

Research Summary

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I have tried to make this summary readable by any mathematician, and the first paragraph of each section mostly readable by any scientist. First I discuss my past research, and then starting on page 10 my current research projects. These include discussion of my published papers relevant to them, namely [12][15][16][18][28].

My field is group theory (the study of symmetry) and its applications outside group theory. For me this has mostly meant applications in algebraic geometry, some by myself and some jointly with several different coauthors. Most notable is a long series of papers concerning the spaces of deformations of various classical geometric objects like curves in the plane, or cubic surfaces. I have also contributed to a number of problems in topology and pure group theory, for example hyperbolic reflection groups and the topology of their associated hyperplane arrangements. My work in algebraic geometry and reflection groups led to my current research project concerning the monster finite simple group, described in section 6.

In 2006 I began moving into a new field: representation theory. In a sense this is still group theory: the action of symmetry groups on vector spaces. But it is a huge field with very specialized tools, many having nothing to do with group theory. I laid out my approach to some of the important objects, “linear algebraic groups,” in a graduate topics course. This was original enough that it led to a research paper, and I’ve been encouraged to develop my notes into a monograph. I am now well advanced in a large research project: the classification of “Lorentzian Lie algebras.” This is the same sort of problem as one of the most influential results in mathematics: the classification of continuous symmetry groups by Killing and Cartan. I expect that my classification will be a valuable supplement to theirs; it will include most of the Lie algebras already recognized as important in mathematics and physics. Please see section 7 for more information.

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1. ALGEBRAIC GEOMETRY.

In almost every problem in algebraic geometry, the objects under study may vary their shapes continuously. The moduli space of whatever objects one has in mind means the set of shapes that can arise by such deformations. One regards two deformations as “the same” if they differ only in some uninteresting way, such as a change of coordinates.

My work in algebraic geometry, largely joint with J. Carlson and D. Toledo, centers on describing moduli spaces as quotients of symmetric spaces by discrete groups. The classical example is that the family of all genus one curves can be described as the points of the open unit disk in the complex plane, with two points identified if they differ by translation by an element of the group $SL_2\mathbb{Z}$. There are other results of this sort, each giving special insight into a particular moduli space. But our results came as a surprise and sparked similar research by other researchers in algebraic geometry.

We studied the moduli spaces of cubic surfaces in (projective) 3-space and cubic threefolds in (projective) 4-space, meaning the hypersurfaces defined by a single homogeneous cubic polynomial in 4 or 5 variables. Two are considered equivalent if they differ by a symmetry of projective space.

Theorem 1 ([20],[21]). *The moduli space of smooth cubic surfaces in projective 3-space $\mathbb{C}P^3$ is isomorphic to the unit ball B^4 in complex 4-space, minus a hyperplane arrangement, modulo the action of a certain discrete group Γ . The isomorphism is an isomorphism of complex analytic orbifolds, and the group Γ is the projective isometry group of a 5-dimensional lattice over $\mathbb{Z}[\sqrt[3]{1}]$, namely the one with the inner product*

$$\langle x|y \rangle = -x_0\bar{y}_0 + x_1\bar{y}_1 + \cdots + x_4\bar{y}_4.$$

Theorem 2 ([26]). *The same result holds for hypersurfaces in $\mathbb{C}P^4$, with B^4 replaced by B^{10} and Γ replaced by the isometry group of a certain 11-dimensional lattice over $\mathbb{Z}[\sqrt[3]{1}]$.*

We actually proved much more than this, giving precise descriptions of Γ , the hyperplane arrangements, and the nature of degenerations to singular hypersurfaces. Indeed, the hard part of the proofs concerned what happens when the hypersurfaces are allowed to acquire singularities. Even the precise meaning of “the moduli space” requires careful

analysis with a tool called geometric invariant theory, which I carried out for cubic threefolds in a separate paper [11]. The proofs in [21] and [26] used Hodge theory, hyperbolic analysis (in the sense of Kobayashi), group theory, and topological arguments on the nature of the locus of non-smooth hypersurfaces.

Our work is a part of a long tradition of ball quotients in algebraic geometry, going back to Schwartz and Picard in the 19th century and continued by Deligne and Mostow [41] and Thurston [72]. Our work in turn inspired further work, notably the work of Dolgachev and Kondō [42] on moduli spaces of certain kinds of K3 surfaces, the work of Kondō on the moduli spaces of curves of genus three [51] and four [52], the work of Heckmann and Looijenga [49] on moduli of del Pezzo surfaces of degree one, and very recent work of Looijenga [57] and Laza [55][56] on moduli of cubic hypersurfaces in $\mathbb{C}P^5$. Also, Looijenga and Swierstra [58] gave an independent proof of theorem 2.

