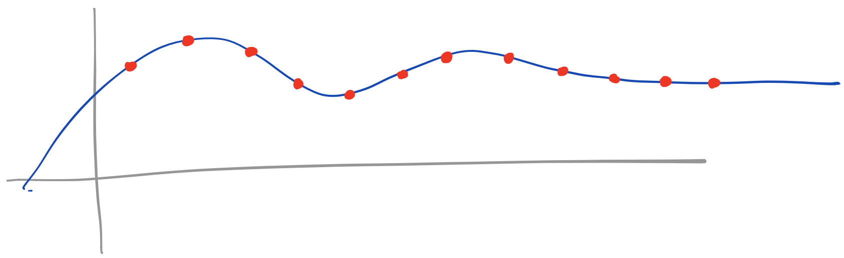


This sequence converges to 0.

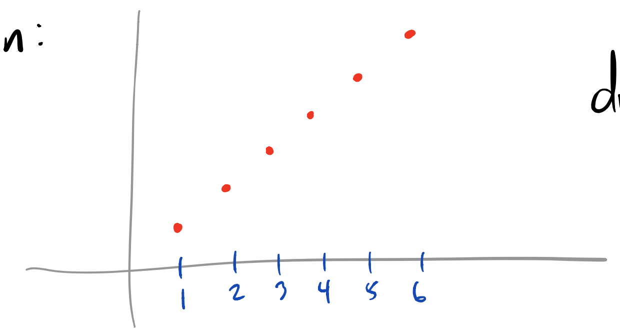
We also write this as $\lim_{n\to\infty} a_n = 0$ i.e. $\lim_{n\to\infty} \frac{1}{n} = 0$.

(That might remind you of $\lim_{x\to\infty} \frac{1}{x} = 0$.

Indeed: <u>if</u> we have a seq. given by a <u>function</u>, $a_n = f(n)$, with $\lim_{x\to\infty} f(x) = L$, then also $\lim_{n\to\infty} a_n = L$, ie $\{a_n\}$ converges to L.)

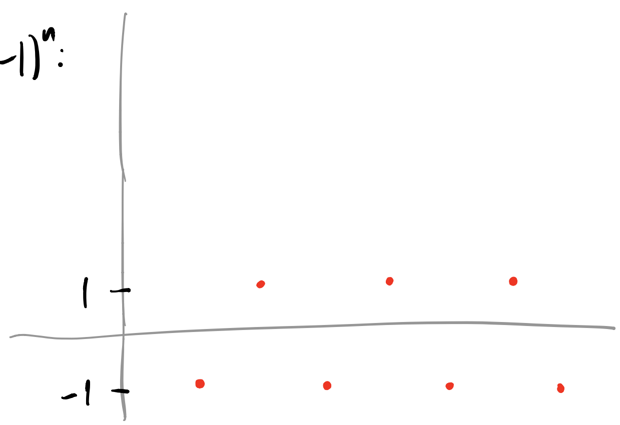


<u>Ex</u> $a_n = n$:

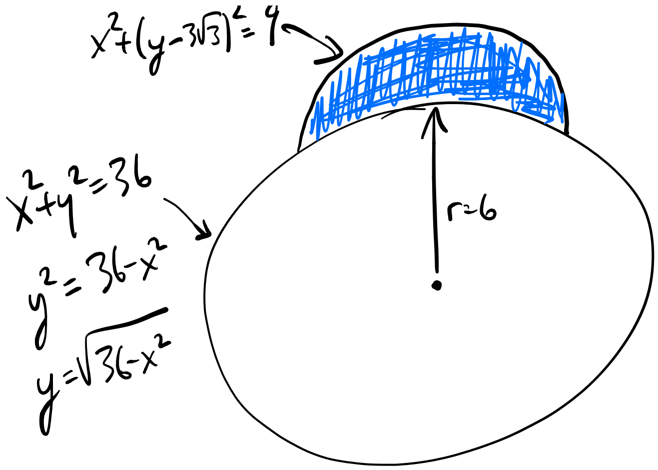


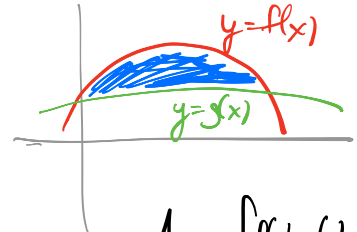
<u>diverges</u>

<u>Ex</u> $a_n = (-1)^n$:



