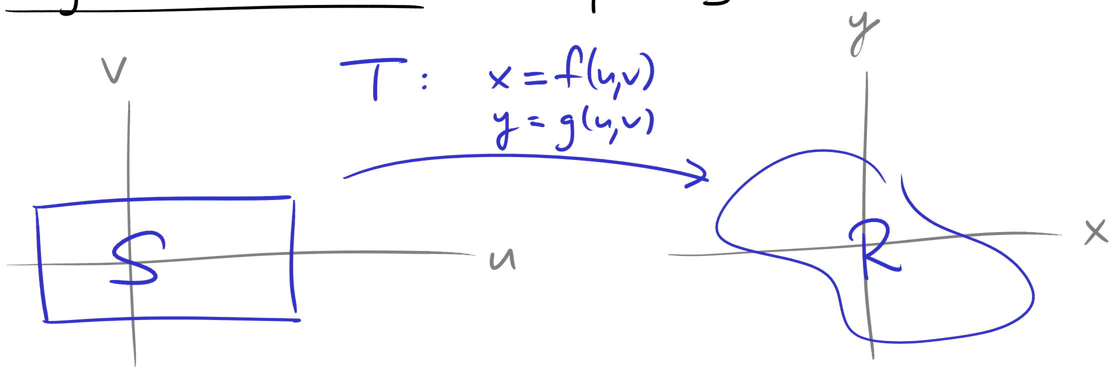
Lecture 25

2 Dec 2014

Final exam review next Wed — time, room TBA

also some review in next (last) lecture

Last time: change of variable formula in multiple integrals



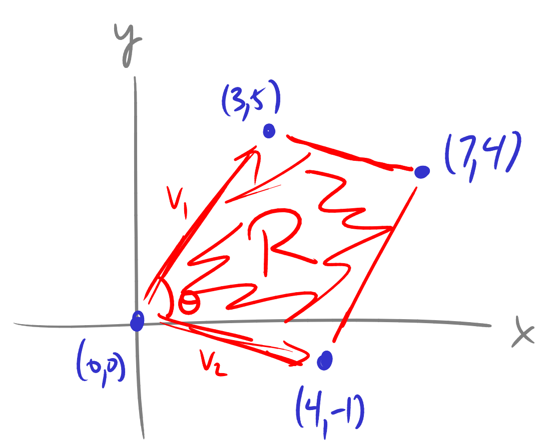
$$\iint_S F(f(u,v), g(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv = \iint_R F(x,y) \underbrace{dx\, dy}_{dA}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}$$

Remark: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} c & d \\ a & b \end{vmatrix} = -\begin{vmatrix} b & a \\ d & c \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$

Ex If R is the parallelogram

find $\iint_R 1 \cdot dA$.



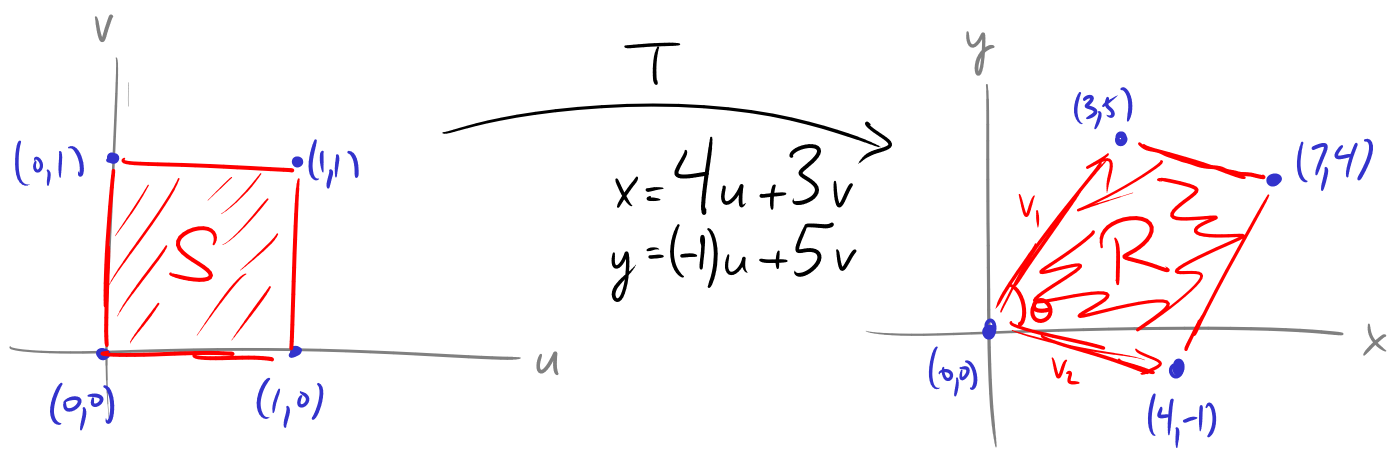
The "easy" way: the area is $\|\vec{v}_1 \times \vec{v}_2\|$ $\quad (= \|\vec{v}_1\| \cdot \|\vec{v}_2\| \cdot \sin\theta)$

