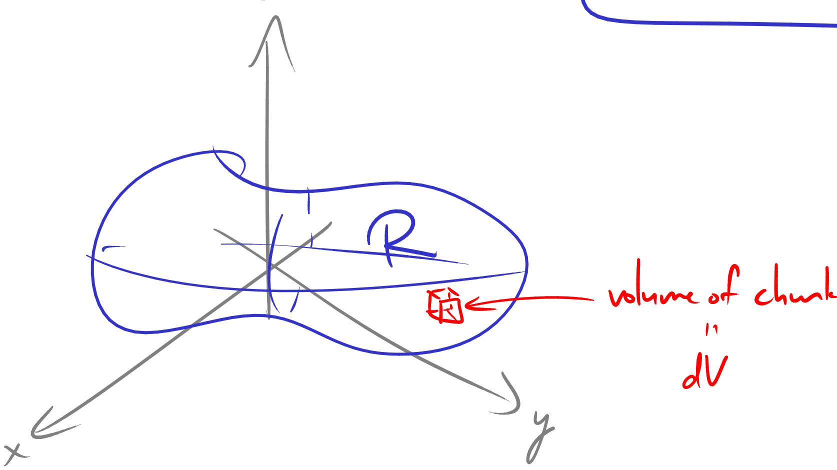
Lecture 26

4 Dec 2014

Last time: volume integrals (triple integrals)

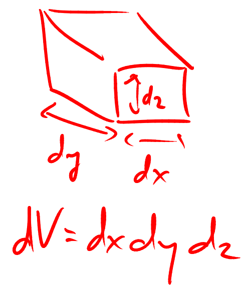
$$\iiint_R f(x,y,z)\,dV$$



Ex $B=\{0\le x\le 1,\ -1\le y\le 2,\ 0\le z\le 3\}$

$$\iiint_B xyz^2\,dV = \int_0^3\int_{-1}^2\int_0^1 xyz^2\,dx\,dy\,dz$$

$$= \cdots = \frac{27}{4}$$

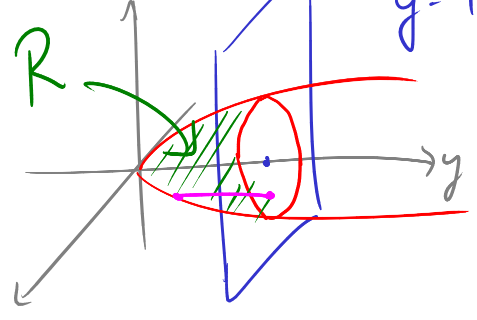


How about more complicated/interesting domains?

Ex $\displaystyle\iiint_R \sqrt{x^2+z^2}\,dV$

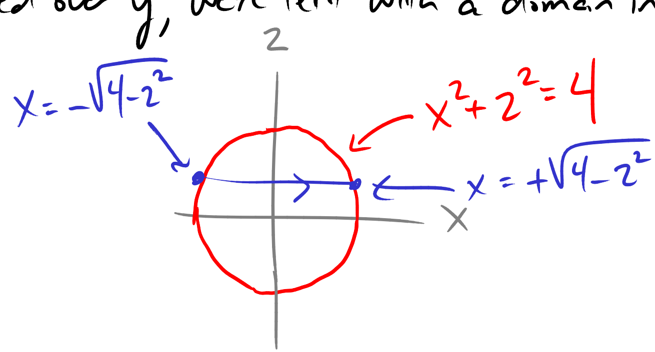
R bounded by $y=x^2+z^2$

$y=4$



$$\int_{-2}^{2}\left[\int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}}\left[\int_{x^2+z^2}^{4}\sqrt{x^2+z^2}\,dy\right]dx\right]dz$$

After we've integrated over y, we're left with a domain in the x-z plane:



Do the integral over y:

$$\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\, dy = (4-x^2-z^2)\sqrt{x^2+z^2}$$

So the original $\int$ becomes

$$\int_{-2}^{2} \int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} (4-x^2-z^2)\sqrt{x^2+z^2}\, dx\, dz$$

Could do this directly, but better to switch to polar coords:

