The Moduli Space of Cubic Threefolds

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Abstract.
We describe the moduli space of cubic hypersurfaces in $\mathbb{C}P^4$ in the sense of geometric invariant theory. That is, we characterize the stable and semistable hypersurfaces in terms of their singularities, and determine the equivalence classes of semistable hypersurfaces under the equivalence relation of their orbit-closures meeting.

§1. Introduction

Mumford's geometric invariant theory provides a construction of complete moduli spaces of families of varieties. In this paper we apply his methods to obtain a concrete description of the moduli space of cubic hypersurfaces in $\mathbb{C}P^4$. More precisely, we work out which cubic threefolds are stable, which are semistable, which of the semistable orbits are minimal, and which semistable threefolds degenerate to which minimal orbits, all in terms of the singularities of the threefolds. Many authors have treated stability and semistability in other settings. Hilbert treated point-sets in the projective line, plane curves of degree ≤ 6 and cubic surfaces [8]. Shah provided much more detailed information about sextic plane curves [12] and analyzed quartic surfaces [13]. Mumford and Tate treated point-sets in projective spaces of arbitrary dimension [10, chap. 3], Miranda treated pencils of cubics in P^2 [9], and Avritzer and Miranda have recently treated pencils of quadrics in P^4 [3]. For further references, see [10]. After writing this paper we learned that Collino made a partial analysis of the stability of cubic threefolds, as background for his work on the fundamental group of the Fano surface of lines on a smooth cubic threefold [6]. In particular, he established our lemma 6.1.

The reason this paper exists is the problem of uniformizing the moduli space by the complex 10-ball, much as the moduli space of cubic surfaces is uniformized by the complex 4-ball [1]. J. Carlson, D. Toledo and the author have constructed a period map that identifies the moduli space of smooth cubic threefolds with a Zariski-dense subset of a quotient of the 10-ball by a discrete group of finite covolume. The current work, together with refinements of the techniques of [1], should provide much more detailed information, such as exactly which discrete group, which periods arise from smooth threefolds, and how degeneration to singular threefolds is reflected in the ball quotient.

We recall the basic definitions of geometric invariant theory in this context; for further background see [10] or [11]. The cubic threefolds are parameterized by $\mathbb{C}P^{34}$, and a threefold T is called semistable if there is an $\mathrm{SL}(5,\mathbb{C})$-invariant hypersurface in $\mathbb{C}P^{34}$ which does not contain T. A semistable threefold is called stable if it has finite symmetry group and its orbit is closed (in the space of semistable threefolds). If just the second of these conditions holds then its orbit is called minimal. We say that one semistable threefold degenerates to another if the second lies in the orbit closure of the first. The moduli space may be described topologically as the quotient of the semistable threefolds by the relation that two are equivalent if their orbit closures meet (in the space of semistable threefolds); every equivalence class contains a unique minimal orbit. The moduli space is a projective variety in a natural way, and contains the orbit space of stable threefolds as an open dense subset.

A hypersurface singularity in 4 variables is called an A_n singularity ($n \geq 1$) if it is locally analytically equivalent to
$$x_1^{n+1} + x_2^2 + x_3^2 + x_4^2 = 0\ ,$$

and a D_4 singularity if it is locally equivalent to

$$x_1^3 + x_2^3 + x_3^2 + x_4^2 = 0 .$$

The quadratic terms of any hypersurface singularity define a quadratic form on the tangent space to $\mathbb{C}P^4$. The kernel of this form determines a linear subspace of $\mathbb{C}P^4$, which we call the null space of the singularity; the dimension of this space is called the nullity of the singularity. The nullity of an A_n $(n > 1)$ singularity is 1 and the nullity of a D_4 singularity is 2.

Theorem 1.1. *A cubic threefold is stable if and only if each of its singularities has type A_1, A_2, A_3 or A_4.*

Semistable threefolds that are not stable are called strictly semistable. The corresponding points of the moduli space turn out to form a rational curve and an isolated point. The isolated point is given by the threefold Δ defined by

$$F_\Delta = x_0x_1x_2 + x_3^3 + x_4^3 ,$$

which has exactly three singularities, each of type D_4. Theorem 5.4 shows that Δ is the only cubic threefold with three D_4 singularities, up to projective equivalence. The rational curve is given by the threefolds $T_{A,B}$ defined by

$$F_{A,B} = Ax_2^3 + x_0x_3^2 + x_1^2x_4 - x_0x_2x_4 + Bx_1x_2x_3 , \tag{1.1}$$

where at least one of A and B is nonzero. By rescaling the variables one sees that if $k \neq 0$ then $T_{A,B}$ is projectively equivalent to $T_{k^2A,kB}$. When we want to refer to the projective equivalence class of $T_{A,B}$ we usually just write T_β, where $\beta = 4A/B^2 \in \mathbb{C} \cup \{\infty\}$. If $\beta \neq 0, 1$ then T_β has just two singularities, both of type A_5. If $\beta = 0$ then T_β acquires a third singularity, of type A_1. If $\beta = 1$ then T_β is the secant variety of a rational normal curve of degree 4, which we call a chordal cubic; its singular locus is the rational normal curve. We show in section 5 that T_β and $T_{\beta'}$ are projectively equivalent if and only if $\beta = \beta'$, and that any cubic threefold with two A_5 singularities is projectively equivalent to T_β for some $\beta \neq 1$.

Theorem 1.2. *The minimal orbits of strictly semistable cubic threefolds are the orbits of Δ and of the T_β, $\beta \in \mathbb{C} \cup \{\infty\}$.*

Theorem 1.3. *A cubic threefold T is strictly semistable if and only if*
(i) T contains a D_4 singularity, in which case T degenerates to Δ, or
(ii) T contains an A_5 singularity, in which case T degenerates to T_β for some $\beta \neq 1$, or
(iii) T contains an A_n singularity $(n \geq 6)$, but none of the planes containing its null line, in which case T degenerates to a chordal cubic, or
(iv) T is a chordal cubic.

Theorem 1.4. *A cubic threefold T is unstable (i.e., not semistable) if and only if*
(i) T has non-isolated singularities and is not a chordal cubic, or
(ii) T contains an isolated singularity of nullity ≥ 3, or
(iii) T contains an isolated singularity of nullity 2 other than a D_4 singularity, or
(iv) T contains an A_n singularity $(n \geq 6)$ and also a plane containing its null line.

In section 2 we explain some singularity theory, which we will use throughout the paper. In section 3 we apply the Hilbert-Mumford criteria for semistability and stability to establish necessary conditions for a cubic threefold to be unstable (resp. not stable). In section 4 we show that the presence of certain singularities makes a threefold unstable (resp. not stable). In section 5 we study the special threefolds Δ and T_β more closely, and in particular we characterize them by their singularities. The final section assembles material from earlier sections to establish the theorems above.