We then studied the moduli space of *real* cubic surfaces, which is closer to everyday intuition (one can visualize them) but quite different in flavor from the complex case.

Theorem 3 ([27]). *The moduli space of smooth cubic surfaces in real projective 3-space is isomorphic to real hyperbolic 4-space H^4 , minus an arrangement of 2- and 3-dimensional subspaces, modulo the action of a certain nonarithmetic discrete group $\Gamma^{\mathbb{R}}$.*

Again we provided much more detailed information than stated here. I have phrased the theorem to bring out the analogies with the complex case, but there are major differences. The biggest difference is the nonarithmetic nature of $\Gamma^{\mathbb{R}}$. It is provably impossible to give a neat description like we did above, as the symmetry group of a lattice. This is related to the fact that the real moduli space has 5 components, which we glued together. Some of our techniques were known (e.g., [65], [40], [75]), but the gluing process was totally new. It provides the first appearance “in nature” of a famous construction of Gromov and Piatetskii-Shapiro [48]. In addition to the main paper [27] and announcement [23], we have written two expository notes [24], [25] treating similar but simpler situations. These include the moduli space of real 6-tuples in the Riemann sphere. Our work was followed up by Chu’s investigation [38] of real 8-tuples in the Riemann sphere.

For my paper with E. Freitag on automorphic forms [28], please see section 7. Finally, in [9] I simplified the Horikawa-Namikawa description of the moduli space of Enriques surfaces [50][62]. A simple lattice-theory trick allowed a cleaner statement of the main theorem, and replaced some difficult computations by almost trivial ones.

2. GROUPS GENERATED BY REFLECTIONS

Reflection across a line in the plane is a simple operation familiar to everyone. When one chooses several lines and combines their reflections, beautiful patterns appear. This is how kaleidoscopes work and is the basis of many of Escher's woodcuts. Escher even used reflections across lines in the hyperbolic (=non-Euclidean) plane. One can also work in higher dimensions, reflecting across planes or hyperplanes. Besides being beautiful, and useful to crystallographers, groups generated by reflections play a fundamental role in Lie theory (=continuous symmetry). Exploiting this connection is part of my current project to classify the "Lorentzian Lie algebras", described in section 7.

My simplest-to-state work in this area is the construction of a large family of Coxeter groups, meaning discrete groups generated by reflections across hyperplanes.

Theorem 4 ([14]). *For every $n \leq 19$, there are infinitely many essentially distinct Coxeter groups acting on hyperbolic space H^n with finite-volume fundamental domain. With the possible exceptions $n = 16, 17$, the number of such groups grows at least exponentially with volume.*

This is surprising, because in large dimensions ($n > 9$ or so) it becomes very hard to construct Coxeter groups with finite-volume fundamental domain. In fact only a single example is known for $n > 19$, and in very large dimension ($n \geq 996$) there are none at all by a famous result of Prokhorov [68]. My method is a simple trick, but using it effectively required some careful computations using the Leech lattice (a famous arcane object).

Complex reflection groups are also very important, but not for Lie-theoretic reasons. The key point is that when one takes a ramified covering space of a complex manifold, the group of covering transformations is often generated by complex reflections. This principle is lurking behind the scenes in most of my algebra-geometric work. In

[8] I constructed many previously unknown reflection groups acting on complex hyperbolic space $\mathbb{C}H^n$. There was some overlap with those previously discovered by Deligne and Mostow [41], but my largest examples (in $\mathbb{C}H^5$ and $\mathbb{C}H^7$) were new. The idea was to study the symmetries of carefully chosen lattices over rings like $\mathbb{Z}[i]$ and $\mathbb{Z}[\sqrt[3]{1}]$.

Later I improved my methods to construct an example in $\mathbb{C}H^{13}$, which is still the record-holder for dimension [7]. The proof used the Leech lattice, and led to my conjecture concerning the monster simple group (see section 6). In these papers I also constructed the first known reflection groups acting on quaternionic hyperbolic space (up to 7 quaternionic dimensions). And in [4] I even found reflection groups acting on the Cayley hyperbolic plane; these are discrete subgroups of the exceptional Lie group $F_{4(-20)}$.

3. TOPOLOGY OF HYPERPLANE COMPLEMENTS

Removing hyperplanes from Euclidean space leaves one with convex sets, which contract to points and hence have no interesting topology. But *complex* hyperplanes have real codimension 2, so removing them from a complex vector space leaves a space which might be very complicated. There is an industry devoted to studying this (see [66] and its references), but my focus is on a very specific part of it: hyperplane arrangements that arise in nature. For me, that means in moduli problems in algebraic geometry.

