The Finite Reflection Groups

We classify the finite reflection groups. Our treatment has several advantages over some other treatments—in particular, we avoid computing determinants and the use of the Perron-Frobenius Theorem. The ideas here can be found spread across several sections of Coxeter’s *Regular Polytopes*. The only thing missing from our treatment is a construction of the finite groups.

By the norm \( v^2 \) of a vector \( v \), we mean \( v^2 = v \cdot v \); some people call this the squared norm of \( v \).

1 Preliminaries

A reflection is an isometry of Euclidean space \( V \) that leaves a hyperplane (its mirror) fixed pointwise and exchanges the two components of its complement. A reflection group is a group generated by reflections. Suppose \( W \) is a finite reflection group. \( W \) stabilizes some point of Euclidean space (say, the centroid of the orbit of any point), which we will take to be the origin. \( W \) contains only finitely many reflections, and the complement in \( V \) of the union of the mirrors falls into finitely many components. We call the closure of any one of these components a Weyl chamber (or just a chamber). A mirror \( M \) is said to bound a chamber \( C \) if \( C \cap M \) has the same dimension as \( M \). The walls of \( C \) are the mirrors that bound \( C \). A root of \( W \) is a vector \( r \) of norm 2 that is orthogonal to some mirror \( M \) of \( W \); we sometimes refer to the reflection across \( M \) as the reflection in \( r \).

We choose one chamber and call it \( D \). For each wall \( M \) of \( D \) we choose the root associated to \( M \) which has positive inner product with each element of the interior of \( D \). We denote these vectors by \( r_1, \ldots, r_n \) and call them the simple roots of \( W \). We write \( R_i \) for the reflection in \( r_i \), which negates \( r_i \) and fixes \( r_j \) pointwise.

**Lemma 1.1.** The \( R_i \) generate \( W \), which acts transitively on its Weyl chambers.

**Proof:** We say that 2 chambers \( C_1, C_2 \) are neighbors if they are both bounded by the same mirror \( M \) and \( C_1 \cap M = C_2 \cap M \); in this case \( C_1 \) and \( C_2 \) are exchanged by the reflection across \( M \). It is easy to see that any two chambers are equivalent under the equivalence relation generated by the relation of neighborliness. (Proof: choose points in the interiors of the 2 chambers in sufficiently general position that the segment joining them never meets an intersection of 2 mirrors. The sequence of chambers that this segment passes through provides a sequence of neighbors.)

If a subgroup \( G \) of \( W \) contains the reflections in the walls of a chamber \( C_1 \), and \( C_2 \) is a neighbor of \( C_1 \), then \( G \) also contains the reflections in the walls of \( C_2 \). Here’s why: letting \( R \) be the reflection across the common wall of \( C_1 \) and \( C_2 \), we have \( R \in G \) and we observe that the reflections in the walls of \( C_2 \) are the conjugates by \( R \) of those in the walls of \( C_1 \).

Letting \( G \) be the group generated by \( R_1, \ldots, R_n \), we see that \( G \) contains the reflections in the walls of the neighbors of \( D \), and of their neighbors, and so on. That is, \( G \) contains all the reflections of \( W \), so equals \( W \). Since any two neighboring chambers are equivalent under \( W \), we also see that \( W \) acts transitively on chambers. \( \square \)

Consider the subgroup \( H \) of \( W \) generated by the reflections in a pair of distinct simple roots \( r_i \) and \( r_j \). In this paragraph we will restrict our attention to the span of \( r_i \) and \( r_j \), since \( H \) acts trivially on \( r_i^+ \cap r_j^+ \). Consider the chambers of \( H \); these are even in number since each reflection of \( H \) permutes them freely. Furthermore, lemma 1.1 shows that they are all equivalent under \( H \). Letting \( 2n_{ij} \) be the number of Weyl chambers, we deduce that the 2 mirrors bounding any chamber
meet at an angle of $\pi/n_{ij}$. Because no mirror of $H$ can cut the Weyl chamber $D$ of $W$, the mirrors of $R_i$ and $R_j$ must bound the same chamber of $H$, so their interior angle is $\pi/n_{ij}$. Picture-drawing in the plane allows us to determine the angle between $r_i$ and $r_j$, and we find

