Many problems in this course will simply be statements. In such a case the problem is always to prove it.

**Problem 1.** There is no integer solution to \( x^2 + y^2 = -73 \). (The math here is easy; the point is to practice writing it up as a formal proof. You may use elementary properties of arithmetic, if you clearly state what you are using.)

**Problem 2.** We showed in class that there is no integer solution to \( x^2 - 2y^2 = 0 \) other than \( x = y = 0 \). Generalize that result by proving that there is no rational solution except \( x = y = 0 \). (Hint: this problem is meant to give you practice writing a proof by contradiction. Try to follow the proof structure of my model. You can use standard properties of rational numbers.)

**Problem 3.** Find all integer solutions to \( x^2 + y^2 + z^2 = 51 \). (Hint: this is meant to provide practice using “without loss of generality”. There are in fact several different levels of “w.l.o.g” that you can use.)

**Problem 4.** We saw in class that division doesn’t play well with congruences; this problem shows that division can sometimes be used. In this problem \( a, b, c \) and \( m \) are integers.

(a) Give an example, different from the one in class, that shows that \( c | a \) and \( c | b \) and \( a \equiv b \mod m \) does not necessarily imply that \( \frac{a}{c} \equiv \frac{b}{c} \mod m \).

(b) If \( c | a \) and \( c | b \) and \( c | m \), then \( a \equiv b \mod m \) **DOES** imply something slightly different, namely \( \frac{a}{c} \equiv \frac{b}{c} \mod \frac{m}{c} \). (This is something we will use repeatedly later on, so please remember it.)

**Problem 5.**

(a) Check that none of \( x = 0, 1, 2, 3, 4 \) satisfy the congruence \( x^2 \equiv 2 \mod 5 \).

(b) Deduce that there is no integer \( x \) satisfying that congruence.

(c) Deduce that there is no integer \( x \) satisfying \( x^2 = 137 \).

**Problem 6.**

(a) Check that no choice of \( x = 0, 1, 2, 3, 4 \) and \( y = 0, 1, 2, 3, 4 \) satisfies the congruence \( x^2 - 5y^2 \equiv 2 \mod 5 \). (This looks like it requires 25 cases, but it is nowhere near that bad.)

(b) Deduce that there are no integers \( x \) and \( y \) satisfying that congruence.

(c) Deduce that there are no integers \( x \) and \( y \) satisfying \( x^2 - 5y^2 = 137 \).

(You probably were not impressed by (c) in the previous problem, because the fact \( \sqrt{137} \notin \mathbb{Z} \) is not new to you. I hope you are more impressed by this (c). In particular, without the buildup of (a) and (b), it is surprising that one can prove something like this. This problem’s “mod 5 obstruction” to the existence of a solution is the best way to understand the lack of solutions to this particular equation. Note also that you can’t use the same argument I did with \( x^2 - y^2 = 2 \) on day 1 in class, since you can’t factor \( x^2 - 5y^2 \).)