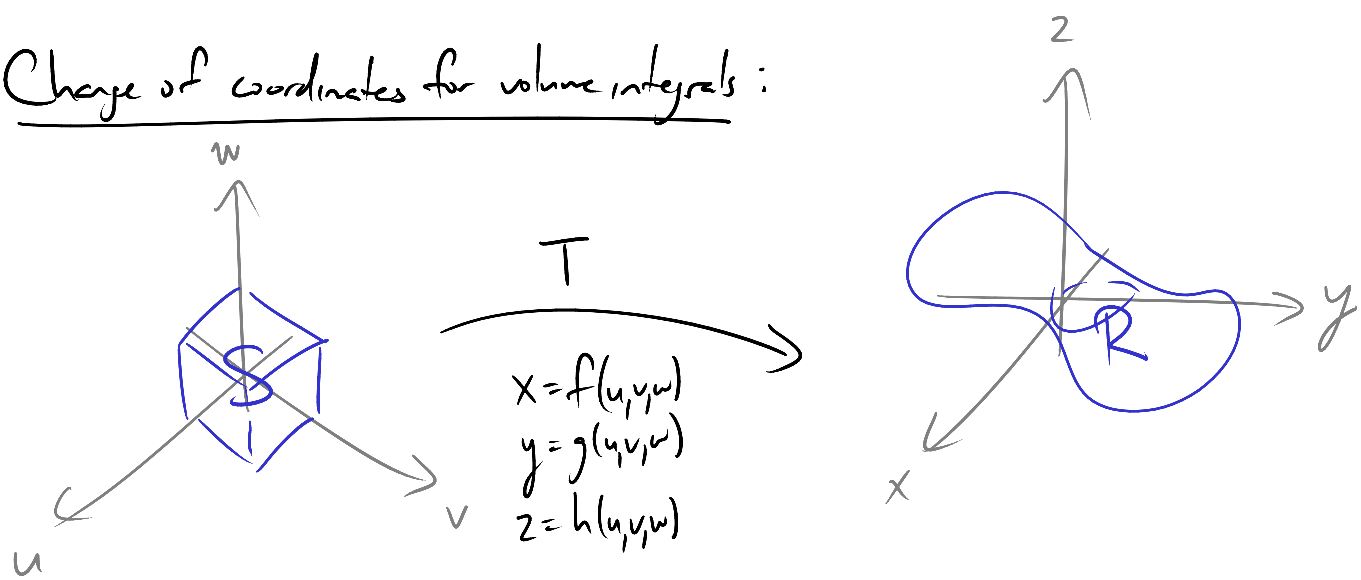
$$x = r\cos\theta$$
$$z = r\sin\theta$$
$$x^2+z^2 = r^2$$

our integral becomes

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{2} (4-r^2)\, r \quad r\, dr\, d\theta$$

$$= 2\pi \int_0^2 4r^2 - r^4\, dr = \cdots = \frac{128\pi}{15}$$

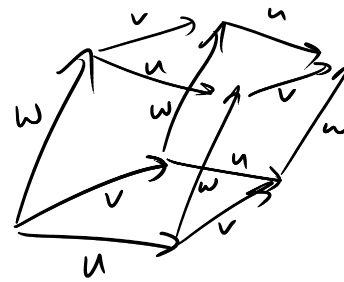
Change of coordinates for volume integrals:



Key fact: $$dx\,dy\,dz = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

where $$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{vmatrix}$$

(And recall: $\begin{vmatrix} v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \end{vmatrix}$ is the volume of



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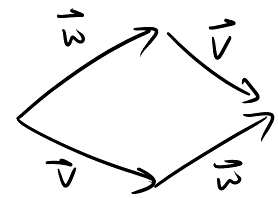
(cf. our proof of the change of coords formula in 2 dim, used $\begin{vmatrix} v_1 & v_2 \\ w_1 & v_2 \end{vmatrix}$ = area of