For example, theorem 1 above describes the moduli space of smooth cubic surfaces in terms of a hyperplane arrangement in the complex 4-ball. Now the complex 4-ball is one guise of complex hyperbolic 4-space, which is a negatively curved manifold. I was able to use this negative curvature, restricted to the complement of the hyperplanes, to prove that the moduli space is aspherical: it has no homotopy groups except for its fundamental group. Using my paper [9], the same conclusion also applies to the moduli space of Enriques surfaces. Both results follow from the geometry of specific hyperplane arrangements and the following theorem, which I proved for this purpose.

Theorem 5 ([6]). *Suppose M is a complete Riemannian manifold with non-positive sectional curvature, and we remove a locally finite family of totally geodesic submanifolds of real codimension 2. Suppose also*

that any intersection of these submanifolds is transverse and orthogonal. Then the complement is aspherical.

The proof depends on CAT(0) geometry, that is, the geometry of non-positively curved metric spaces more general than manifolds. The orthogonality condition is obviously very strong, and if it could be weakened then the theorem would be more useful. It is natural to hope that my method would then apply to many more moduli spaces (e.g., K3 surfaces and smooth cubic 4-folds), to the discriminant complements of many singularities, and to the well-known “ $K(\pi, 1)$ problem for Artin groups.” I have a nearly-finished paper that improves theorem 5 considerably, and I hope to address some of these applications in the future.

In [10] I developed a simple graphical calculus for many Artin groups (for experts: those of type B_n , C_n , D_n , \tilde{A}_n , \tilde{B}_n , \tilde{C}_n and \tilde{D}_n). These are analogues of the classical braid group, and come from hyperplane complements that arise naturally in algebraic geometry and Lie theory. This calculus resembles the usual manipulation of pictures of braids. One reason I introduced it was to attack the word problem for the Artin groups of types \tilde{B}_n and \tilde{D}_n , but for now this remains an open problem.

Finally, Carlson, Toledo and I addressed whether theorem 1 above was optimal—perhaps there is some other description of the moduli space as a quotient of the complex 4-ball by a discrete group, without having to remove the hyperplane arrangement? We proved that this is impossible:

Theorem 6 ([22]). *Suppose \mathcal{H} is an arrangement of mutually orthogonal hyperplanes in complex hyperbolic space $\mathbb{C}H^{n>1}$, and is invariant under a torsion-free discrete group Γ with finite-volume fundamental domain. Then the fundamental group of $(\mathbb{C}H^n - \mathcal{H})/\Gamma$ is not a lattice in any Lie group.*

4. GEOMETRIC GROUP THEORY

To first approximation, geometric group theory means the study of groups that act as symmetries of spaces built by gluing polyhedra together. (Usually one restricts to “finitely presented” groups and “proper cocompact” actions.) The philosophy is that all such actions

are the same, and one expresses this precisely in terms of something called a quasi-isometry. Many properties of these spaces (e.g., curvature and isoperimetric inequalities, suitably defined) are quasi-isometry invariant. This means that studying the geometry of a complex of polyhedra yields intrinsic information about the group. My papers in this field are largely independent, so I will discuss each separately.

A homological characterization of hyperbolic groups [29]: S. Gersten and I gave a homological characterization of the class of word-hyperbolic groups (made famous by Gromov). Our result is that a finitely presented group is word hyperbolic if and only if it has no H_1 and no reduced H_2 , where H_* means ℓ^1 -homology. (One uses cellular homology and allows infinite sums of cells, so long as the sums are ℓ^1 .) This is important because the usual definitions of word-hyperbolic groups are very geometric and not at all algebraic—our result provides a bridge.

An isoperimetric inequality for the Heisenberg groups [2]: I proved that a Heisenberg group \mathcal{H} of dimension > 3 satisfies a quadratic isoperimetric inequality. If one is geometrically minded, this says that loops in the continuous Heisenberg group bound disks of area quadratic in the length of the loop. If one is combinatorially-minded, there is a rephrasing of this idea in terms of converting words in the discrete Heisenberg group that represent the identity into the trivial word using \mathcal{H} 's relations only quadratically many times. This theorem had been asserted by Gromov; some accepted his proof and some did not. Mine was a completely new approach, which has been developed further by R. Young [76] and V. Magnani [59].

Triangles of Baumslag-Solitar groups [19]: I studied the groups with presentation

$$\langle x, y, z \mid y^{-1}x^a y = x^b, z^{-1}y^c z = y^d, x^{-1}z^e x = z^f \rangle,$$

where a, \dots, f are nonzero integers. The basic question is when they are infinite, and the motivation is that they are the simplest positively-curved triangles of groups, in the sense of Gersten and Stallings [70]. I extended known results of Mennicke [61], Post [67] and others, and discovered that the main result of [63] is wrong. My result is that the group collapses (often to a finite group) if any of a, \dots, f divides its “partner” parameter, except in a few completely understood cases.

