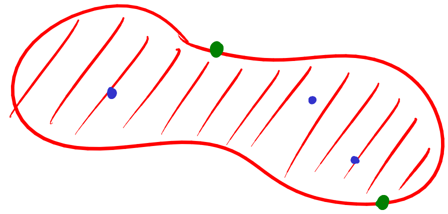
Lecture 20

11 Nov 2014

Last time: max/min of functions of <u>two variables</u> $f(x,y)$

On a closed bounded domain D:

to find max/min of $f(x,y)$

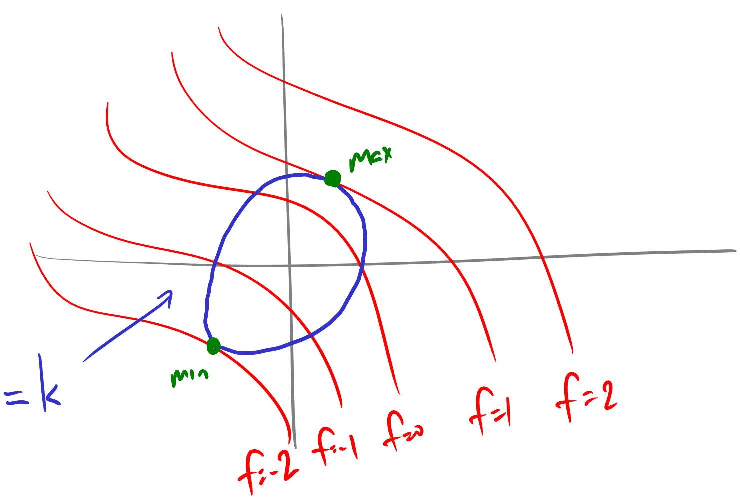


1. find critical points of $f(x,y)$ in the <u>interior</u> of D, ⟵ by solving $\vec{\nabla} f(x,y) = 0$
2. find max/min of $f(x,y)$ on <u>boundary</u> of D ⟵ <u>how</u>? Today's lecture!
3. look at all points (x,y) you found in 1, 2 — <u>biggest</u> $f(x,y)$ is <u>max</u>, <u>smallest</u> $f(x,y)$ is <u>min</u>

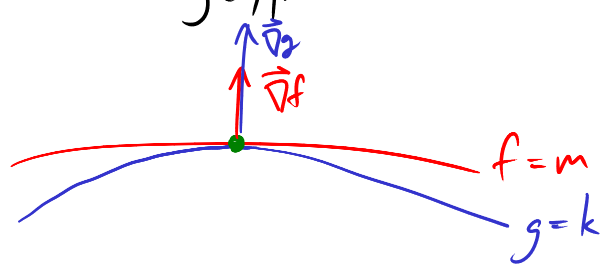
<u>Ex</u> of the kind of problem you could run into in step 2:

Find max/min of the function $f(x,y)$ subject to a <u>constraint</u> $g(x,y)=k$.

[e.g. $f(x,y) = x^2+2y^2$
constraint $x^2+y^2=1$
[so here $g(x,y)=x^2+y^2$]]



<u>Notice</u>: at the max and min of f subject to the constraint, the contour-lines of $f(x,y)$ are <u>tangent</u> to the constraint curve $g(x,y)=k$.



How to <u>find</u> the places where this happens?

They are places where $\vec{\nabla} f$ and $\vec{\nabla} g$ are either parallel or anti-parallel, so that $\vec{\nabla} f = \lambda \vec{\nabla} g$

for some unknown λ ("Lagrange multiplier")

So, strategy for finding max/min of $f(x,y)$ subject to constraint $g(x,y)=k$:
[if they exist]

① Find (x,y,λ) such that $\vec{\nabla} f(x,y) = \lambda \vec{\nabla} g(x,y)$
and $g(x,y) = k$

(3 eq. in 3 unknowns)

② Among these (x,y,λ) take the biggest $f(x,y)$ — max
smallest $f(x,y)$ — min

Ex Find max/min of $f(x,y) = x^2+2y^2$ subject to constraint $x^2+y^2=1$.

$f(x,y) = x^2+2y^2 \qquad g(x,y) = x^2+y^2$

$\vec{\nabla} f = \langle 2x, 4y \rangle \qquad \vec{\nabla} g = \langle 2x, 2y \rangle$

$\vec{\nabla} f = \lambda \vec{\nabla} g$ means $\langle 2x, 4y \rangle = \langle \lambda \cdot 2x, \lambda \cdot 2y \rangle$

$$\left.\begin{aligned} 2x &= 2\lambda x \\ 4y &= 2\lambda y \\ x^2+y^2 &= 1 \end{aligned}\right\} \longrightarrow \text{solve these for } (x,y,\lambda)$$

$2x = 2\lambda x \implies$ either $\lambda = 1$, or $x = 0$.