$$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 5 & 0 \\ 4 & -1 & 0 \end{vmatrix} = \vec{i} \cdot \begin{vmatrix} 5 & 0 \\ -1 & 0 \end{vmatrix} - \vec{j} \cdot \begin{vmatrix} 3 & 0 \\ 4 & 0 \end{vmatrix} + \vec{k} \cdot \begin{vmatrix} 3 & 5 \\ 4 & -1 \end{vmatrix}$$

$$= 0\vec{i} + 0\vec{j} + (-23)\vec{k}$$

so $\|\vec{v}_1 \times \vec{v}_2\| = 23$ is the area.

The "hard" way, by change of variables:



to find this formula for (x,y): set $x = Au + Bv$

we know $(u,v) = (1,0) \longrightarrow (x,y) = (4,-1)$

plug this in: $(u,v) = (1,0) \longrightarrow 4 = x = A \cdot 1 + B \cdot 0 = A$

so $\underline{\underline{A = 4}}$

Using this change of vars: our integral becomes

$$\iint_R 1 \cdot dA = \int_0^1\int_0^1 1 \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv \quad \text{and} \quad \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} 4 & 3 \\ -1 & 5 \end{vmatrix} = 23$$

so get $\int_0^1\int_0^1 23\,du\,dv = \underline{\underline{23}}$ ✓

Could use this change of vars similarly to compute

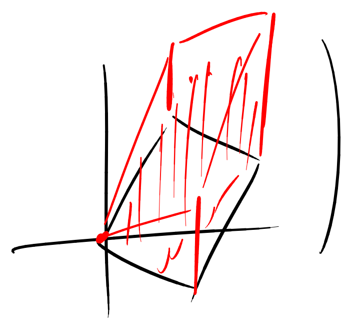
$$\iint_R x \, dA = \int_0^1 \int_0^1 (4u+3v) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du \, dv$$

$$= 23 \int_0^1 \int_0^1 4u+3v \, dv \, du$$

$$= \cdots$$

$$= 23\left(2+\frac{3}{2}\right) = \frac{161}{2}$$

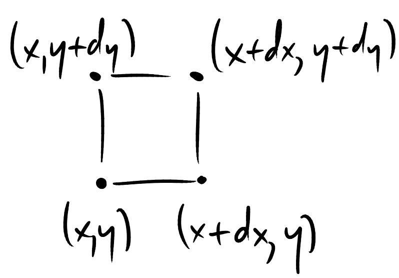
(Geometrically: this is the <u>volume</u> of a 3-d thing whose height over a point (x,y) in R is given by x.)