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§2. Some singularity theory

One of our fundamental tools is a recognition principle for simple hypersurface singularities, given by Bruce and Wall in [5]. The hypersurface singularities A_n ($n \geq 1$), D_n ($n \geq 4$), E_6, E_7 and E_8 in m variables are given by

$$\begin{aligned}
A_n &: \quad x_1^{n+1} + x_2^2 + \cdots + x_m^2 \\
D_n &: \quad x_1^{n-1} + x_1 x_2^2 + x_3^2 + \cdots + x_m^2 \\
E_6 &: \quad x_1^4 + x_2^3 + x_3^2 + \cdots + x_m^2 \\
E_7 &: \quad x_1^3 x_2 + x_2^3 + x_3^2 + \cdots + x_m^2 \\
E_8 &: \quad x_1^5 + x_2^3 + x_3^2 + \cdots + x_m^2 \, .
\end{aligned}$$

They are quasihomogeneous with weights $(\frac{1}{n+1}, \frac{1}{2}, \ldots, \frac{1}{2})$, $(\frac{1}{n-1}, \frac{n-2}{2(n-1)}, \frac{1}{2}, \ldots, \frac{1}{2})$, $(\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \ldots, \frac{1}{2})$, $(\frac{2}{9}, \frac{1}{3}, \frac{1}{2}, \ldots, \frac{1}{2})$ and $(\frac{1}{5}, \frac{1}{3}, \frac{1}{2}, \ldots, \frac{1}{2})$, respectively. A power series f is called semiquasihomogeneous (SQH) with given weights $(w_1, \ldots, w_n)$ if f has no terms of weighted degree < 1, and the terms of weighted degree 1 define an isolated singularity.

Theorem 2.1. *If an analytic function $f(x_1, \ldots, x_m)$ is SQH with respect to the weights of one of the singularities given above, then f has a singularity of that type at the origin.* □

Bruce and Wall treat the case of 3 variables by reducing it to results of Arnol′d, and the proof in the general case is the same. The theorem lets one recognize many singularities immediately. However, there are some cases where one can apply it only after a local coordinate change. As an example we investigate the nature of the singularity

$$f(x_1, \ldots, x_4) = x_3^2 - x_2 x_4 + x_1^2 x_4 + K x_1 x_2 x_4 + x_2^2 L(x_3, x_4)$$

where K is a generic constant and L is a generic linear form. We would like to apply the theorem, and we must clearly assign weight $\frac{1}{2}$ to each of x_2, x_3 and x_4. However, the largest n for which all the terms of f have degree ≥ 1 with respect to the weights $\left(\frac{1}{n}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is $n = 4$, and then the terms of degree 1 do not define an isolated singularity. The problem is the $x_1^2 x_4$ term, and the solution is to kill it with the substitution $x_2 \mapsto x_2 + x_1^2$, which yields

$$f = x_3^2 - x_2 x_4 + K x_1 x_2 x_4 + K x_1^3 x_4 + \left(x_2^2 + 2x_1^2 x_2 + x_1^4\right) L(x_3, x_4) \, .$$

Now we take weights $\left(\frac{1}{6}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, but again the degree 1 terms define a nonisolated singularity. The problem is now the $x_1^3 x_4$ term, which we kill by $x_2 \mapsto x_2 + K x_1^3$. This yields

$$f = x_3^2 - x_2 x_4 + K^2 x_1^4 x_4 + x_1^4 L(x_3, x_4) + (\text{terms of degree} > 1)$$

with respect to the weights $\left(\frac{1}{8}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. For generic K and L, the degree 1 terms define an isolated singularity, so f has type A_7.

If a given isolated singularity has nullity 1 then this process terminates, so in practice it is easy (if tedious) to identify such singularities. The reader may enjoy checking that the threefold defined by

$$x_2^3 + x_0x_3^2 + x_1^2x_4 - x_0x_2x_4 - 2x_1x_2x_3 + Kx_4^3$$

has an A_{11} singularity at $P = [1, 0, 0, 0, 0]$ if $K \neq 0$. (This threefold is a linear combination of a chordal cubic and the cube of a hyperplane that meets the rational normal curve at only one point.)

An important invariant of a singularity is its milnor number, called its multiplicity in [2]. If $f : \mathbb{C}^n \to \mathbb{C}$ has a singularity at a point p, then the milnor number at p is defined as the vector space dimension of the quotient of the local ring of $\mathbb{C}^n$ at p by the Jacobian ideal of f. The milnor number of an A_n, D_n or E_n singularity is the subscript n. More generally, a singular point is an isolated singularity if and only if it has finite milnor number. An important property of the milnor number is its semicontinuity: if a function f_0 is a limit of functions f_t, and each f_t has a singularity of milnor number $\geq n$ at p, then so does f_0. For more information, see [2]. It follows immediately that if $f(x_1, \ldots, x_m)$ has nullity 1 and has only terms of degree ≥ 1 with respect to the weights $\left(\frac{1}{n}, \frac{1}{2}, \ldots, \frac{1}{2}\right)$, and the terms of degree 1 fail to define an isolated singularity, then the singularity is worse than A_{n-1}, in the sense that the milnor number is at least n. We also have:

Lemma 2.2. *Suppose*

$$f(x_1, \ldots, x_m) = C(x_1, x_2) + Q(x_3, \ldots, x_m) + (\text{terms of degree} > 1)$$

with respect to the weights $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{2}, \ldots, \frac{1}{2}\right)$, where C (resp. Q) is a cubic (resp. quadratic) form. If C has a multiple root then f has milnor number ≥ 5. If Q is nondegenerate and C has only simple roots then f has type D_4.

Proof: The second claim is part of theorem 2.1. To prove the first claim one shows that for f generic among those for which C has a multiple root, the singularity has type D_5. □

§3. Necessary conditions for instability

In this section we identify certain singularities that a cubic threefold must have if it fails to be semistable (resp. stable). In our analysis of a singular point of a cubic threefold T with defining form F, we will almost always choose coordinates $x_0, \ldots, x_4$ so that the singularity lies at $P = [1, 0, 0, 0, 0]$. We define L_{ij} as the line defined by $x_k = 0$ for all $k \neq i, j$. We will write $f(x_1, \ldots, x_4)$ for the local defining equation for T at P obtained by substituting $x_0 = 1$ in F.

If X is a subset of the 35 cubic monomials in $x_0, \ldots, x_4$ then we say that a cubic form F has type X if all the monomials of F with nonzero coefficients lie in X. The 35 monomials may be arranged in the obvious way on a 4-dimensional simplex. A picture of the 4-simplex with the 35 monomials marked by empty or filled circles, such as any of the figures 3.1(a)–(f), denotes the set of monomials marked by filled circles. The mnemonic is that an empty circle looks like 0. We sometimes say that a cubic form has some type by referring to a figure, for example "F has type 3.1(a)." We apply the same terminology to T.

Lemma 3.1. *A cubic threefold is unstable if and only if it is projectively equivalent to a threefold of one of the types 3.1(a)–(f). (See figures 3.1(a)–(f).)*