Spotting infinite groups [5]: It is standard that a group with more generators than relations is infinite. I showed that under a non-triviality condition, a relation which is an n th power “counts as only $1/n$ th of a relation” for purposes of this count. I also found lower bounds for the abelianizations of subgroups of such a group, improving results of R. Thomas [71]. Inspired by my paper, G. Bergman developed the idea in much greater generality [33]

Hyperbolic surfaces with prescribed symmetry groups [13]: It is well-known but nontrivial that every finite group is the full symmetry group of some compact Riemann surface. I showed that if one drops compactness, then the same result holds for any countable group whatsoever. (It turns out that a totally different proof had been given a few years earlier by Winkelmann [74]).

5. PAPERS DIFFICULT TO CATEGORIZE

A Banach space determined by the Weil height [31]: There is a natural metric space structure on the multiplicative group of $\bar{\mathbb{Q}}$ modulo torsion, defined in terms of the absolute logarithmic Weil height. J. Vaaler and I identified its metric completion: it is a hyperplane in L^1 of the space of places of $\bar{\mathbb{Q}}$.

Monodromy groups of Hurwitz-type problems [30]: C. Hall and I solved the “Hurwitz monodromy problem” for simply-branched degree 4 covers of $\mathbb{C}P^1$, answering one case of a question of Eisenbud, Elkies, Harris and Speiser [43]. For a fixed number $2n$ of branch points, there is a space parametrizing all genus $n + 1 - d$ curves equipped with a degree d projection to $\mathbb{C}P^1$ having $2n$ branch points. It is called the Hurwitz space with parameters (d, n) , and it is a finite-sheeted covering space over the space of distinct $2n$ -tuples in $\mathbb{C}P^1$. For $d = 4$, we showed that the fundamental group of the base of this covering acts on the fiber as the largest possible group; we even identified the action exactly.

A new approach to rank one linear algebraic groups [17]: There is an ugly step in the basic structure theory of linear algebraic groups, requiring a lot of machinery before one can even characterize SL_2 . I found a direct argument, using a non-obvious induction.

Ideals in the integral octaves [3]: The non-associative “field” of octaves (or Cayley numbers) has a unique-up-to-symmetry natural ring of integers, introduced by L. E. Dickson in the early 20th century. I determined all its ideals, improving results of K. Mahler [60].

Identifying models of the octave projective plane [1]: J. Tits and H. Freudenthal introduced geometry over the octaves in the 1950’s as a way to explain the appearance of the exceptional Lie groups. I noticed that there were two quite different models of the octave projective plane in the literature, but no proof that they were the same. So I provided one.

Current Research

6. THE MONSTER FINITE SIMPLE GROUP.

Any finite group decomposes into groups which cannot be decomposed further, called simple groups. A decades-long effort by more than a hundred mathematicians led to the classification of the simple groups, and the list is fascinating. Most of the groups fall into infinite families having similar structure, but there are 26 groups that just don't fit. They are called the sporadic groups and the monster M is the largest one, of order $\sim 8 \times 10^{54}$. It has already appeared in unexpected places in mathematics, for example Borchers earned fame by explaining its mysterious role in the classical theory of automorphic functions on the hyperbolic plane. The monster is a good candidate for the most singular object in mathematics.

I may have come across the monster in another unexpected setting, and I am working on resolving whether it is actually present. I noticed a formal resemblance between one known description of the monster, and an infinite group Γ that I found when working on complex hyperbolic reflection groups (section 2). This led to a precise conjecture that seemed wildly speculative but has proved to satisfy a number of consistency conditions. I am now taking it seriously. The conjecture and its motivation appear in [15], and a briefer exposition in [16]. If the conjecture is true then there must be a reason why it *should* be true, and an exciting possibility is for complex hyperbolic space $\mathbb{C}H^{13}$, modulo Γ , to be a moduli space with algebra-geometric meaning. Here is a precise statement of the conjecture:

Conjecture 7 ([15]). *Let L be the unique lattice of signature $(13, 1)$ over $\mathbb{Z}[\sqrt[3]{1}]$ whose dual lattice coincides with $\frac{1}{\sqrt{-3}}L$. Let Γ be the isometry group of L , so that it acts by symmetries on $\mathbb{C}H^{13}$. Let \mathcal{H} be the union of the hyperplanes fixed by the complex reflections in Γ , so the image of \mathcal{H} in B^{13}/Γ is a complex hypersurface. Let N be the subgroup of the fundamental group $\pi_1((B^{13} - \mathcal{H})/\Gamma)$ generated by small loops that encircle this hypersurface twice. Then the quotient of $\pi_1((B^{13} - \mathcal{H})/\Gamma)$ by N is the semidirect product of $M \times M$ by $\mathbb{Z}/2$.*

Motivating the conjecture properly would require too large a technical detour, so I will just point out that the key player, Γ , is the record-holding complex hyperbolic reflection group mentioned in section 2, and understanding Γ better is probably essential for progress on the conjecture. I did not know at first that Γ is generated by reflections—this is due to T. Basak [32], whose thesis (U.C. Berkeley, 2006) was the first work on my conjecture. His proof required substantial computer computation, which I was later able to avoid with a more conceptual argument [18]. He also found a number of pleasing coincidences, which are what led me to take my conjecture seriously.

