Problem 1. Find the prime factorizations of the following numbers.
(a) 111, 111
(b) 196, 560
(c) 16!
(d) \(\binom{30}{10}\) (meaning the binomial coefficient \(30! / 20! 10!\))

Problem 2. Find the gcd and lcm of each pair. Careful—there is at least one trap.
(a) \(2 \cdot 3^2 5^3\) and \(2^7 3^5 5^3 7^2\)
(b) \(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13\) and \(17 \cdot 19 \cdot 21 \cdot 23 \cdot 29\)
(c) \(2^5 5^7 11^{13}\) and \(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13\)
(d) \(47^{1179111101}\) and \(41^{1183111101}\)

Problem 3. Suppose \(s\) and \(t\) are coprime integers, both odd, and that \(s > t \geq 1\).
At the end of this problem you will have proven that the three numbers
\(a = st\), \(b = (s^2 - t^2) / 2\) and \(c = (s^2 + t^2) / 2\) have gcd equal to 1. The importance of these
triples \((a, b, c)\) is explained in the next problem. Expect to use prime factorization
several times in this problem.
(a) \(a\) is obviously an integer; show that \(b\) and \(c\) are too. (This is needed for the
rest of the problem to make sense.)
(b) Show that 2 does not divide all three of \(a\), \(b\) and \(c\).
(c) Suppose \(p\) is an odd prime. Show that it cannot divide both \(b\) and \(c\). (In
particular, if it cannot divide all three of \(a\), \(b\) and \(c\).)
(d) \(a\), \(b\), \(c\) have gcd equal to 1.

Problem 4. We continue in the situation of the previous problem.
(a) show that \(a^2 + b^2 = c^2\). Triples of integers \((a, b, c)\) satisfying this equation
are called Pythagorean triples. So you now have a way to generate Pythagorean
triples.
(b) Show \(a, b, c > 0\). (This is a minor point; informally it means these triples
have “already been cleaned up” by having any negative signs flipped.)
(c) Explicitly write out the Pythagorean triple corresponding to \(s = 15\), \(t = 13\),
and check on your calculator that it really works.
(d) The most famous Pythagorean triple is \((3, 4, 5)\). What are the \(s, t\ corre-
sponding to it?
(e) I once stumbled across the Pythagorean triple \((99, 20, 101)\) accidentally while
laying out a small deck outside my home. Repeat (d) for this triple.
(In fact it turns out that all Pythagorean triples arise this way. And the formulas
for \(a\), \(b\) and \(c\) in terms of \(s\) and \(t\) did not just fall from the sky, even though it looks
like that from how I wrote them down. There is lovely geometry behind this, which
we might cover if the class is interested. Truth in advertising: my use of the word
“all” requires some “without loss of generality” preparation.)

The next two problems use a number system I will call the fivish integers \(\mathbb{F}\).
This isn’t really their name; in fact they are such an ordinary number system that
they don’t really have a name. The definition is similar to the Gaussian integers,
but different. A fivish integer means an ordered pair \((x, y)\) of ordinary integers.
The sum and product are defined by \((x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)\) and
\((x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - 5y_1 y_2, x_1 y_2 + x_2 y_1)\). One can check that these operations
satisfy the commutative, associative and distributive laws. (You don’t have to check
this, but if you are curious you will find it very similar to the Gaussian case.)

**Problem 5.** (a) For $n \in \mathbb{Z}$ one usually thinks of $(n, 0)$ as being the same as $n$; verify
that this is reasonable by checking that $(n, 0) \cdot (x, y) = (nx, ny)$ for all $(x, y) \in \mathbb{F}$.

(b) One abbreviates $(0, 1)$ as $\sqrt{-5}$ (NOT $i\sqrt{5}$); verify this is reasonable by check-
ing that $(0, 1)^2 = -5$.

(c) You are now free to write a fivish integer $(x, y)$ as $x + y\sqrt{-5}$, and compute with
them in the obvious way. Use this to check that $2 \cdot 3 = 6 = (1 + \sqrt{-5}) \cdot (1 - \sqrt{-5})$.

**Problem 6.** (a) If $\alpha = (x, y)$ is a fivish integer, then we define its “norm” $N(\alpha)$
as $x^2 + 5y^2$. WARNING: this is DIFFERENT from the norm we used for Gauss-
ian integers. The fivish integers are are different from the Gaussian integers, so
you shouldn’t be surprised that their norms are different. Show that the norm is
multiplicative: if $\alpha, \beta \in \mathbb{F}$ then $N(\alpha\beta) = N(\alpha)N(\beta)$.

(b) We define divisibility in $\mathbb{F}$ is just like in $\mathbb{Z}$: if $\alpha, \beta \in \mathbb{F}$ then we say that $\alpha|\beta$
if there exists $\gamma \in \mathbb{F}$ such that $\alpha\gamma = \beta$. Find all the divisors of 2 in $\mathbb{F}$. (Hint: (a).)

(c) Repeat (b) with $3, 1 + \sqrt{-5}$ and $1 - \sqrt{-5}$ in place of 2.

Remarks: although we have not defined “prime” in $\mathbb{F}$, I am sure that after doing
(b) and (c) you will think it’s reasonable to call 2, 3 and $1 \pm \sqrt{-5}$ primes in $\mathbb{F}$. The
point of this problem is to see that the two factorizations of 6 in part (c) of the
previous problem are factorizations into primes. So $\mathbb{F}$ is a number system in which
a particular number can have two essentially different prime factorizations.