Last time: functions as power series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1+x+x^2+x^3+\cdots \quad \text{for } |x|<1 \quad (*)$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = 1-x^2+x^4-x^6+\cdots \quad \text{for } |x|<1 \quad (**)$$

by substituting $x \to -x^2$ in $(*)$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1} = 1+2x+3x^2+\cdots$$

by taking $\frac{d}{dx}$ of both sides of $(*)$

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -x-\frac{x^2}{2}-\frac{x^3}{3}+\cdots$$

by taking $-\int dx$ of both sides of $(*)$

Q 1) Derive a power series for $\tan^{-1}x$ by integrating $(**)$.

2) Use this to give a formula for π.

1) $$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n$$

$$\int \frac{1}{1+x^2}\,dx = \int \sum_{n=0}^{\infty} (-x^2)^n\,dx$$

$$\tan^{-1}x + C = \sum_{n=0}^{\infty} \int (-1)^n x^{2n}\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

to get C, plug in $x=0$: then $\tan^{-1}0 + C = 0$

$0 + C = 0$
$C = 0$

$$\rightarrow \tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

interval of conv: $(-1, 1]$

2) plug in $x = 1$:

$$\tan^{-1}(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \cdots$$

$\tan^{-1}(1) = \pi/4$

so $$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \cdots\right)$$

conditionally convergent!

Q Find the power series representing $f(x) = \ln\left(\frac{1+x}{1-x}\right)$ centered at $x = 0$.

1st method

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$
$$= \left(-\sum_{n=1}^{\infty} \frac{(-x)^n}{n}\right) - \left(-\sum_{n=1}^{\infty} \frac{x^n}{n}\right)$$
$$= -\sum_{n=1}^{\infty} \frac{(-x)^n}{n} + \sum_{n=1}^{\infty} \frac{x^n}{n}$$
$$= \sum_{n=1}^{\infty} \frac{x^n}{n} - \frac{(-x)^n}{n}$$

$$\ln(1-x) = -\sum \frac{x^n}{n}$$
$\downarrow x \to -x$
$$\ln(1+x) = -\sum \frac{(-x)^n}{n}$$