While on the subject of sporadic finite simple groups, I will mention my paper [12]. The Leech lattice is a lattice in 24-dimensional Euclidean space, already mentioned in section 2. Its symmetry group (modulo center) is another of the sporadic simple groups, the Conway group Co_1 . It is much smaller than the monster, having order $\sim 4 \times 10^{18}$, but it plays a key role in the construction and structure of M . I solved an algorithmic problem in the lattice: given two lattice vectors, is there an isometry carrying one to the other? The try-every-isometry algorithm is useless because of the size of Co_1 ; my algorithm uses the special geometry of the lattice to answer the question much more quickly.

Originally I planned to use the algorithm for classifying certain classes of integral quadratic forms, for describing the (infinite-volume) fundamental domain of the isometry group of $\mathbb{Z}^{24,1}$, and for studying the “fake monster Lie algebra”. M. Dutour has taken on the problem of $\mathbb{Z}^{24,1}$.

7. CLASSIFICATION OF LORENTZIAN LIE ALGEBRAS

Lie algebras are the basis of the study of continuous symmetry. The landmark result in the field is the Killing-Cartan classification of finite-dimensional simple Lie algebras: every finite-dimensional Lie algebra decomposes into indecomposable pieces, all of which are known. This classification suffices for the study of things like rigid bodies rotating in space, but there are some natural infinite-dimensional symmetry groups, to which their theorem doesn’t apply. For example, the ways that a fluid can rearrange itself in a tank can be described in terms of

the infinite-dimensional group of all self-transformations of the inside of the tank. Other infinite-dimensional Lie algebras, called Kac-Moody algebras, have also played a major role in physics and mathematics.

Kac-Moody algebras are “like” the simple Lie algebras in the Killing-Cartan classification (except the 1-dimensional algebra), in the sense that everything about them is governed by their root lattices and Weyl groups. They are more general because the root lattice is allowed to be semi-definite or indefinite rather than Euclidean. Kac classified the algebras in the semi-definite case, in terms of affine Weyl groups. These algebras are the famous affine Lie algebras, and their associated “loop groups” play a major role in modern theoretical physics. I am studying the next class: hyperbolic KM algebras. Many of them are beautiful and important, but the KM construction is so general most of them probably are not. Gritsenko and Nikulin [45][46] have identified the subclass of “interesting” hyperbolic KM algebras, by defining what they call Lorentzian Lie algebras. I am working on the classification:

Problem 8. *Classify all Lorentzian Kac-Moody algebras, in the sense of Gritsenko and Nikulin [46].*

The first part of the problem is to find all the possibilities for the root lattice of a Lorentzian Lie algebra. It should be a lattice in Minkowski space, which implies that its symmetry group acts on hyperbolic space. The Weyl group is generated by hyperbolic reflections, and one part of the Gritsenko-Nikulin definition is that this group should have a finite-volume fundamental domain. (I am over-simplifying: a Lorentzian Lie algebra need only satisfy a slightly weaker condition, and need only be a “generalized” KM algebra, rather than a KM algebra. Please accept this sleight of hand for purposes of this summary.)

Therefore I am working on classifying the lattices with these properties. The problem is also important in the study of arithmetic groups and Coxeter groups, independent of its application to KM algebras:

Problem 9. *Enumerate (up to scale) all integral lattices L in $(n + 1)$ -dimensional Minkowski space for which the reflections of L generate the symmetry group of L , up to finite index.*

By work of Esselmann [44] we know that such L exist only for $n \leq 19$ and $n = 21$. Also, the classification problem is mostly solved for $n \leq 4$,

by work of Nikulin [64] and Scharlau-Walhorn [69]. I have a well-developed plan to complete the classification in all dimensions. The ingredients are bare-handed manipulation of the fundamental domains, tricks from the theory of integral quadratic forms, and the use of a computer to apply what is known as Vinberg’s algorithm. Similar tools have been used by Esselmann, Nikulin, Scharlau and Walhorn, but I believe I have found the right combination of tools to finish the problem.