If $\lambda=1$: $4y=2y$ so $y=0$.

$x^2=1$ so $x=\pm 1$.

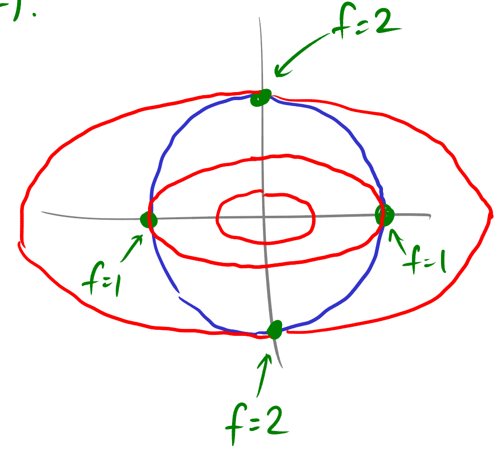
So, 2 possibilities: $(x,y,\lambda)=(1,0,1)$ or $(-1,0,1)$

If $x=0$: $y^2=1$ so $y=\pm 1$.

If $y=+1$: $4=2\lambda$ so $\lambda=2$.

If $y=-1$: $-4=-2\lambda$ so $\lambda=2$.

So, 2 possibilities: $(x,y,\lambda)=(0,1,2)$ or $(0,-1,2)$.



To find max, min:

$f(1,0)=1$ $\quad f(-1,0)=1$

$f(0,1)=2$ $\quad f(0,-1)=2$

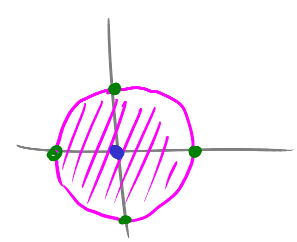
So max is 2, min is 1.

Ex Find max, min of $f(x,y)=x^2+2y^2$ on the domain $D=\{x^2+y^2\le 1\}$.

① Crit pts in interior: $\vec{\nabla}f(x,y)=0$

$\langle 2x,4y\rangle=\langle 0,0\rangle$

$x=0 \quad y=0$



So, one crit pt: $(x,y)=(0,0)$ $\quad f(x,y)=0$

② Max/min on boundary: from last problem $(x,y)=(1,0)$ $(-1,0)$ $(0,1)$ $(0,-1)$

$f=1 \quad f=1 \quad f=2 \quad f=2$

So, minimum at $(0,0)$ $\quad f(0,0)=0$

max at $(0,1)$ or $(0,-1)$ $\quad f(0,1)=2 \quad f(1,0)=2$

For $f(x,y,z)$ the method is similar.

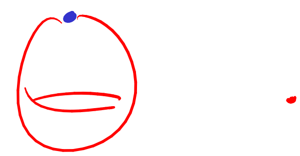
Ex Find the point on the sphere $x^2+y^2+z^2=4$ closest to/furthest from the point $(9,0,0)$.

Distance from (x,y,z) to $(9,0,0)$ is $\sqrt{(x-9)^2+y^2+z^2}$

Finding min/max for this is the same as finding min/max for

$$f(x,y,z) = (x-9)^2+y^2+z^2$$

Our constraint is $g(x,y,z) = x^2+y^2+z^2=4$



$$\vec{\nabla}f = \langle 2(x-9),\ 2y,\ 2z \rangle$$

$$\vec{\nabla}g = \langle 2x,\ 2y,\ 2z \rangle$$

$$\vec{\nabla}f = \lambda\vec{\nabla}g: \quad \left.\begin{aligned} 2(x-9) &= \lambda\cdot 2x \\ 2y &= \lambda\cdot 2y \\ 2z &= \lambda\cdot 2z \\ \text{and,}\quad x^2+y^2+z^2 &= 4 \end{aligned}\right\} \longrightarrow \text{solve these for } (x,y,z,\lambda)$$

$$2z = \lambda\cdot 2z$$

$z=\lambda z$

$0=z(\lambda-1)$ so $z=0$ or $\lambda=1$

If $z=0$: $x^2+y^2=4$, $\quad 2y=\lambda\cdot 2y$

$y=\lambda y \qquad 0=\lambda y-y=y(\lambda-1)$

so $y=0$ or $\lambda=1$

If $y=0$: $\quad x^2=4 \quad x=\pm 2$

so get $(x,y,z)=(2,0,0)$ or $(-2,0,0)$

If $\lambda=1$: our eq. are

$$2(x-9)=2x \longleftarrow 2x-18=2x$$

i.e. $-18=0$ contradiction

$$2y=2y$$
$$2z=2z$$
$$x^2+y^2+z^2=4$$

So the only candidates are $(x,y,z)=(2,0,0)$ or $(-2,0,0)$

$$\uparrow \qquad\qquad \uparrow$$

$f=7^2=49 \qquad f=11^2=121$

min $\qquad$ max

Ex Find min/max of $f(x,y,z)=x^4+y^4+z^4$

with the constraint $x^2+y^2+z^2=1$

$$\vec{\nabla} f=\langle 4x^3, 4y^3, 4z^3\rangle \qquad \vec{\nabla} g=\langle 2x, 2y, 2z\rangle$$