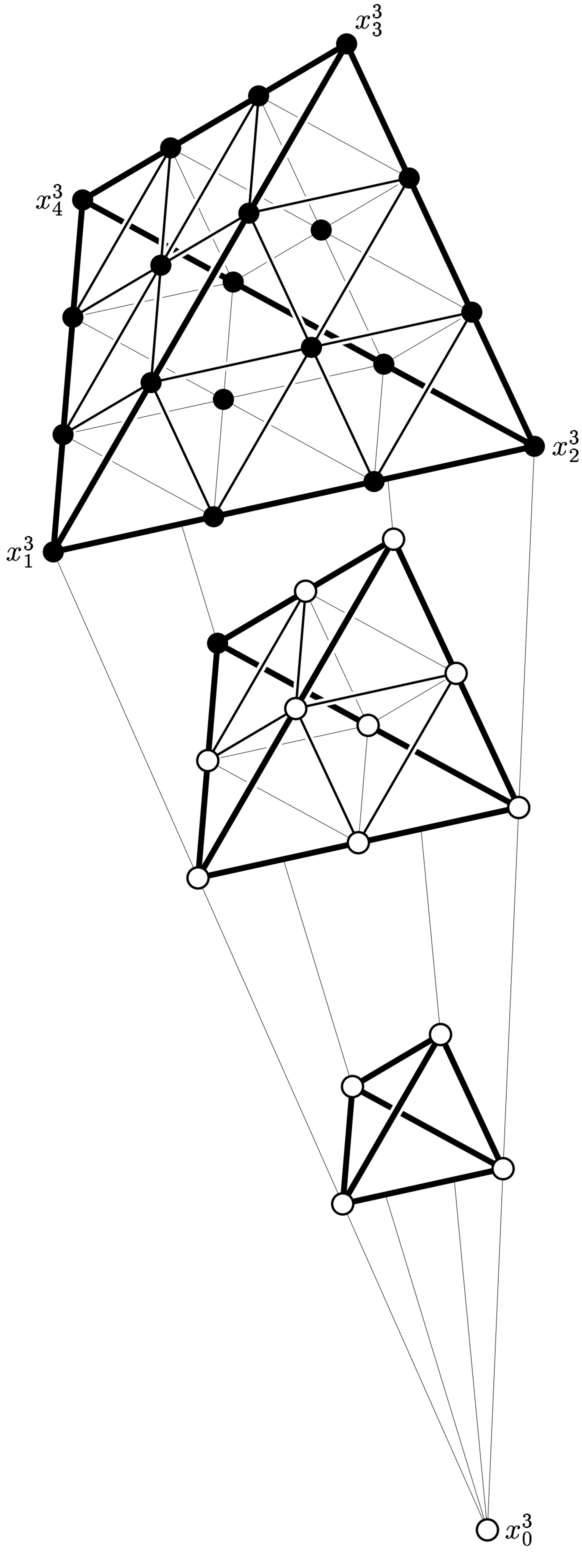

Figure 3.1(a). See lemma 3.1.
H_v for $v = \left(\frac{25}{8}, -\frac{5}{36}, -\frac{25}{72}, -\frac{55}{72}, -\frac{15}{8}\right)$.

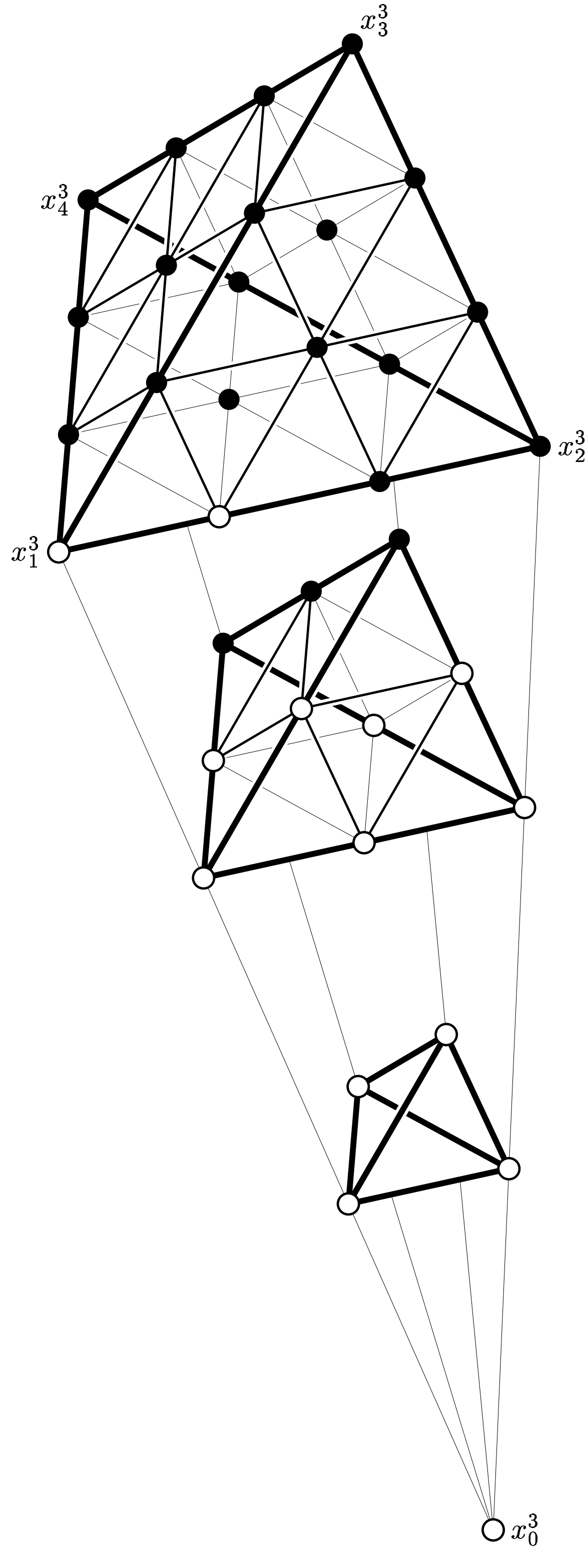

Figure 3.1(b). See lemma 3.1.
H_v for $v = \left(\frac{5545}{1716}, \frac{2155}{3432}, -\frac{1435}{3432}, -\frac{1435}{858}, -\frac{3035}{1716}\right)$.

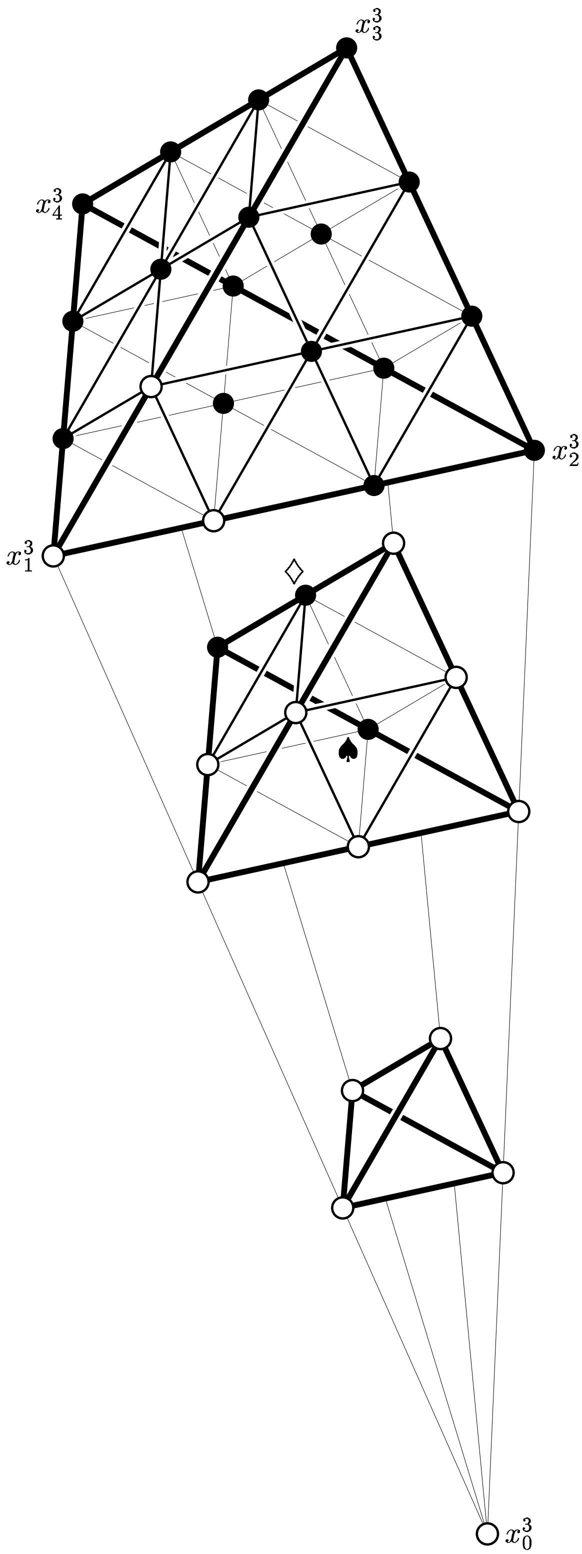

Figure 3.1(c). See lemma 3.1.
H_v for $v = \left(\frac{97}{36}, \frac{121}{144}, -\frac{157}{288}, -\frac{197}{288}, -\frac{83}{36}\right)$.

