Lecture 21

13 Nov 2014

Last time: max/min with constraints (method of Lagrange multipliers)

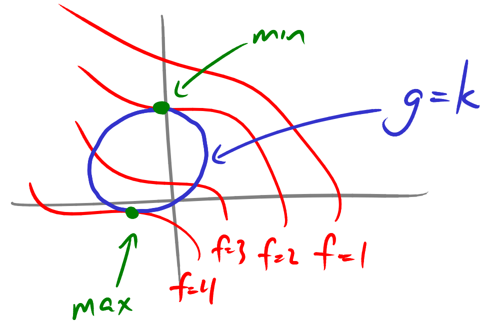
e.g. find max/min of $f(x,y)$ with constraint $g(x,y)=k$

$\longrightarrow$ involved solving $\vec{\nabla} f = \lambda \vec{\nabla} g$

which means contour-line of f

is tangent to constraint curve $g=k$

(contour-line of g)



With 2 constraints $\quad g(x,y,z)=k$

$h(x,y,z)=c$

we wrote $\quad \vec{\nabla} f = \lambda \vec{\nabla} g + \mu \vec{\nabla} h \qquad (\star)$

Why does this work?

"(x,y) is a local max/min of f subject to the constraints"

is the same as

"$D_{\vec{u}} f(x,y) = 0$ as long as $\vec{u}$ is tangent to all the constraint surfaces"

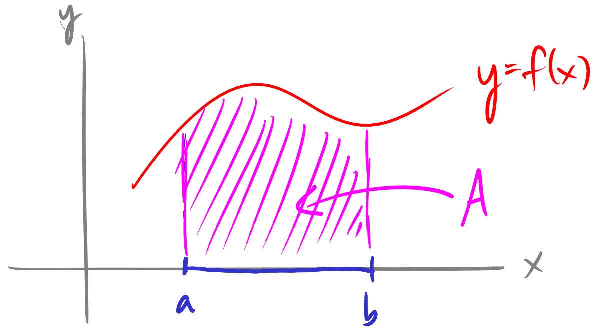
Now, suppose $(\star)$ is true. And suppose $\vec{u}$ is tangent to both constraint surfaces: $\vec{u} \perp \vec{\nabla} g$, $\vec{u} \perp \vec{\nabla} h$ i.e. $\vec{u} \cdot \vec{\nabla} g = 0$

$\vec{u} \cdot \vec{\nabla} h = 0$

Then $D_{\vec{u}} f = \vec{u} \cdot \vec{\nabla} f = \lambda \, \vec{u} \cdot \vec{\nabla} g + \mu \cdot \vec{u} \cdot \vec{\nabla} h$

$= \lambda \cdot 0 + \mu \cdot 0 = \underline{\underline{0}}$ ✓

Multiple integrals – Double and iterated integrals (Ch 15.1, 15.2)



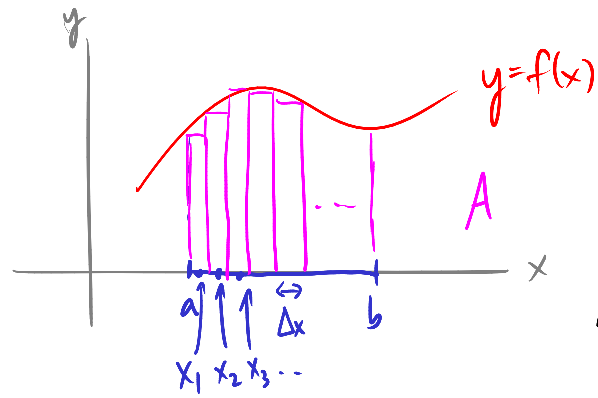
Say $f(x)$ continuous
and $f(x)>0$ for $a \le x \le b$

