From HW: Does $\sum_{n=0}^{\infty} \frac{3}{\ln(n+5)} \cos\left(\frac{n\pi}{2}\right)$ converge?

$$n = 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad \cdots$$

$$\frac{3}{\ln 5} + 0 - \frac{3}{\ln 7} + 0 + \frac{3}{\ln 9} + 0 - \frac{3}{\ln 11} + 0 + \cdots$$

$$= \frac{3}{\ln 5} - \frac{3}{\ln 7} + \frac{3}{\ln 9} - \frac{3}{\ln 11} + \cdots$$ alternating series

So, use alt. series test: the seq. $\frac{3}{\ln 5}, \frac{3}{\ln 7}, \frac{3}{\ln 9}, \ldots$ is decreasing and has limit $= 0$

So the sum converges

$$\lim_{x \to \infty} \frac{3}{\ln x} = 0 \qquad \lim_{n \to \infty} \frac{3}{\ln(2n+3)} = 0$$

Absolute Convergence

$\sum_{n=1}^{\infty} a_n$ Call $\sum_{n=1}^{\infty}$ absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

Q1) Is $\sum_{n=1}^{\infty} (-1)^n/\sqrt{n}$ absolutely convergent? Is it convergent?

Q2) Is $\sum_{n=1}^{\infty} (-1)^n/n^2$ abs. convergent? Is it convergent?

A1) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent by p-test $\left(\frac{1}{2} < 1\right)$ $\longrightarrow$ $\sum \frac{(-1)^n}{\sqrt{n}}$ is not absolutely conv

but $\sum \frac{(-1)^n}{\sqrt{n}}$ is convergent by alt. series test $\left(\frac{1}{\sqrt{n}}\right.$ is decreasing, limit is $\left.0\right)$

A2) $\sum \frac{1}{n^2}$ is convergent by p-test $(2 > 1)$ $\longrightarrow$ $\sum \frac{(-1)^n}{n^2}$ is absolutely conv

but $\sum \frac{(-1)^n}{n^2}$ is convergent by alt series test $\left(\frac{1}{n^2}\right.$ is decreasing, limit is $\left.0\right)$

Fact: if $\sum a_n$ is absolutely convergent then it is convergent.

If $\sum a_n$ is convergent but not absolutely convergent we call it conditionally convergent.

Ex $\sum \frac{(-1)^n}{\sqrt{n}}$ is conditionally convergent.

→ 3 possibilities:

- absolutely convergent } convergent
- conditionally convergent }
- divergent

Q Is $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$:

- abs. conv?
- cond. conv?
- divergent?

A $\sum_{n=1}^{\infty} \frac{|\cos(n)|}{n^2}$ $\quad 0 \leq \frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

so $\sum_{n=1}^{\infty} \frac{|\cos(n)|}{n^2}$ abs. conv. (Comparison Test)

So $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}$ is absolutely convergent

(so it's convergent)

Q $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n + 2}$

- abs conv?
- cond conv?
- div?

- First check for absolute conv: does $\sum_{n=2}^{\infty} \frac{1}{\ln n + 2}$ converge?

TFD: $\lim_{n \to \infty} \frac{1}{\ln n + 2} = 0 \implies$ no info

$\ln n < n$

$\int$ test: $\int_1^{\infty} \frac{dx}{\ln x + 2}$ looks hard

but, $\boxed{\frac{1}{\ln n + 2} > \frac{1}{n}}$ and $\sum \frac{1}{n}$ diverges

$\left(n = 1000 \qquad \ln(1000) \approx 7 \qquad \frac{1}{9} > \frac{1}{1000} \right)$

So: $\sum_{n=2}^{\infty} \frac{1}{\ln n + 2}$ diverges by comparison test

- Next check for conditional convergence: does $\sum \frac{(-1)^n}{\ln n + 2}$ converge?

  Yes, by alt. series test $\left(\frac{1}{\ln n + 2} \text{ decreasing and limit} = 0 \right)$

So, $\sum \frac{(-1)^n}{\ln n + 2}$ converges conditionally.

Ratio Test

1) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

2) If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \underset{(\text{or } \infty)}{L} > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

(If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$ then this test gives no info.)

Ex $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$. Ratio test: $a_n = (-1)^n \frac{n^3}{3^n}$

$$a_{n+1} = (-1)^{n+1} \frac{(n+1)^3}{3^{n+1}}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} \frac{(n+1)^3}{3^{n+1}}}{(-1)^n \frac{n^3}{3^n}} \right| = \lim_{n\to\infty} \frac{(n+1)^3}{n^3} \cdot \frac{3^n}{3^{n+1}}$$

$\frac{3^n}{3^{n+1}} = \frac{3^n}{3^n \cdot 3} = \frac{1}{3}$

$\frac{(n+1)^3}{n^3} = \left(\frac{n+1}{n} \right)^3$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^3 \cdot \frac{1}{3}$$

$$= 1 \cdot \frac{1}{3} = \frac{1}{3} < 1$$

So $\sum (-1)^n \frac{n^3}{3^n}$ is absolutely convergent.

Q Does $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$ converge?

Ratio Test: $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}/(n+1)!}{n^n/n!}$$

$$= \frac{\overbrace{(n+1)^{n+1}}^{(n+1)(n+1)^n}}{n^n} \cdot \underbrace{\frac{n!}{(n+1)!}}_{\frac{n(n-1)(n-2)\cdots 1}{(n+1)n(n-1)(n-2)\cdots 1}}$$

$$= (n+1)\cdot\left(\frac{n+1}{n}\right)^n \cdot \frac{1}{n+1}$$

$$= \left(\frac{n+1}{n}\right)^n$$

$$= \left(1+\frac{1}{n}\right)^n$$

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \underline{\underline{e > 1}}$$

So, $\sum \frac{n^n}{n!}$ <u>diverges</u> by Ratio Test.