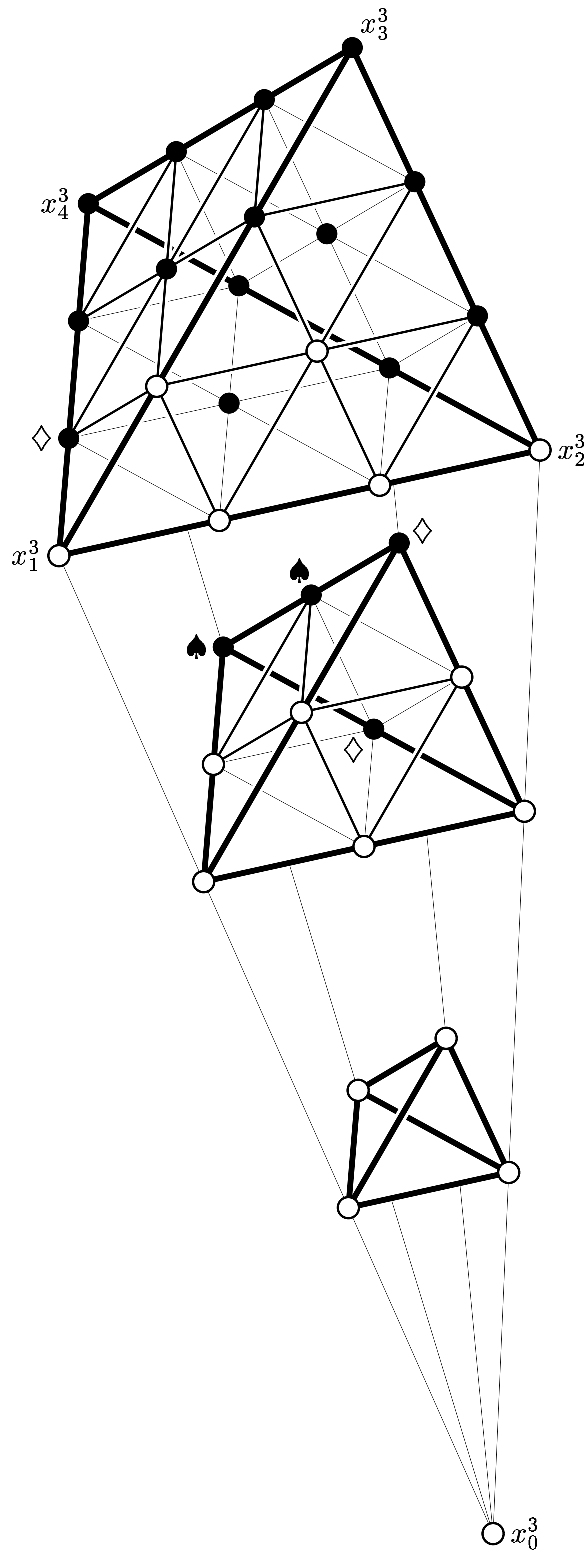

Figure 3.1(d). See lemma 3.1.
H_v for $v = \left(\frac{1117}{504}, \frac{1333}{1008}, \frac{223}{504}, -\frac{1207}{1008}, -\frac{1403}{504}\right)$.

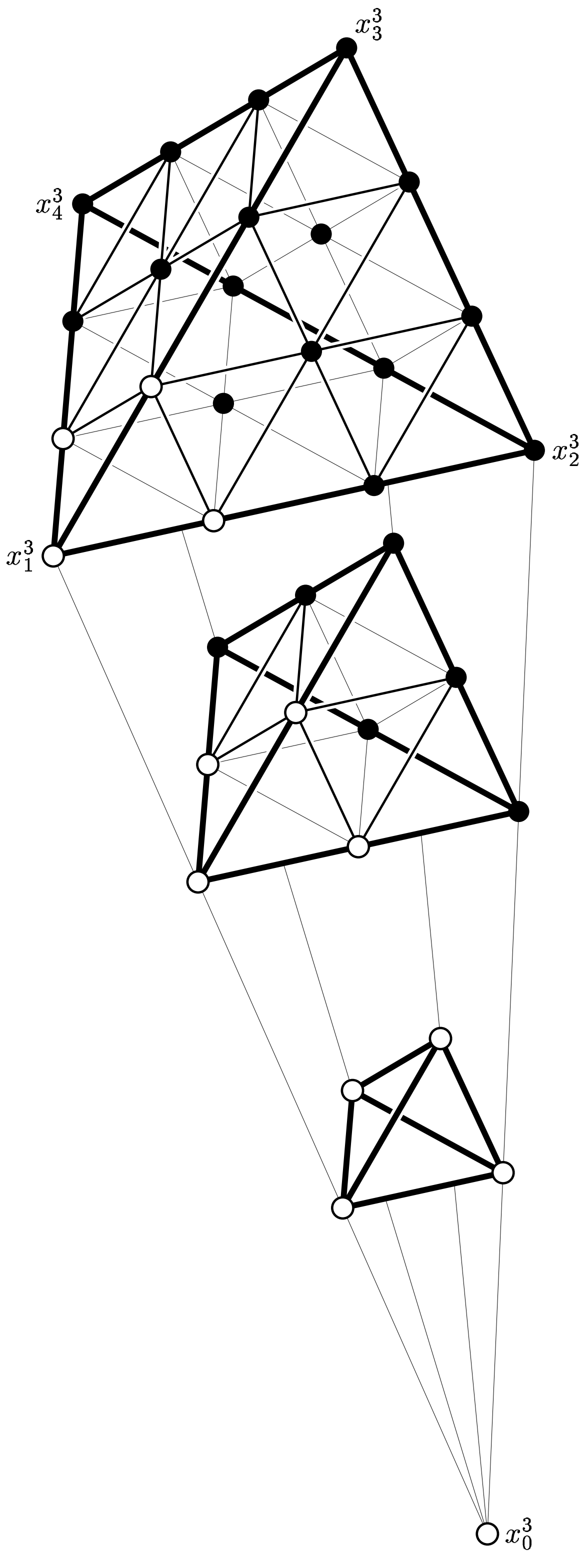

Figure 3.1(e). See lemma 3.1.
H_v for $v = \left(\frac{491}{168}, \frac{421}{168}, -\frac{533}{336}, -\frac{593}{336}, -\frac{349}{168}\right)$.