After classifying the root lattices, a little more work will be required to enumerate the possible root systems, since several different root systems can have the same root lattice. Then I’ll have a list of KM algebras, and it will remain to figure out which ones satisfy the most important property in the Gritsenko-Nikulin definition. This is the existence of automorphic forms whose zero sets lie along certain hypersurfaces, a requirement imposed by analogy with Borchers’ results on the “fake monster Lie algebra” [35]. This condition is very natural in the representation theory of the KM algebra.

In fact Borchers’ entire theory of automorphic forms [36][37] grew from his work on this single example. In one of the first applications of his theory, E. Freitag and I found the ring of automorphic forms for the group Γ from theorem 1, acting on the complex 4-ball [28]. This gave an explicit embedding of the moduli space of cubic surfaces into complex projective 9-space. (Actually, we treated “marked” cubic surfaces and a finite-index subgroup of Γ .) It also inspired further work in the field, for example Kondō’s similar descriptions of the moduli spaces of Enriques surfaces [53] and quartic plane curves [54]. My work on this won’t be directly needed for problem 8, but it highlights the connection to my algebraic geometry research.

Solutions to problems 8 and 9 will be significant advances in our knowledge. The question of classification of arithmetic Coxeter groups has been in the air since Vinberg’s 1972 work on $\mathbb{Z}^{n,1}$ in [73], and hyperbolic KM algebras have promised much deep mathematics since Borchers’ use of them [34] for solving the Conway-Norton moonshine conjectures [39]. Much of the work of Gritsenko and Nikulin is motivated by the fact that the automorphic forms associated to these KM algebras are related to algebraic geometry, via mirror symmetry and

the discriminant hypersurfaces in moduli spaces of K3 surfaces and other Calabi-Yau manifolds.

Finally, having a comprehensive list may enable people to spot the influence of a Lie algebra they might otherwise have overlooked. For example, physicists were famously led to introducing the exceptional Lie algebra $E_8 \times E_8$ into string theory when they noticed that it had the right dimension (496) and rank (16) for their purpose. They had been using $SO(32)$, and might have missed the role of E_8 had Killing and Cartan not already classified the simple Lie algebras.

REFERENCES

- [1] D. Allcock, Identifying models of the octave projective plane, *Geom. Dedicata* **65** (1997) 215-217.
- [2] D. Allcock, An isoperimetric inequality for the Heisenberg groups, *GAFSA* **8** (1998) 219-233.
- [3] D. Allcock, Ideals in the integral octaves, *J. Alg.* **220** (1999) 396-400.
- [4] D. Allcock, Reflection groups on the octave hyperbolic plane *J. Alg.* 213 (1999) 467-498.
- [5] D. Allcock, Spotting infinite groups, *Math. Proc. Camb. Phil. Soc.* **125** (1999) 39-42.
- [6] D. Allcock, Asphericity of moduli spaces via curvature, *J. Diff. Geom.* **55** (2000) 441-451.
- [7] D. Allcock, The Leech lattice and complex hyperbolic reflections, *Invent. Math.* **140** (2000) 283-301.
- [8] D. Allcock, New complex- and quaternion-hyperbolic reflection groups, *Duke Math. J.* **103** (2000) 303-333.
- [9] D. Allcock, The period lattice for Enriques surfaces *Math. Ann.* **317** (2000) 483-488.
- [10] D. Allcock, Braid pictures for Artin groups, *T.A.M.S.* **354** (2002), 3455-3474.
- [11] D. Allcock, The moduli space of cubic threefolds, *J. Alg. Geom.* **12** (2003) 201-223.
- [12] D. Allcock, Orbits in the Leech lattice, *Experimental Math.* **14** (2005) 491-509.
- [13] D. Allcock, Hyperbolic surfaces with prescribed infinite symmetry groups, *Proc. Amer. Math. Soc.* **134** (2006) 3057-3059.
- [14] D. Allcock, Infinitely many hyperbolic Coxeter groups through dimension 19, *Geometry & Topology* 10 (2006) 737-758.
- [15] D. Allcock, A monstrous proposal, to appear in *Groups and symmetries: From the Neolithic Scots to John McKay*, ed. J. Harnad.
- [16] D. Allcock, Complex hyperbolic geometry and the monster simple group (conjectural), to appear in *Oberwolfach Reports*.
- [17] D. Allcock, A new approach to rank one linear algebraic groups, *J. Algebra* **321** (2009) 2540-2544.
- [18] D. Allcock, On the Y_{555} complex reflection group, *J. Alg.* **322** (2009) 1454-1465.
- [19] D. Allcock, Triangles of Baumslag-Solitar groups, preprint 2008.