Then $A = \int_a^b f(x)\, dx$

Defined by chopping the interval $I = [a,b]$ into pieces:

say N equal-size pieces

$$\Delta x = \frac{b-a}{N}$$



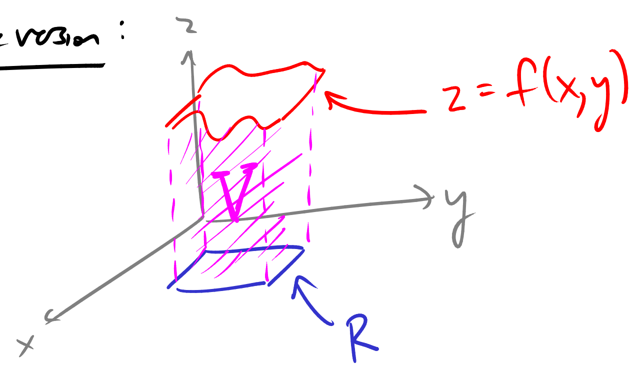
sum up areas of rectangles

$$A \approx (\Delta x) f(x_1) + (\Delta x) f(x_2) + \cdots + (\Delta x) f(x_N)$$

$$= \sum_{i=1}^{N} f(x_i)\, \Delta x$$

Taking the limit $N \to \infty$ gives the exact A.

2-variable version:

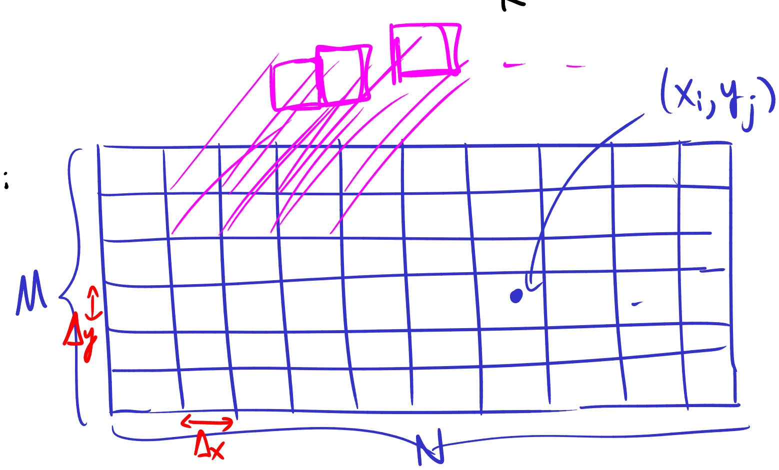


If $f(x,y)$ is continuous
and $f(x,y)>0$ for all
(x,y) lying in the rectangle R:

$$V = \iint_R f(x,y)\, dA$$

How to define it?

Chop R into little boxes:



$$V \approx \sum_{i=1}^{N} \sum_{j=1}^{M} \Delta x \, \Delta y \, f(x_i, y_j) = \sum_{i=1}^{N} \sum_{j=1}^{M} f(x_i, y_j) \, \Delta A$$

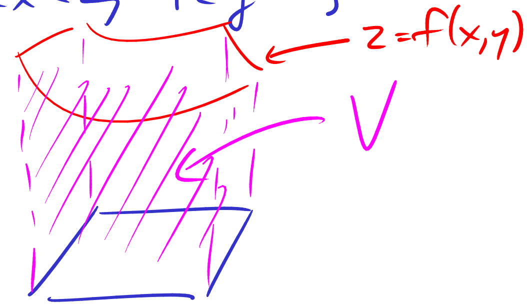
$(\Delta A = \Delta x \, \Delta y)$

("2-dimensional Riemann sum")

Take N and M to ∞:

$$V = \lim_{N \to \infty} \lim_{M \to \infty} \sum_{i=1}^{N} \sum_{j=1}^{M} \Delta x \, \Delta y \, f(x_i, y_j)$$