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Ex $x = v + w^2$
$y = w + u^2$
$z = u + v^2$

then $$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 0 & 1 & 2w \\ 2u & 0 & 1 \\ 1 & 2v & 0 \end{vmatrix}$$

$$= 0 \cdot \begin{vmatrix} 0 & 1 \\ 2v & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} 2u & 1 \\ 1 & 0 \end{vmatrix} + 2w \cdot \begin{vmatrix} 2u & 0 \\ 1 & 2v \end{vmatrix}$$

$$= 0 + 1 + 8uvw$$

So, $$dx\,dy\,dz = |1 + 8uvw|\,du\,dv\,dw$$

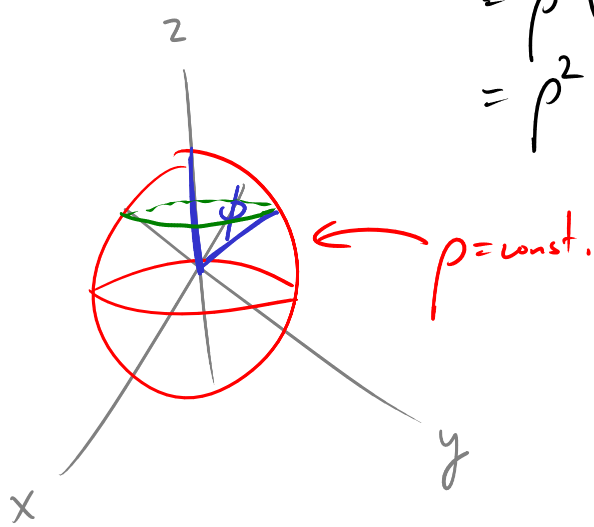
Ex Spherical coordinates $\quad (u,v,w) = (\rho, \theta, \phi)$

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

Interpretation: $x^2+y^2+z^2 = \rho^2(\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta + \cos^2\phi)$

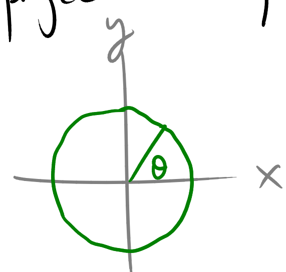
$$= \rho^2(\sin^2\phi + \cos^2\phi)$$
$$= \rho^2$$

so $\rho = \sqrt{x^2+y^2+z^2}$ distance from origin



and ϕ = angle from z-axis

θ = position on the circle projected to x-y plane



$$0 \le \rho \le \pi$$
$$0 \le \theta \le 2\pi$$
$$0 \le r$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\theta & 0 & -\rho\sin\phi \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & 0 & -\rho\sin\phi \\ \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \end{vmatrix}$$

$$= \cdots = -\rho^2 \sin\phi$$

thus $dV = dx\,dy\,dz = \rho^2 \sin\phi \; d\rho\, d\theta\, d\phi$

<u>Final exam</u>: Sat Dec 13 9am–12pm 25 problems

weighted toward material from <u>after</u> midterm 2

Given $f(x,y)$ how to find local minima?

$$f(x,y) = 1 - x^2 - 4y^3 + 4y$$

Critical points: $\vec{\nabla} f = 0$

$$\vec{\nabla} f = \langle -2x,\ -12y^2+4 \rangle = \langle 0,0 \rangle$$

$\longrightarrow$ $x=0,\ 12y^2-4=0$ $\quad 12y^2=4$, $\ y=\pm\frac{1}{\sqrt{3}}$

$\longrightarrow$ crit pts at: $(x,y) = \left(0, -\frac{1}{\sqrt{3}}\right)$

$(x,y) = \left(0, +\frac{1}{\sqrt{3}}\right)$

2nd deriv test: need $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -2 & 0 \\ 0 & -24y \end{vmatrix} = 48y$

$f_{xx} = -2 \qquad f_{xy} = 0 \qquad f_{yy} = -24y$

At $(x,y) = (0, +\frac{1}{\sqrt{3}})$: $D = 48 \cdot \frac{1}{\sqrt{3}} = \frac{48}{\sqrt{3}} > 0$

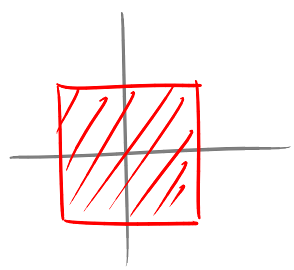
Since $D > 0$ and $f_{xx} < 0$, this is a local max