$$r_i \cdot r_j = -2\cos(\pi/n_{ij}).$$

We have already made the choice $r_i \cdot r_j = 2$, so we set $n_{ij} = 1$ to be consistent with (1.1). Note that the integers $n_{ij}$ determine $W$: the mutual inner products of any set of vectors in Euclidean space determines them (up to isometry), so the $n_{ij}$ determine the $r_i$, which determine the $R_i$, which by lemma 1.1 determine $W$.

A Coxeter diagram (sometimes just called a diagram) is a simplicial graph with each edge labeled by an integer $> 2$. The Coxeter diagram $\Delta_W$ of $W$ is the diagram whose vertices are the $r_i$, with $r_i$ and $r_j$ joined by an edge marked with the integer $n_{ij}$ when $n_{ij} > 2$. This definition depends on our choice $D$ of Weyl chamber, but the transitivity of $W$ on its chambers shows that a different choice of chamber leads to essentially the same diagram. We may recover the $n_{ij}$ from $\Delta_W$, so $\Delta_W$ determines $W$. For simplicity, when drawing a Coxeter diagram one omits the numeral 3 from edges that would be so marked.

## 2 Controlling $\Delta$.

**Lemma 2.1.** Suppose $v \in V$ with $v = \sum_{i=1}^{n} v_ir_i$. If $v_i \geq 0$ and are not all 0, then $v^2 > 0$.

**Proof:** Since each $r_i$ has positive inner product with each element of the interior of $C$, we does $v$. Thus $v \neq 0$ and so $v^2 > 0$.

A subdiagram of a Coxeter diagram $\Delta$ is a diagram whose vertex set is a subset of that of $\Delta$, whose edge set consists of all edges of $\Delta$ joining pairs of these vertices, and whose edges are marked by the same numbers as in $\Delta$. If $\Delta$ and $\Delta'$ are Coxeter diagrams with the same vertex set and with edge markings $m_{ij}$ and $n_{ij}$, respectively, then we say that $\Delta'$ is an increasement of $\Delta$ if $n_{ij} \geq m_{ij}$ for all $i$ and $j$. In terms of the diagrams, $\Delta'$ is a (strict) increasement of $\Delta$ if $\Delta'$ can be obtained from $\Delta$ by increasing edge labels or adding edges.

**Lemma 2.2.** No diagram appearing in table 1 or table 2, nor any increasement of one, may appear as a subdiagram of $\Delta_W$.

**Proof:** Let $\Delta$ be a diagram from one of the tables, and $\Delta'$ an increasement of $\Delta$ that is a subdiagram of $\Delta_W$. Identifying the vertices of $\Delta$ and $\Delta'$ with (some of) the simple roots $r_i$, we may construct the vector $v = \sum_i v_i r_i$, where $v_i$ is the (positive) number adjacent to the vertex $r_i$ on the table. One may compute the norm of $v$ from knowledge of the edge labels $n_{ij}$ of $\Delta' \subseteq \Delta_W$. If the edge labels of $\Delta$ are $m_{ij}$ then

$$v^2 = \sum_{ij} -2v_i v_j \cos(\pi/n_{ij}) \leq \sum_{ij} -2v_i v_j \cos(\pi/m_{ij}),$$

the last inequality holding because $\Delta'$ is an increasement of $\Delta$. In each case, computation reveals that the right hand side of (2.1) is at most 0, contradicting lemma 2.1. For reference, $-2\cos(\pi/n)$ equals 0, $-1$, $-\sqrt{2}$, $-\phi$ and $-\sqrt{3}$, for $n = 2, 3, 4, 5$ and 6, respectively, and $\phi = (1 + \sqrt{5})/2 = 1.618\ldots$ is the golden mean.