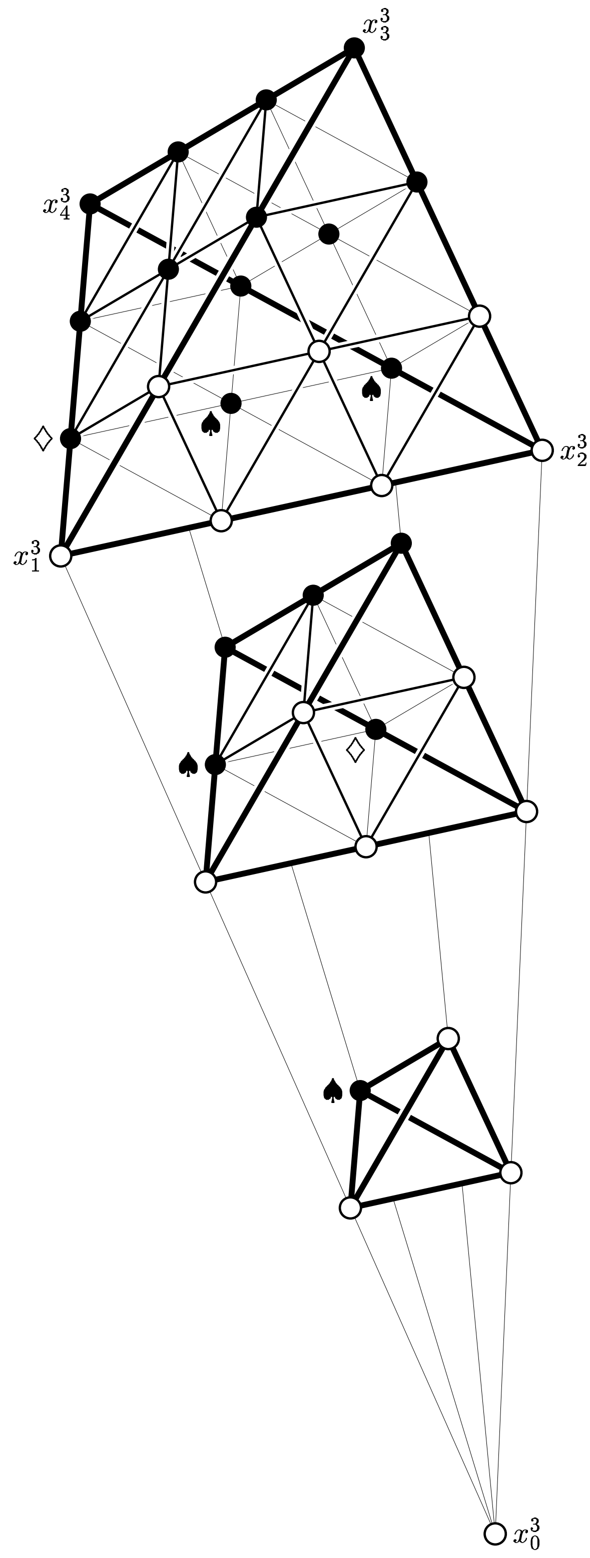

Figure 3.1(f). See lemma 3.1.
H_v for $v = \left(\frac{45}{28}, \frac{505}{336}, \frac{145}{112}, -\frac{85}{84}, -\frac{95}{28}\right)$.

Proof: The Hilbert-Mumford criterion for semistability [10, theorem 2.1] says that T is unstable if and only if there exists a coordinate system with respect to which all the monomials of F with nonzero coefficients lie strictly to one side of a hyperplane through the center of the 4-simplex. We rephrase this criterion as follows. We regard each cubic monomial $x_0^{a_0} \cdots x_4^{a_4}$ as the vector $(a_0, \ldots, a_4) \in \mathbb{R}^5$. For every nonzero vector $v = (v_0, \ldots, v_4) \in \mathbb{R}^5 - \{0\}$ satisfying $v_0 + \cdots + v_4 = 0$, we define H_v to be the set of cubic monomials having negative inner product with v under the usual inner product. Then T is unstable if and only if it is equivalent to a threefold of type H_v for some v.

To reduce this infinity of types to only six, we write Y for the union of the orthogonal complements of the 35 monomials, and observe that we may ignore those v that lie in Y, for near such a v we may find $v' \notin Y$ with $H_v \subseteq H_{v'}$. Furthermore, we may restrict attention to one v from each S_5-orbit, so that we may take $v_0 \geq \cdots \geq v_4$. This defines a Weyl chamber for S_5, namely the cone spanned by the vectors $(1,1,1,1,-4)$, $(2,2,2,-3,-3)$, $(3,3,-2,-2,-2)$ and $(4,-1,-1,-1,-1)$. By multiplying v by a positive number we may suppose that v lies in their convex hull, which is a 3-simplex S. Finally, if both v and v' lie in $S - Y$ then $H_v = H_{v'}$ if and only if v and v' lie in the same component of $S - Y$. Therefore, to enumerate the possible H_v it suffices to compute the set of polyhedra into which Y divides S, and then choose a vector v from the interior of each polyhedron. By a computer calculation of less than a minute (see below) there are 72 of these polyhedra, so 72 types of cubic forms represent all unstable forms. It turns out that many of these are special cases of each other (i.e., $H_v \subseteq H_{v'}$). It is easy for the computer to eliminate such redundancy, and every one of the 72 types is a special case of one of the types 3.1(a)–(f). By construction, every form in one of these families is unstable, and the lemma follows. □

Remarks on the computation: To perform the subdivision of S by Y, we wrote a computer program in C++, using software for arbitrary-precision rational arithmetic developed by the GNU project (GMP version 2.0.2 [7]). For $k \geq 0$ we define a k-polytope in $\mathbb{Q}^n$ to be a bounded set whose affine span has dimension k and which is the intersection of finitely many closed rational half-spaces. A facet is one of its $(k-1)$-dimensional faces. We encode a 0-polytope (a point) by its coordinates, and for $k > 0$ we encode a k-polytope by the set of its facets, which of course are themselves encoded as $(k-1)$-polytopes. We will describe an algorithm for checking if the intersection of a k-polytope with a closed rational half-space is k-dimensional, and in this case for computing the intersection. With such an algorithm, to divide S by Y one simply divides S by the first hyperplane, then each of the resulting pieces by the next hyperplane, and so on. It is easy to compute the set of vertices of a k-polytope K; the vectors given in figures 3.1(a)–(f) were obtained by averaging the vertices of their polyhedra.

Suppose that K is a k-polytope and L is a rational hyperplane; we will show how to check if $K \cap L$ has dimension at least $k-1$, and in this case how to compute $K \cap L$. When $k = 0$ or 1 the problem is trivial, so suppose $k > 1$. First we check if $K \subseteq L$ (by checking whether the vertices of K lie in L), in which case $K \cap L = K$ and we are done. If $K \not\subseteq L$ then we check whether any facet of K lies in L, in which case $K \cap L$ equals that facet and we are done. Otherwise, we check whether L meets the interior of K, by checking if it separates some pair of vertices. If it does not then $K \cap L$ has dimension $< k-1$ and we are done. If it does then $\dim(K \cap L) = k-1$ and the facets of $K \cap L$ are the intersections $K' \cap L$ that have dimension $k-2$, where K' varies over the facets of K.