Ex $z = 1 + y^3 + xy$ $\quad R = \{0 \le x \le 2, \ 1 \le y \le 5\}$

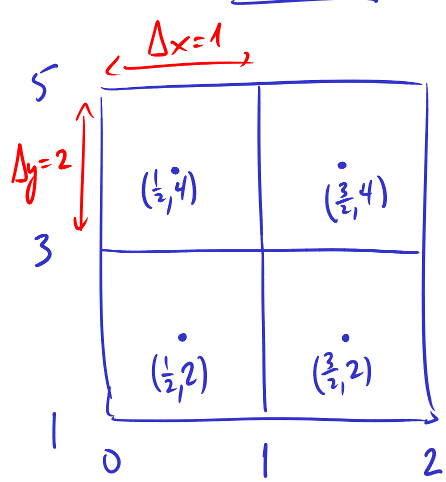


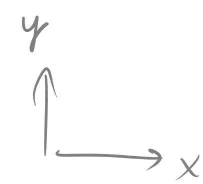
$f(x,y) = 1 + y^3 + xy$

What is V, approximately?

Take $N = 2$, $M = 2$:

$\Delta A = \Delta x \Delta y = 1 \cdot 2 = 2$



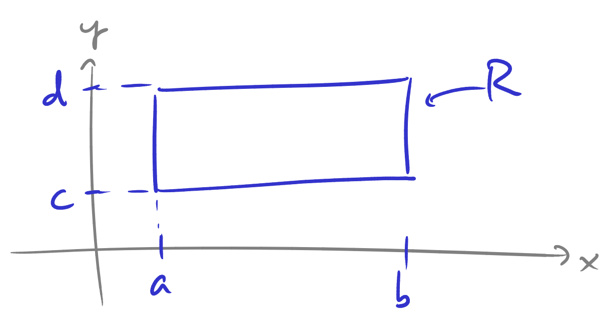


$$V \approx \sum \Delta A \cdot f(x_i, y_j)$$

$$= 2 \cdot f(\tfrac{1}{2}, 2) + 2 \cdot f(\tfrac{3}{2}, 2) + 2 \cdot f(\tfrac{1}{2}, 4) + 2 \cdot f(\tfrac{3}{2}, 4)$$

$$= 2 \cdot (10 + 12 + 67 + 71) = 2 \cdot 160 = \underline{\underline{320}}$$

How to get V exactly?



Iterated integral:

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

$$= \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy$$

Ex Find the exact value of $\iint_R f(x,y)\,dA$ $\quad R = \{0 \le x \le 2,\ 1 \le y \le 5\}$

$f(x,y) = 1 + y^3 + xy$

$$\iint_R f(x,y)\,dA = \int_1^5 \left[\int_0^2 1 + y^3 + xy \,dx \right] dy$$

$$= \int_1^5 \left[x + xy^3 + \tfrac{1}{2}x^2 y \Big|_{x=0}^{x=2} \right] dy$$

$$= \int_1^5 (2 + 2y^3 + 2y) - (0 + 0y^3 + \tfrac{1}{2}0^2 y)\,dy$$

$$= \int_1^5 2 + 2y^3 + 2y \,dy$$

$$= 2y + \tfrac{1}{2}y^4 + y^2 \Big|_{y=1}^{y=5}$$

$$= \ldots$$

$$= \underline{\underline{344}}$$

Remark We could also do the integrals in the other order: $\quad R = \{0 \le x \le 2,\ 1 \le y \le 5\}$

$$\int_0^2 \left[\int_1^5 1 + y^3 + xy \,dy \right] dx$$

$$= \int_0^2 \left(y + \tfrac{1}{4}y^4 + \tfrac{1}{2}y^2 x \Big|_{y=1}^{y=5} \right) dx$$

$$= \cdots$$

$$= \int_0^2 160 + 12x \, dx$$

$$= \cdots$$

$$= \underline{344}$$

<u>Ex</u> $\displaystyle\int_1^3 \int_1^5 \frac{\ln(y)}{xy} \, dy \, dx = \int_1^3 \left[\int_1^5 \frac{\ln(y)}{xy} \, dy \right] dx$