At $(x,y) = (0, -\frac{1}{\sqrt{3}})$: $D = 48 \cdot \frac{-1}{\sqrt{3}} < 0$

this is a saddle point

$\Rightarrow$ If we want to find minimum value of f

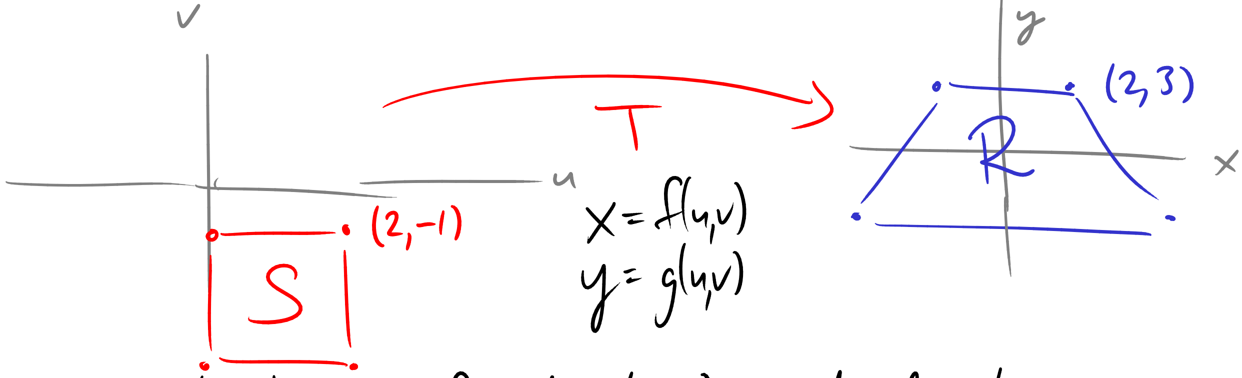
on $\{|x| \leq 1, |y| \leq 1\}$



then it will have to be somewhere on the boundary

Finding limits after a change of variables:

Simple cases: linear change of vars acting on linear domain



$x = f(u,v)$
$y = g(u,v)$

f, g linear: find the (u,v) coords of vertices

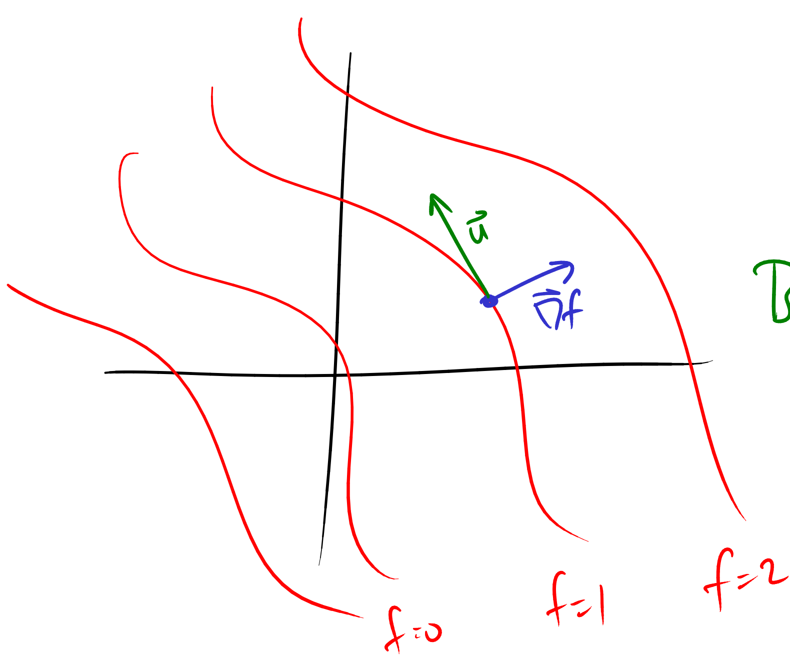
e.g. if $x = 3u + 4v$
$y = u - v$

vertex $(x,y) = (2,3)$

maps to (u,v): $2 = 3u + 4v$
$3 = u - v$

$14 = 7u \quad u = 2 \quad v = -1$

$(u,v) = (2,-1)$



$\vec{u}$ tangent to contour line:

then $D_{\vec{u}} f = 0$

But $D_{\vec{u}} f = \lim_{h \to 0} \frac{f(\vec{r} + \vec{u}) - f(\vec{r})}{h}$

$= \vec{u} \cdot \vec{\nabla} f$

So $\vec{u} \perp \vec{\nabla} f$