Finally, suppose K is a k-polytope and L^+ is a closed rational half-space with bounding hyperplane L. We will show how to check if $K \cap L^+$ has dimension k, and if so how to compute $K \cap L^+$. When $k = 0$ or 1 the problem is trivial, so suppose $k > 1$. We first check whether K lies in L^+; if it does then $K \cap L^+ = K$ and we are done. Otherwise, we check whether L meets the interior of K. If it does not then $\dim(K \cap L^+) < k$ and we are done. If it does then $\dim(K \cap L^+) = k$ and the facets of $K \cap L^+$ are $K \cap L$ and the intersections $K' \cap L^+$ that have dimension $k-1$, where K' varies over the facets of K.

Next we will reduce the six types to four, and then find geometric features of the remaining four types.

Lemma 3.2. *A cubic form of type 3.1(f) is projectively equivalent to one of type 3.1(d), and one of type 3.1(c) is projectively equivalent to one of type 3.1(b).*

Proof: Studying figure 3.1(f) shows that linear substitutions fixing x_3 and x_4 while mixing x_0, x_1 and x_2 together preserve the family. We focus on the $x_4 \cdot \text{quadratic}(x_0, x_1, x_2)$ part of F. By a linear substitution we may suppose that these terms reduce to $x_4 \cdot (Kx_1^2 + K'x_0x_2)$ for some constants K and K'. We have marked these terms by diamonds ($\diamondsuit$) and the four killed terms by spades ($\spadesuit$). Since the lower two spades vanish, F has type 3.1(d).

If F has type 3.1(c) then the proof is similar: mixing x_2 and x_3 preserves the family and mixes the coefficients of the two marked monomials together. We can suppose that the one marked with a spade vanishes, and then F has type 3.1(b). $\square$

Theorem 3.3. *If a cubic threefold T is unstable, then either*
(i) T contains a double line, or
(ii) T contains a singularity of nullity ≥ 3, or
(iii) T contains a singularity of nullity 2 and milnor number ≥ 5, or
(iv) T contains a singularity of nullity 1 and milnor number ≥ 7, and also a plane containing the null line of the singularity.

Proof: It suffices to show that if T is defined by a cubic form F of one of the types 3.1(a), 3.1(b), 3.1(d) and 3.1(e) then T satisfies one of the conditions listed. If F has type 3.1(a), then $f = Kx_0x_4^2 + \text{cubic}(x_1, \ldots, x_4)$, so P has nullity ≥ 3. If F has type 3.1(e) then T contains L_{01} as a double line. If F has type 3.1(b) then its nullity at P is at least 2 by inspection of the diagram. Also,

$$f = \text{singular cubic}(x_1, x_2) + \text{quadratic}(x_3, x_4) + (\text{terms of degree} > 1)$$

with respect to the weights $(\frac{1}{3}, \frac{1}{3}, \frac{1}{2}, \frac{1}{2})$, so that the milnor number is at least 5 by lemma 2.2.

We claim next that a generic F of type 3.1(d) has an A_7 singularity at P. By genericity we may assume that the $x_1^2x_4$, $x_0x_2x_4$ and $x_0x_3^2$ terms (all marked $\diamondsuit$) are nonzero, and by rescaling the variables we may assume that the coefficients are 1, -1 and 1, respectively. By a linear substitution $x_3 \mapsto x_3 + \lambda x_4$ we use the $x_0x_3^2$ term to kill the $x_0x_3x_4$ term, and then by $x_2 \mapsto x_2 + \lambda x_4$ we use the $x_0x_2x_4$ term to kill the $x_0x_4^2$ term. Each killed term is marked with a spade. Then

$$\begin{aligned} f(x_1, \ldots, x_4) = x_3^2 - x_2x_4 + x_1^2x_4 + Kx_1x_2x_4 + x_2^2L(x_3, x_4) \\ + x_1Q(x_3, x_4) + x_2Q'(x_3, x_4) + C(x_3, x_4) \end{aligned} \tag{3.1}$$

where K is a constant and L (resp. Q, Q', C) is a linear (resp. quadratic, quadratic, cubic) form. For generic K and L, the singularity has type A_7; the computation is essentially the worked example of section 2.

Now, if F has type 3.1(d) then P has milnor number ≥ 7, so if the nullity at P is more than 1 then T falls into category (ii) or (iii). If the nullity is 1 then the null line is L_{01} because the $x_0 \cdot \text{quadratic}(x_1, \ldots, x_4)$ terms of F are free of x_1. Then T falls into category (iv) because T contains the plane $x_3 = x_4 = 0$. $\square$

At this point it would be natural to prove that a threefold exhibiting one of these features is unstable. We postpone this to the next section because we are about to study the non-stable cubic threefolds, and the argument for the converse of theorem 3.3 also proves the converse of theorem 3.5.

Lemma 3.4. *A cubic threefold that is not stable is either unstable or projectively equivalent to a threefold of one of the types 3.2(a)–(d). (See figures 3.2(a)–(d).)*

Proof: This is similar to the proof of lemma 3.1. The Hilbert-Mumford criterion for stability [10, theorem 2.1] asserts that T is not stable if and only if there exists a coordinate system with respect to which all the monomials of F with nonzero coefficients lie in a closed half-space whose bounding hyperplane passes through the center of the 4-simplex. For each $v \in \mathbb{R}^5 - \{0\}$ having coordinate sum zero, we define $\bar{H}_v$ to be the set of monomials having nonpositive inner product with v. Then T is not stable if and only if it is equivalent to a threefold of type $\bar{H}_v$ for some v. Recall the 3-dimensional simplex S and its tessellation by Y. We choose one point from the interior of each 0-, 1-, 2- and 3-dimensional face of the tessellation, and let v vary over these points. It is obvious that every non-stable T is equivalent to a threefold of type $\bar{H}_v$ for one of these v. A computer calculation along the lines of the previous one yields 481 possibilities for v. For most of them, the threefolds of type $\bar{H}_v$ are obviously unstable: $\bar{H}_v$ lies in one of the sets given in figures 3.1(a)–(f). After eliminating these cases, only 6 possibilities for v remain. After eliminating those that are special cases others, only the four we have shown remain. A solid block indicates a monomial that lies on the bounding hyperplane. $\square$

Theorem 3.5. *If a cubic threefold is not stable then either*
(i) it has a singularity of nullity ≥ 2, *or*
(ii) it has a singularity of nullity 1 *and milnor number* ≥ 5.

Proof: It suffices to show that every unstable cubic threefold and every threefold of one of the types 3.2(a)–(d) has one of these features. For unstable threefolds this follows immediately from theorem 3.3. If F has type 3.2(b) or 3.2(d) then the nullity at P is obviously ≥ 2. If F has type 3.2(a) then we use the method of lemma 3.2: by mixing x_0 and x_1 together we can kill the $x_0^2x_4$ term, reducing F to type 3.2(b). (One can also reduce type 3.2(d) to type 3.2(b) by mixing together x_3 and x_4.)

To complete the proof, we claim that a generic F of type 3.2(c) has an A_5 singularity at P. To see this, apply the proof of theorem 3.3 before (3.1). Then we have

$$\begin{aligned} f = {} & x_3^2 - x_2x_4 + x_1^2x_4 + Kx_2^3 + x_1x_2L(x_3,x_4) \\ & + x_2^2L'(x_3,x_4) + x_1Q(x_3,x_4) + x_2Q'(x_3,x_4) + C(x_3,x_4)\ . \end{aligned}$$

After the substitution $x_2 \mapsto x_2 + x_1^2$ we have

$$f = x_3^2 - x_2x_4 + Kx_1^6 + x_1^3L(x_3,x_4) + (\text{terms of degree} > 1) \tag{3.2}$$

with respect to the weights $(\frac{1}{6}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. For generic K and L the degree 1 terms define an isolated singularity, and the claim follows from theorem 2.1. $\square$

§4. Sufficient conditions for instability

In this section we show that cubic threefolds with certain sorts of singularities are unstable (resp. not stable). We begin with a treatment of the threefolds with a singularity of nullity ≥ 2. It turns out that the threefold Δ defined by

$$F_\Delta = x_0x_1x_2 + x_3^3 + x_4^3$$

plays a central role. Calculation shows that Δ has three singularities of type D_4 and no others. It follows from theorem 3.3 that Δ is semistable, and Δ is strictly semistable because it has infinite symmetry group.