<u>Inside part:</u> $\displaystyle\int_1^5 \frac{\ln(y)}{xy} \, dy = \frac{1}{x} \int_1^5 \frac{\ln(y)}{y} \, dy$ $\quad u = \ln y$, $du = \frac{dy}{y}$

$$= \frac{1}{x} \int_0^{\ln(5)} u \, du$$

$$= \frac{1}{x} \left(\frac{1}{2} u^2 \Big|_{u=0}^{u=\ln(5)} \right)$$

$$= \frac{1}{2x} (\ln 5)^2$$

<u>outside integral:</u> $\displaystyle\int_1^3 \frac{1}{2x} (\ln 5)^2 \, dx = \frac{(\ln 5)^2}{2} \int_1^3 \frac{dx}{x}$

$$= \frac{(\ln 5)^2}{2} \ln x \Big]_{x=1}^{x=3}$$

$$= \underline{\underline{\frac{(\ln 5)^2}{2} \ln 3}}$$

<u>Remark:</u> Here $f(x,y) = \dfrac{\ln y}{xy} = \dfrac{1}{x} \cdot \dfrac{\ln y}{y}$

only uses x $\qquad$ only uses y

So $\displaystyle\int_1^3\int_1^5 f(x,y)\,dy\,dx = \int_1^3\int_1^5 \frac{1}{x}\cdot\frac{\ln y}{y}\,dy\,dx$

$$= \int_1^3 \frac{1}{x}\cdot\left[\int_1^5 \frac{\ln y}{y}\,dy\right]dx$$

$$= \left[\int_1^5 \frac{\ln y}{y}\,dy\right]\cdot\left[\int_1^3 \frac{1}{x}\,dx\right]$$

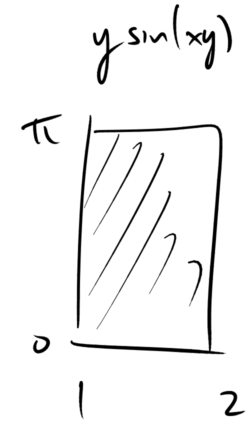
$$= (\ln 5)^2/2 \cdot (\ln 3)$$

In general, whenever
$$f(x,y) = g(x)\,h(y)$$
then
$$\int_a^b\int_c^d f(x,y)\,dy\,dx = \left[\int_c^d h(y)\,dy\right]\cdot\left[\int_a^b g(x)\,dx\right]$$

Ex $\displaystyle\iint_R y\sin(xy)\,dA \qquad R = \{1 \leq x \leq 2,\ 0 \leq y \leq \pi\}$

Here it's much easier to do $\int dx$ first

so, take $\displaystyle\int_0^{\pi}\left[\int_1^2 y\sin(xy)\,dx\right]dy$



$$= \int_0^{\pi}\left(-\cos(xy)\Big]_{x=1}^{x=2}\right)dy$$

$$= \int_0^{\pi} -\cos(2y)+\cos(y)\,dy$$

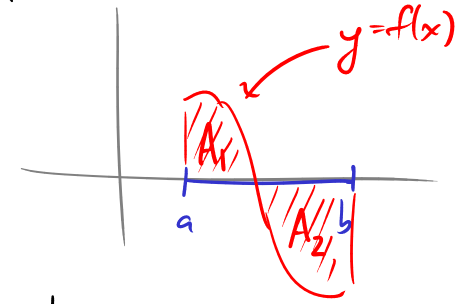
$$= -\tfrac{1}{2}\sin(2y) + \sin(y)\Big|_0^{\pi}$$