- [20] D. Allcock, J. Carlson and D. Toledo, A complex hyperbolic structure for moduli of cubic surfaces, *CRAS* **326** (1998) 49-54.
- [21] D. Allcock, J. Carlson and D. Toledo, The complex hyperbolic geometry of the moduli space of cubic surfaces, *J. Alg. Geom.* **11** (2002) 659-724.
- [22] D. Allcock, J. Carlson and D. Toledo, Orthogonal complex hyperbolic arrangements, *Symposium in Honor of C. H. Clemens, Contemp. Math.* **312** (2002) 1-8.
- [23] D. Allcock, J. Carlson and D. Toledo, Real cubic surfaces and real hyperbolic geometry, *CRAS* **337** (2003) 185-188.
- [24] D. Allcock, J. Carlson and D. Toledo, Nonarithmetic uniformization of some real moduli spaces, *Geometriae Dedicata* **122** (2006) 159–169. Erratum: **122** (2006) 171.
- [25] D. Allcock, J. Carlson and D. Toledo, Hyperbolic geometry and the moduli space of real binary sextics; in *Algebra and Geometry around Hypergeometric Functions*, R.P. Holzapfel, A.M. Uludag, M. Yoshida (editors), Progress in Mathematics series v. 260, Birkauer 2007, 1-22.
- [26] D. Allcock, J. Carlson and D. Toledo, The complex hyperbolic geometry of the moduli space of cubic threefolds; to appear in *Memoirs of the AMS*.
- [27] D. Allcock, J. Carlson and D. Toledo, Nonarithmetic uniformization of the moduli space of real cubic surfaces, to appear in *Ann. Sci. de L'É.N.S.*
- [28] D. Allcock and E. Freitag, Cubic surfaces and Borcherds products *Comm. Math. Helv.* **77** (2002) 270–296.
- [29] D. Allcock and S. Gersten, A homological characterization of hyperbolic groups, *Invent. Math.* **135** (1999) 723-742.
- [30] D. Allcock and C. Hall, Monodromy groups of Hurwitz-type problems, submitted 2008.
- [31] D. Allcock and J. Vaaler, A Banach space determined by the Weil height, *Acta Arithmetica* **136** (2009), 279–298.
- [32] T. Basak, The complex Lorentzian Leech lattice and the bimonster, *J. Alg.* **309** (2007) 32–56.
- [33] G. Bergman, Infinite groups arising from partial presentations of finite groups, *Math. Proc. Camb. Phil. Soc.* **127** (1999) 425-439.
- [34] R. Borcherds, Monstrous moonshine and monstrous Lie superalgebras, *Invent. Math.* **109** (1992) 405-444.
- [35] R. Borcherds, The monster Lie algebra, *Adv. Math.* **83** (1990) 30-47.
- [36] R. Borcherds, Automorphic forms on $O_{s+2,2}(\mathbb{R})$ and infinite products. *Invent. Math.* **120** (1995), no. 1, 161–213.
- [37] R. Borcherds, Automorphic forms with singularities on Grassmannians, *Inv. Math.* **132** (1998) 491-562.
- [38] K. Chu, On the geometry of the moduli space of real binary octics, preprint, arXiv:0708.0419.
- [39] J. H. Conway and S. Norton, Monstrous moonshine, *Bull. London Math. Soc.* **11** (1979), no. 3, 308–339.
- [40] A. Degtyarev and V. Kharlamov, Topological classification of real Enriques surfaces, *Topology* **35** (1996) 711–729.
- [41] P. Deligne and G. D. Mostow, Monodromy of hypergeometric functions and non-lattice integral monodromy, *Publ. Math. I.H.E.S.* **63** (1986) 5–89.