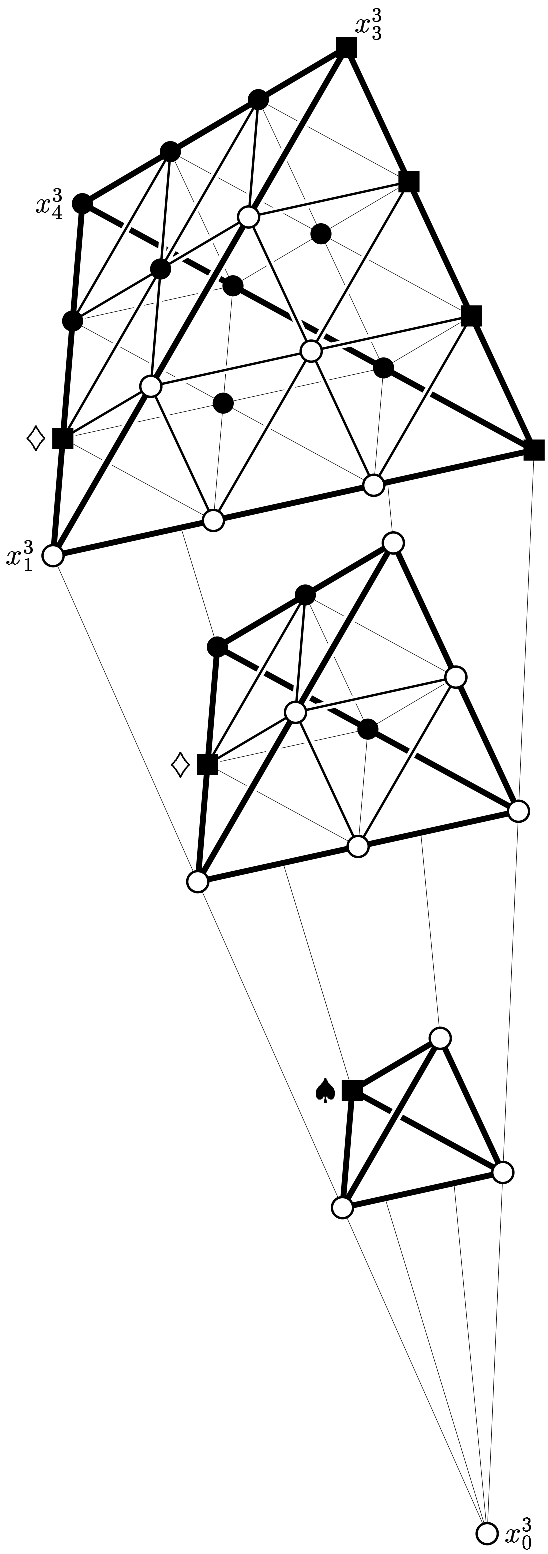

Figure 3.2(a). See lemma 3.4.
$\bar{H}_v$ for $v = \left(\frac{5}{3}, \frac{5}{3}, 0, 0, -\frac{10}{3}\right)$.

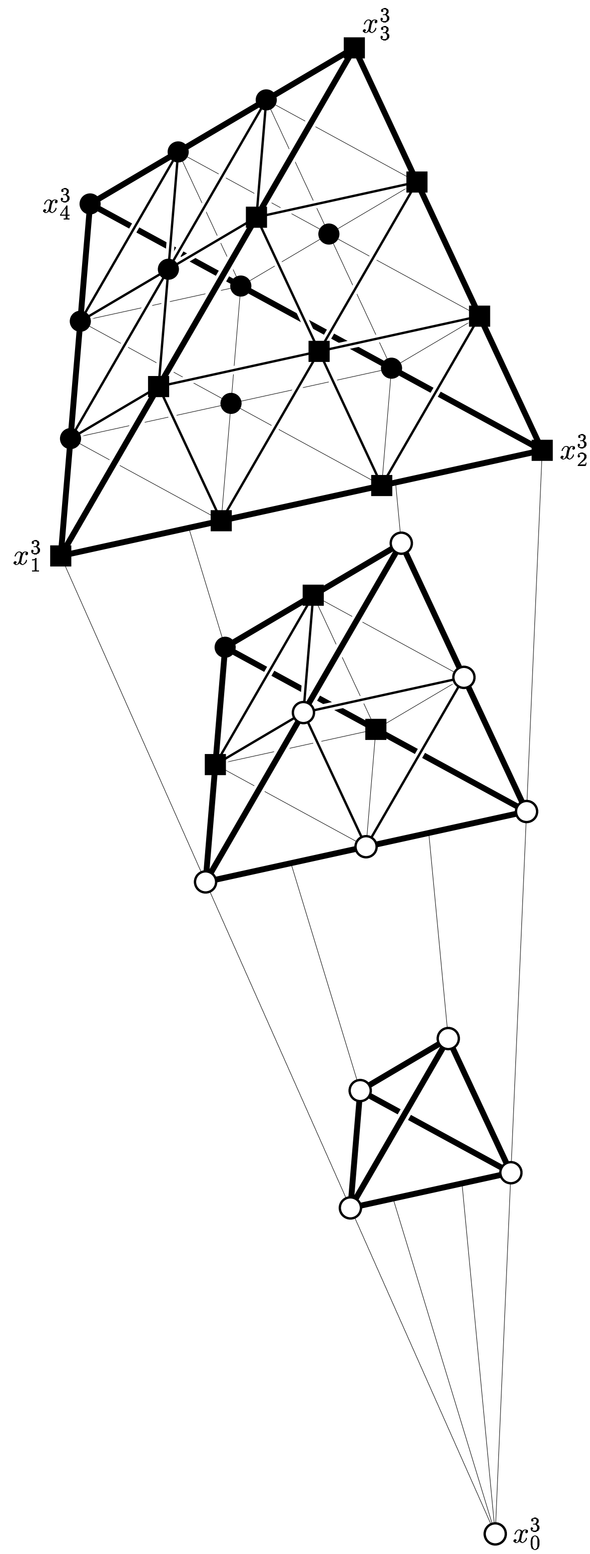

Figure 3.2(b). See lemma 3.4.
$\bar{H}_v$ for $v = \left(\frac{5}{2}, 0, 0, 0, -\frac{5}{2}\right)$.