$\vec{\nabla} f=\lambda \vec{\nabla} g$

$$4x^3=2\lambda x$$
$$4y^3=2\lambda y$$
$$4z^3=2\lambda z$$
$$x^2+y^2+z^2=1$$

$4x^3=2\lambda x \longrightarrow$ either $4x^2=2\lambda$ or $x=0$

$4y^3=2\lambda y \longrightarrow$ either $4y^2=2\lambda$ or $y=0$

$4z^3=2\lambda z \longrightarrow$ either $4z^2=2\lambda$ or $z=0$

e.g. if $x=0$ and $y=0$ then have left $4z^2=2\lambda$ $\quad z^2=1 \longrightarrow$ $(0,0,+1)$ $f=1$

$(0,0,-1)$ $f=1$

$x=0$ and $z=0 \longrightarrow$ $(0,1,0)$ $f=1$

$(0,-1,0)$ $f=1$

$y=0$ and $z=0 \longrightarrow$ $(1,0,0)$ $f=1$

$(-1,0,0)$ $f=1$

maximum

If only $x=0$ then $\quad 4y^2=2\lambda \quad$ so $y^2=z^2$

$4z^2=2\lambda \quad$ and $y^2+z^2=1$

so $2y^2=1 \quad y=\pm\frac{1}{\sqrt{2}}$

$z=\pm\frac{1}{\sqrt{2}}$

$\longrightarrow \left(0, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$ $f=\frac{1}{2}$

Similarly $\left(\pm\frac{1}{\sqrt{2}}, 0, \pm\frac{1}{\sqrt{2}}\right)$ $f=\frac{1}{2}$ $\quad \left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, 0\right)$ $f=\frac{1}{2}$

If none of (x,y,z) is 0 then $\quad 4x^2=2\lambda \quad$ so $x^2=y^2=z^2$

$4y^2=2\lambda \quad$ but $x^2+y^2+z^2=1$

$4z^2=2\lambda \quad$ so $3x^2=1 \quad x=\pm\frac{1}{\sqrt{3}}$

$y=\pm\frac{1}{\sqrt{3}}$

$z=\pm\frac{1}{\sqrt{3}}$

$\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$ $f=\frac{1}{3}$

$\uparrow$

minimum

The case of multiple constraints

To find max/min of $f(x,y,z)$ subject to two constraints

$g(x,y,z)=k$

$h(x,y,z)=c$

we use a similar rule: one Lagrange multiplier for each constraint

solve

$$\vec{\nabla} f = \lambda \cdot \vec{\nabla} g + \mu \cdot \vec{\nabla} h$$

$$g(x,y,z) = k$$

$$h(x,y,z) = c$$

5 eq. in
5 unknowns

<u>Ex</u> Find max/min of $f(x,y,z) = x + 2y$

subject to $x + y + z = 1$ $\quad g(x,y,z) = x+y+z$

$y^2 + z^2 = 4$ $\quad h(x,y,z) = y^2 + z^2$

$$\vec{\nabla} f = \lambda \vec{\nabla} g + \mu \vec{\nabla} h$$

$\vec{\nabla} f = \langle 1, 2, 0 \rangle$ $\quad \vec{\nabla} g = \langle 1,1,1 \rangle$ $\quad \vec{\nabla} h = \langle 0, 2y, 2z \rangle$

$$\left.\begin{aligned} 1 &= \lambda \\ 2 &= \lambda + \mu \cdot 2y \\ 0 &= \lambda + \mu \cdot 2z \\ x + y + z &= 1 \\ y^2 + z^2 &= 4 \end{aligned}\right\} \text{ solve for } (x,y,z,\mu,\lambda)$$

$\lambda = 1$

$2 = 1 + \mu \cdot 2y \longrightarrow \mu = \frac{1}{2y}$

$0 = 1 + \frac{z}{y} \longrightarrow z = -y$

$x = 1$

$y^2 + z^2 = 4 \longrightarrow 2y^2 = 4 \longrightarrow y = \pm\sqrt{2}$

$(x,y,z) = (1, \sqrt{2}, -\sqrt{2})$ $\quad f = 1 + 2\sqrt{2}$ max

or $(1, -\sqrt{2}, \sqrt{2})$ $\quad f = 1 - 2\sqrt{2}$ min