$$= 0$$

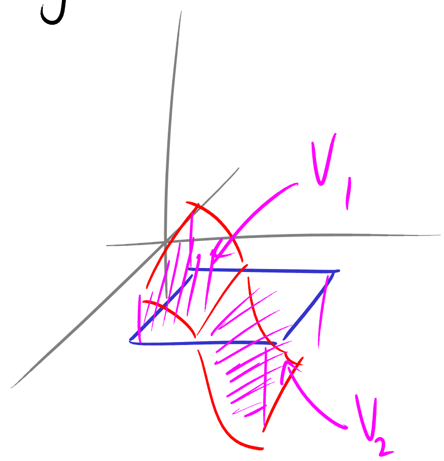
Remark: If f is allowed to be negative the integral

$$\iint_R f(x,y)\,dA$$

gives a signed volume:



$$\int_a^b f(x)\,dx = A_1 - A_2$$



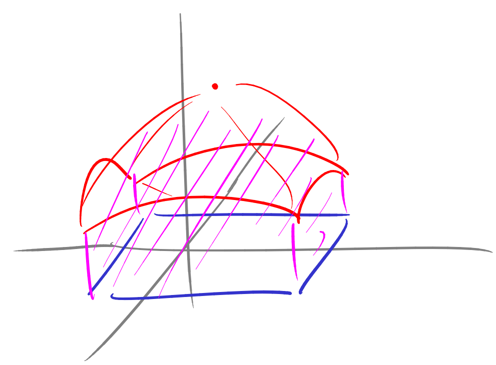
$$\iint f(x,y)\,dA = V_1 - V_2$$

Ex Compute the volume of the region lying under the paraboloid

$$\frac{x^2}{4} + \frac{y^2}{9} + z = 1$$

and above the rectangle $R = [-1,1] \times [-2,2] = \{-1 \leq x \leq 1,\ -2 \leq y \leq 2\}$

$$z = \underbrace{1 - \frac{x^2}{4} - \frac{y^2}{9}}_{f(x,y)}$$

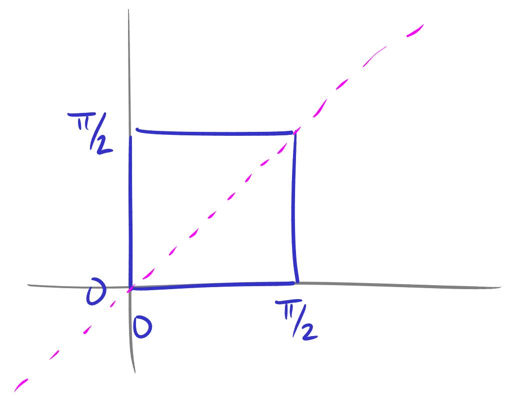


On R, $\frac{x^2}{4} \leq \frac{1}{4}$ $\quad \frac{y^2}{9} \leq \frac{4}{9}$

so $1 - \frac{x^2}{4} - \frac{y^2}{9} \geq 1 - \frac{1}{4} - \frac{4}{9} > 0$ so $f(x,y) > 0$

$$V = \int_{-2}^{2}\int_{-1}^{1} 1 - \frac{x^2}{4} - \frac{y^2}{9}\, dx\, dy = \cdots = \frac{166}{27}$$

Ex $\displaystyle\iint_R \sin(x-y)\, dA$ $\quad R = \{0 \le x \le \tfrac{\pi}{2},\ 0 \le y \le \tfrac{\pi}{2}\}$



$$= \int_0^{\pi/2}\left[\int_0^{\pi/2} \sin(x-y)\, dx\right] dy$$

$$= \int_0^{\pi/2}\left(-\cos(x-y)\Big]_{x=0}^{x=\pi/2}\right) dy$$

$$= \int_0^{\pi/2} -\cos\left(\tfrac{\pi}{2}-y\right) + \cos(-y)\, dy$$

$$= \cdots$$

$$= 0$$

Why did we get 0? One way to understand it:

$$\sin(x-y) = -\sin(y-x)$$

and the domain is symmetrical under