- [42] I. Dolgachev, B. van Geemen and S. Kondō, A complex ball uniformization of the moduli space of cubic surfaces via periods of $K3$ surfaces, *J. Reine Angew. Math.* **588** (2005) 99–148.
- [43] D. Eisenbud, N. Elkies, J. Harris, R. Speiser, “On the Hurwitz scheme and its monodromy,” *Compositio Math.* **77** (1991), no. 1, 95–117.
- [44] F. Esselmann, Über die maximale Dimension von Lorentz-Gittern mit co-endlicher Spiegelungsgruppe, *J. Number Theory* 61 (1996), no. 1, 103–144.
- [45] V. Gritsenko and V. Nikulin, Automorphic forms and Lorentzian Kac-Moody algebras I, *Internat. J. Math.* 9 (1998), no. 2, 153–199.
- [46] V. Gritsenko and V. Nikulin, Automorphic forms and Lorentzian Kac-Moody algebras II, *Internat. J. Math.* 9 (1998), no. 2, 201–275.
- [47] V. Gritsenko and V. Nikulin, On the classification of Lorentzian Kac-Moody algebras, *Russian Math. Surveys* 57 (2002), no. 5, 921–979.
- [48] M. Gromov and I. Piatetski-Shapiro, Nonarithmetic groups in Lobachevsky spaces, *Publ. Math. I.H.E.S.* **66** (1988) 93–103.
- [49] G. Heckmann and E. Looijenga, The moduli space of rational elliptic surfaces, in *Algebraic geometry 2000, Azumino (Hotaka)*, 185–248; Adv. Stud. Pure Math., 36, Math. Soc. Japan, Tokyo, 2002.
- [50] E. Horikawa, On the periods of Enriques surfaces I, *Math. Ann.*, 234:73–88, 1978.
- [51] S. Kondō, A complex hyperbolic structure for the moduli space of curves of genus three, *J. Reine Angew. Math.* **525** (2000) 219–232.
- [52] S. Kondō, The moduli space of curves of genus 4 and Deligne-Mostow’s complex reflection groups, in *Algebraic geometry 2000, Azumino (Hotaka)*, 383–400; Adv. Stud. Pure Math., 36, Math. Soc. Japan, Tokyo, 2002.
- [53] S. Kondō, The moduli space of Enriques surfaces and Borcherds products, *J. Alg. Geom.* **11** (2002) 601–627.
- [54] S. Kondō, Moduli of plane quartics, Göpel invariants and Borcherds products, preprint.
- [55] R. Laza, Moduli space of cubic fourfolds (the GIT compactification), to appear in *J. Algebraic Geom.*
- [56] R. Laza, Moduli space of cubic fourfolds via the period map (the period map), to appear in *Ann. Math.*
- [57] E. Looijenga, The period map for cubic fourfolds, to appear in *Inv. Math.*
- [58] E. Looijenga and R. Swierstra, The period map for cubic threefolds, *Compositio Math.* **143** 10371049.
- [59] V. Magnani, Contact equations, Lipschitz extensions and isoperimetric inequalities, arXiv:0711.5003.
- [60] K. Mahler. On ideals in the Cayley-Dickson algebra. *Proc. Roy. Irish Acad.* **48** (1943) 123–133.
- [61] J. Mennicke. Einige endliche Gruppen mit drei Erzeugenden und drei Relationen. *Arch. Math.* 10 (1959) 409–418.
- [62] Y. Namikawa, Periods of Enriques surfaces. *Math. Ann.* **270** (1985) 201–222.
- [63] B. H. Neumann, Some group presentations, *Can. J. Math.*, **30** (1978) 838–850.
- [64] V. Nikulin, On the classification of hyperbolic root systems of rank three. *Proc. Steklov Inst. Math.* **230** (2000) 1–241.
- [65] V. Nikulin, Involutions of integer quadratic forms and their applications to real algebraic geometry. *Izv. Akad. Nauk SSSR Ser. Mat.* **47** (1983) 109–188.

- [66] P. Orlik and H. Terao, *Arrangements of hyperplanes*, Grundlehren der Mathematischen Wissenschaften 300, Springer-Verlag, Berlin, 1992.
- [67] M. Post, Finite three-generator groups with zero deficiency, *Communications in Algebra*, **6** (1978) 1289–1296.
- [68] M. Prokhorov, Absence of discrete reflection groups with a non-compact polyhedron of finite-volume in Lobachevsky spaces of large dimension, *Math. USSR Izv.* **28** (1987) 401–411.
- [69] R. Scharlau and C. Walhorn, Integral lattices and hyperbolic reflection groups, *Journées Arithmétiques, 1991 (Geneva)*. *Astrisque* No. 209 (1992), 15–16, 279–291.
- [70] J. Stallings, Non-positively curved triangles of groups, *Group theory from a geometrical viewpoint (Trieste, 1990)*, eds. É. Ghys, A. Haefliger and A. Verjovsky, World Sci., 1991, 491–503.
- [71] R. Thomas, Cayley graphs and group presentations, *Proc. Camb. Phil. Soc.* **103**(1988), 385–7.
- [72] W. Thurston, Shapes of polyhedra and triangulations of the sphere, *The Epstein birthday schrift*, 511–549 (electronic), *Geom. Topol. Monogr.* **1**, Geom. Topol. Publ., Coventry, 1998.
- [73] E. Vinberg, The groups of units of certain quadratic forms. *Mat. Sb. (N.S.)* **87**(129) (1972), 18–36.
- [74] J. Winkelmann, Realizing countable groups as automorphism groups of Riemann surfaces, *Documenta Math.* **7** (2002) 413–417.
- [75] M. Yoshida, A hyperbolic structure on the real locus of the moduli space of marked cubic surfaces, *Topology* **40** (2001) 469–473.
- [76] R. Young, Scaled relators and Dehn functions for nilpotent groups, arXiv: math.GR/0601297