Exam 3 next Tue, Dec 5 7–9pm Jester A121A

covers: 1) multiple integration $\iint$

2) series up to + not including power series

HW 8 (multiple integration part) — HW 12

HW13 will be covered on Final Exam, not on Exam 3

Functions as Power Series

Using the formula for sum of a geometric series:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad \begin{pmatrix} a = 1 \\ r = x \end{pmatrix}$$

(for $|x|<1$)

ie $$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

Ex $\frac{1}{0.98}$ to two decimal places: $\frac{1}{1-0.02} = 1 + 0.02 + (0.02)^2 + \cdots$

$x = 0.02$ ≈ 1.02

Q Find a representation of the function $f(x) = \frac{1}{1+x^2}$

as a power series centered at 0, ← $\sum_{n=0}^{\infty} (-1)^n x^{2n}$

and find its interval of convergence.

The idea: already know $\frac{1}{1-y} = 1+y+y^2+y^3+\cdots = \sum_{n=0}^{\infty} y^n$

then set $y = -x^2$

get $\frac{1}{1+x^2} = 1-x^2+x^4-x^6+\cdots = \underline{\underline{\sum_{n=0}^{\infty}(-x^2)^n}}$

<u>Interval of convergence:</u>

easier way — this is geometric series, $r = -x^2$

so converges just if $|r|<1$ i.e. $|-x^2|<1$

i.e. $|x|<1$

i.e. $x \in \underline{\underline{(-1,1)}}$

harder way — <u>ratio test:</u> $\lim_{n\to\infty} \frac{|(-x^2)^{n+1}|}{|(-x^2)^n|} = \lim_{n\to\infty} |x|^2 = |x|^2$

so converges for $|x|<1$

diverges for $|x|>1$

for $x=1$: $\sum (-1)^n$ diverges (TFD)

$x=-1$: $\sum (1)$ diverges (TFD)

<u>Q</u> Write $\frac{1}{x+7}$ as a power series centered at 0. | $\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$

$$\frac{1}{x+7} = \frac{1}{7}\cdot\frac{1}{\frac{x}{7}+1}$$

$$= \frac{1}{7}\cdot\frac{1}{1-(-\frac{x}{7})} \qquad y = -\frac{x}{7}$$

$$= \frac{1}{7} \cdot \sum_{n=0}^{\infty} \left(-\frac{x}{7}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{7^{n+1}} \cdot (-x)^n \qquad = \sum_{n=0}^{\infty} \frac{(-1)^n}{7^{n+1}} x^n$$

Remark: We could also say

$$\frac{1}{x+7} = \frac{1}{1-(-x-6)} = \sum_{n=0}^{\infty} (-x-6)^n = \sum_{n=0}^{\infty} (-1)^n (x+6)^n$$

this is another power series for the same function but centered at $a=-6$, not 0

Q Write the function $\dfrac{x^4}{x+7}$ as a power series centered at 0.

$$\frac{x^4}{x+7} = x^4 \cdot \frac{1}{x+7} = x^4 \cdot \overbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{7^{n+1}} x^n}^{\text{from last problem}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{7^{n+1}} x^{n+4}$$

Could also rewrite this:

say $m = n+4$, $m-4 = n$ $\longrightarrow$ $\displaystyle\sum_{m=4}^{\infty} \frac{(-1)^{m-4}}{7^{m-3}} x^m$

$$= \sum_{m=4}^{\infty} \frac{(-1)^m}{7^{m-3}} x^m$$

Q1) Find power series centered at 0 for $f(x) = \dfrac{1}{(1-x)^2}$.

$\left(\text{Trick: use } \dfrac{1}{(1-x)^2} = \dfrac{d}{dx}\left(\dfrac{1}{1-x}\right)\right)$

2) What is $\dfrac{1}{(0.98)^2}$ to 2 decimal places?

1) $$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right)$$

$$= \sum_{n=0}^{\infty} n x^{n-1}$$

but the first term is $0 \cdot x^{-1}$ so can skip it:

$$= \sum_{n=1}^{\infty} n x^{n-1}$$

$n-1=m$
$n=m+1$

$$= \sum_{m=0}^{\infty} (m+1) x^m$$

$$= 1 + 2x + 3x^2 + 4x^3 + \cdots$$

2) $$\frac{1}{(0.98)^2} = \frac{1}{(1-0.02)^2} \qquad x = 0.02$$

$$\approx 1 + 2(0.02) = 1.04$$

Remark when we differentiate or integrate a power series, the radius of convergence is not changed.

So e.g. $\dfrac{1}{(1-x)^2} = \sum (m+1) x^m$

has same radius of conv. as $\frac{1}{1-x} = \sum x^n$

$(R=1)$

Q Find power series centered at 0 for: A) $\ln(1-x)$ $\quad \int \frac{1}{1-x}\,dx = -\ln(1-x)+C$

B) $\ln(2-x)$

A) $\int \frac{1}{1-x}\,dx = -\ln(1-x) + C$

$$\left(\int \frac{1}{1-x}\,dx\right) - C = -\ln(1-x)$$

$$-\int \frac{1}{1-x}\,dx + C = \ln(1-x)$$

$$-\int \sum_{n=0}^{\infty} x^n\,dx + C = \ln(1-x)$$

$$-\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C = \ln(1-x)$$

plug in $x=0$: $\quad -0 + C = \ln(1-0) = 0 \longrightarrow C = 0$

so $\quad \ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$ $\quad$ if $|x|<1$

$$= -\sum_{m=1}^{\infty} \frac{x^m}{m}$$

B) $\ln(2-x) = \ln\left(2\left(1-\frac{x}{2}\right)\right) = \ln 2 + \ln\left(1-\frac{x}{2}\right)$

$$= \ln 2 + \left(-\sum_{m=1}^{\infty} \frac{(x/2)^m}{m}\right)$$

$$= \ln 2 + \sum_{m=1}^{\infty} -\frac{1}{2^m \cdot m} x^m$$

$$\frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)^2 = \left(1+x+x^2+\cdots\right)^2$$

$$= \left(1+x+x^2+\cdots\right)\left(1+x+x^2+\cdots\right)$$

$$= 1+x+x+x^2+x^2+x^2+\cdots$$

$$= 1+2x+3x^2+\cdots$$