The computations are not even very tedious. For $\Delta = H_3$ or $H_4$ they are simplified by using the fact $\phi^2 = \phi + 1$. In all other cases (i.e., with $\Delta$ from table 1), the right hand side of (2.1) vanishes; to prove this one may compute inner products with the $n_{ij}$ replaced by the $m_{ij}$ and show that $v$ is orthogonal to each $r_i$. Almost all cases are resolved by the following observation: if all the edges of $\Delta$ incident to $r_i$ are marked 3 then $v \cdot r_i = 0$ just if twice the $r_i$ label equals the sum of the labels of its neighbors.
Table 1. A list of “affine” Coxeter diagrams. The numbers next to the vertices are used in the proof of lemma 2.2. A diagram $X_n$ has $n + 1$ vertices.
Table 2. Two examples of “hyperbolic” Coxeter diagrams. The numbers next to the vertices are used in the proof of lemma 2.2; $\phi = (1 + \sqrt{5})/2$ is the golden mean.

Table 3. A complete list of possible connected components of Coxeter diagrams of finite reflection groups. (See theorem 3.1.) A diagram $x_n$ has $n$ vertices.
3 The Classification

In light of the fact that $\Delta_W$ determines $W$, the following theorem classifies the finite reflection groups.

**Theorem 3.1.** If $W$ is a finite reflection group, then $\Delta_W$ is a disjoint union of copies of the Coxeter diagrams appearing in table 3.

**Proof:** (This is the usual combinatorial argument.) Let $\Delta$ be a connected component of $\Delta_W$. $\Delta$ can contain no cycles, else the subdiagram spanned by the vertices of a shortest cycle would be an increasement of $A_n$ for some $n$. We will express this sort of reasoning by statements like “By $A_n$, $\Delta$ contains no cycles.”

Suppose that an edge of $\Delta$ has marking $p \geq 4$. By $B_n$, $\Delta$ contains just one edge so marked. By $BD_n$, $\Delta$ has no branch points, so $\Delta$ is a simple chain of edges. By $G_2$, if $p > 5$ then the edge is the whole of $\Delta$, so $\Delta$ is $i_2(p)$. If $p = 5$ then by $H_3$ the edge must be at an end of $\Delta$, and then by $H_4$, $\Delta$ must have fewer than 4 edges. We deduce that if $p = 5$ then $\Delta$ is $i_2(5)$, $h_3$ or $h_4$. If $p = 4$ and the edge is not at an end of $\Delta$ then by $F_4$ we have $\Delta = f_4$. If $p = 4$ and the edge is at an end of $\Delta$ then $\Delta = b_n$ for some $n$.

It remains to consider the case in which all edge labels are 3. If $\Delta$ has no branch points then $\Delta = a_n$ for some $n$. By $D_4$, each branch point of $\Delta$ has valence 3, and by $D_n$ (for $n > 4$), $\Delta$ has at most one branch point. Therefore it suffices to consider $\Delta$ with exactly one branch point, of valence 3. By a ‘leg’ of $\Delta$ we mean one of the 3 subgraphs of $\Delta$ consisting of the edges of the path in $\Delta$ joining the branch point to one of the 3 endpoints of $\Delta$; the length of the leg is the number of these edges. Let $\ell_1$, $\ell_2$, $\ell_3$ be the lengths of the legs, with $\ell_1 \leq \ell_2 \leq \ell_3$. By $E_6$, $\ell_1 = 1$. If we also have $\ell_2 = 1$ then $\Delta = d_n$ for some $n$. If $\ell_2 > 1$ then by $E_7$ we have $\ell_2 = 2$ and then by $E_8$ we have $\ell_3 < 5$, so $\Delta$ is one of $e_6$, $e_7$ and $e_8$. \qed