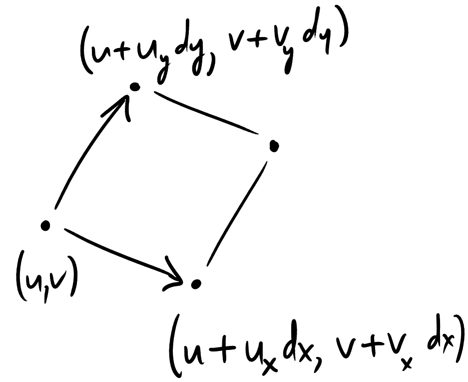
Why does the change of variable formula work?

Key point is to justify the relation $\quad dx\, dy = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$

The idea:



<u>corresponds to</u>

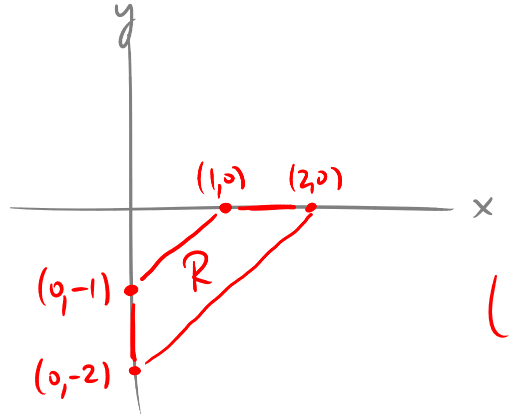


"area $= dx\, dy$"

"$du\, dv = \text{area} = \begin{vmatrix} u_x dx & v_x dx \\ u_y dy & v_y dy \end{vmatrix} = \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} dx\, dy$"

Ex

$$\iint_R e^{(x+y)/(x-y)}\,dA$$



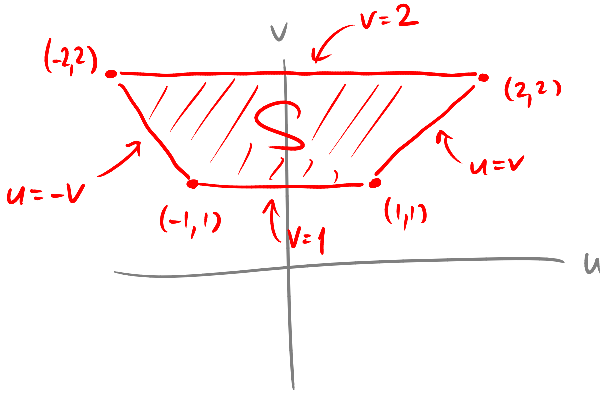
(not parallelogram this time)

Take $u = x+y$ to simplify the function.
$v = x-y$

To compute Jacobian, need to solve for (x,y): $\quad u+v = 2x$, $u-v = 2y$ ie $x = \frac{1}{2}(u+v)$, $y = \frac{1}{2}(u-v)$

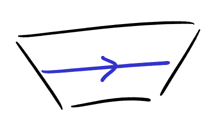
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

The domain becomes:



(x,y)	(u,v)
$(1,0)$	$(1,1)$
$(2,0)$	$(2,2)$
$(0,-1)$	$(-1,1)$
$(0,-2)$	$(-2,2)$

Integrate over S using horizontal slices:



$$\int_1^2 \int_{-v}^{v} e^{u/v} \cdot \frac{1}{2}\,du\,dv$$

$$= \frac{1}{2}\int_1^2 v e^{u/v} \Big]_{u=-v}^{u=v}\,dv$$

$$= \frac{1}{2}\int_1^2 v\left(e - e^{-1}\right)\,dv$$

$$= \frac{1}{2}(e - e^{-1}) \int_1^2 v \, dv$$

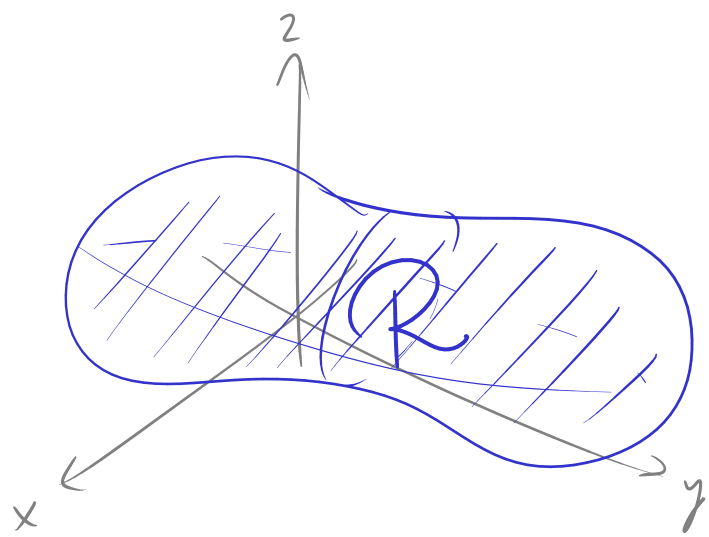
$$= \frac{1}{2}(e - e^{-1}) \left. \frac{v^2}{2} \right]_1^2$$

$$= \frac{1}{2}(e - e^{-1}) \left(2 - \frac{1}{2}\right) = \underline{\underline{\frac{3}{4}(e - e^{-1})}}$$

Remark: $\dfrac{\partial(x,y)}{\partial(u,v)} = 1 \Big/ \dfrac{\partial(u,v)}{\partial(x,y)}$ — occasionally a good check

$$\left[\text{e.g. here } \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \right]$$

Volume integrals

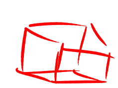


R a region in 3-dim space

function $f(x,y,z)$

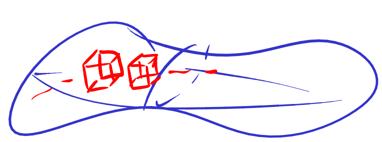
Triple integral $\displaystyle\iiint_R f(x,y,z) \, dV$

$dV = dx\,dy\,dz$



defined as limit of sums $\displaystyle\sum_i f(\underbrace{x_i, y_i, z_i}_{\text{point inside } i\text{-th cube}}) (\Delta x_i \, \Delta y_i \, \Delta z_i)$

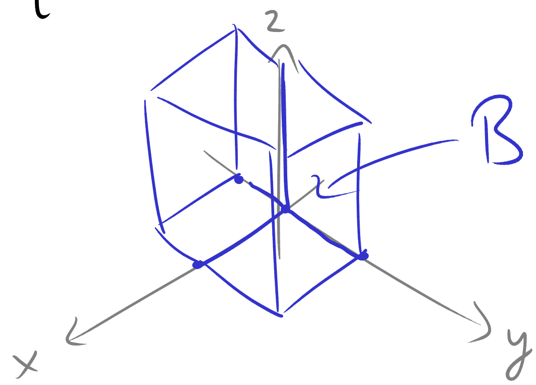
↑ volume of i-th cube



as $\Delta x_i, \Delta y_i, \Delta z_i \longrightarrow 0$.

As with double integrals, can do triple ints by iterated slicing.

<u>Ex</u> $\displaystyle\iiint_B xyz^2 \, dV \qquad B = \{0 \le x \le 1, \ -1 \le y \le 2, \ 0 \le z \le 3\}$



$$= \int_0^3 \left[\int_{-1}^2 \left[\int_0^1 xyz^2 \, dx \right] dy \right] dz$$

$$= \int_0^3 \left[\int_{-1}^2 \frac{1}{2} yz^2 \, dy \right] dz$$

$$= \int_0^3 \frac{1}{2} z^2 \cdot \left(\left[\frac{1}{2} y^2 \right]_{-1}^2 \right) dz$$

$$= \int_0^3 \frac{1}{2} z^2 \cdot \frac{3}{2} \, dz = \frac{3}{4} \int_0^3 z^2 \, dz = \frac{3}{4} \cdot 9 = \underline{\underline{\frac{27}{4}}}$$