<u>diverges</u>





$$A=\int f(x)-g(x)\,dx$$

$$\int_0^2 \frac{5}{(x-1)^{1/3}}\,dx$$

convergent or not?

vertical asymptote at $x=1$

$\Rightarrow$ split it up

$$\int_0^1 \frac{5}{(x-1)^{1/3}}\,dx + \int_1^2 \frac{5}{(x-1)^{1/3}}\,dx$$

p-test $\Rightarrow$ both are convergent $\left(\frac{1}{3}<1\right)$

for vertical asymptotes

$$\int_c^a \frac{1}{(x-c)^p} \text{ is convergent if } p<1$$

divergent if $p\geq 1$

$$\int_1^4 \frac{\sqrt{x}}{x-9}\,dx \qquad u=\sqrt{x}$$

$$f(x,y)=\frac{1}{2}y\tan^{-1}\left(\frac{x}{y}\right)$$

$f_{xy} = ?$

$$f_x = \frac{1}{2}y \cdot \frac{1}{\left(1+\left(\frac{x}{y}\right)^2\right)} \cdot \frac{1}{y}$$

$$= \frac{1}{2} \frac{1}{1+\left(\frac{x}{y}\right)^2}$$

$$= \frac{1}{2} \frac{1}{\frac{x^2+y^2}{y^2}} = \frac{1}{2} \frac{y^2}{x^2+y^2}$$

...

$$\int_0^{\ln 2} \frac{3}{e^x+4}\,dx$$

$u = e^x$

$du = e^x\,dx$

$\frac{du}{u} = dx$

$$= \int_1^2 \frac{3}{u+4} \frac{du}{u}$$

$$= \int_1^2 \frac{3}{u^2+4u}\,du$$

$$\frac{3}{u^2+4u} = \frac{A}{u} + \frac{B}{u+4}$$

$$3 = A(u+4) + Bu$$

$u=-4 \rightarrow 3 = -4B \quad B = -\frac{3}{4}$

$u=0 \rightarrow 3 = 4A \quad A = \frac{3}{4}$

$$\int_1^2 \frac{3}{4} \cdot \frac{1}{u} - \frac{3}{4} \cdot \frac{1}{u+4}\,du$$

$$= \frac{3}{4}\ln u - \frac{3}{4}\ln(u+4)\Big|_1^2$$

$$= \frac{3}{4}\ln 2 - \left(\frac{3}{4}\ln 6 - \frac{3}{4}\ln 5\right)$$

$$= \frac{3}{4}(\ln 2 - \ln 6 + \ln 5) = \underline{\underline{\frac{3}{4}\ln\frac{5}{3}}}$$

$$\int_0^1 \frac{2\ln 7x}{\sqrt{x}}\,dx = \cdots = \lim_{t\to 0} 4(\ln 7 - 2) - 4(\sqrt{t}\ln 7t - 2\sqrt{t})$$

$u = \ln 7x$ $\quad v = 2\sqrt{x}$

$du = \frac{dx}{x}$ $\quad dv = \frac{dx}{\sqrt{x}}$

what's $\lim_{t\to 0} \underbrace{\sqrt{t}}_{\to 0}\ \underbrace{\ln 7t}_{\to -\infty}$?

$\sqrt{t} = t^{1/2}$

$= \frac{1}{t^{-1/2}}$

$$= \lim_{t\to 0} \frac{\ln 7t^{\to -\infty}}{t^{-1/2}{}_{\to\infty}} = \lim_{t\to 0} \frac{1/t}{-\frac{1}{2}t^{-3/2}}$$

$$= \lim_{t\to 0} -2t^{1/2} = \underline{\underline{0}}$$

indeterminate: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0\cdot\infty$

$$\int \tan^{-1}x\,dx =$$ do by IBP

$u = \tan^{-1}x$ $\quad v = x$

$du = \frac{1}{1+x^2}dx$ $\quad dv = dx$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} \qquad \frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$$

$$\int_0^{1/2} \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx \qquad x = \sin\theta$$

$$dx = \cos\theta\, d\theta$$

$$\int \frac{\theta}{\sqrt{1-\sin^2\theta}} \cos\theta\, d\theta$$

$$= \int \theta\, d\theta \quad = \frac{1}{2}\theta^2 = \frac{1}{2}(\sin^{-1}x)^2$$