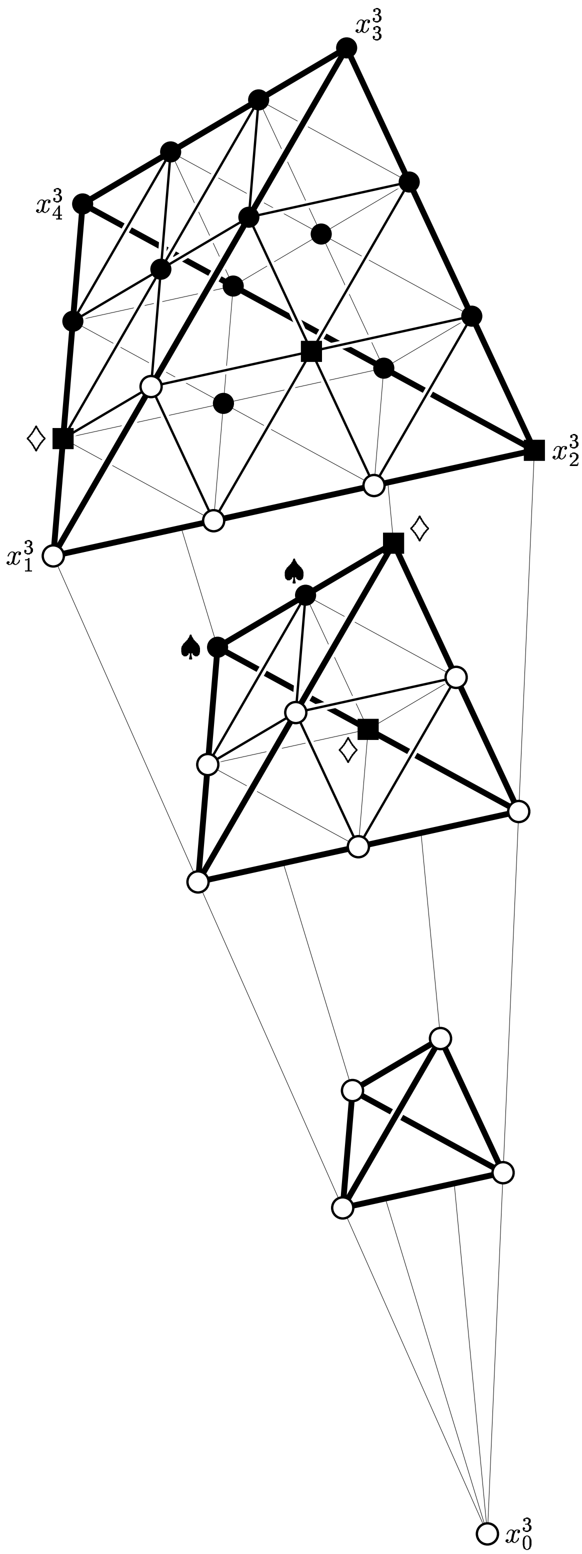

Figure 3.2(c). See lemma 3.4.
$\bar{H}_v$ for $v = \left(\frac{5}{2}, \frac{5}{4}, 0, -\frac{5}{4}, -\frac{5}{2}\right)$.

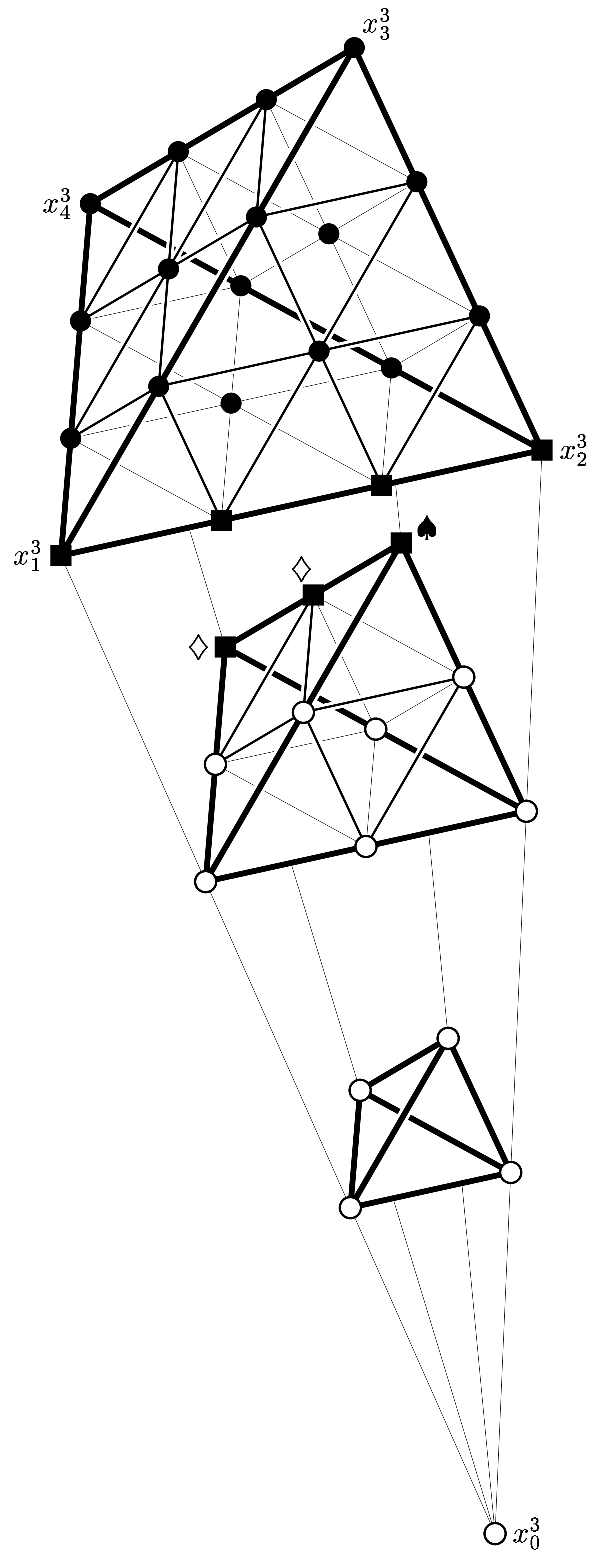

Figure 3.2(d). See lemma 3.4.
$\bar{H}_v$ for $v = \left(\frac{10}{3}, 0, 0, -\frac{5}{3}, -\frac{5}{3}\right)$.

Theorem 4.1. *The orbit of Δ is minimal. A cubic threefold with a D_4 singularity is strictly semistable and degenerates to Δ. A cubic threefold with any other singularity of nullity ≥ 2 is unstable.*

Proof: We prove the last two claims first. Suppose that T is a cubic threefold with a singularity of nullity ≥ 2 placed at P. If the nullity is ≥ 3 then we may choose coordinates so that

$$F = Kx_0x_4^2 + \text{cubic}(x_1, \ldots, x_4) .$$

Such an F has type 3.1(a) and is unstable by lemma 3.1. Now suppose the nullity is 2; we may write

$$F = x_0x_3x_4 + C(x_1, x_2) + \text{other cubic terms}(x_1, \ldots, x_4) .$$

With respect to the weights $(\frac{1}{3}, \frac{1}{3}, \frac{1}{2}, \frac{1}{2})$ we have

$$f(x_1, \ldots, x_4) = x_3x_4 + C(x_1, x_2) + (\text{terms of degree} > 1) .$$

If the singularity is not of type D_4 then C has a multiple root by lemma 2.2. By a linear coordinate change we may suppose that the x_1^3 and $x_1^2x_2$ terms vanish; then F has type 3.1(b) and is unstable. If the singularity has type D_4 then C has three distinct roots, and by a linear change of variables we may suppose $C = x_1^3 + x_2^3$. Then F degenerates to $x_0x_3x_4 + x_1^3 + x_2^3$ under the 1-parameter group

$$(x_0, \ldots, x_4) \mapsto (\lambda^{-2}x_0, x_1, x_2, \lambda x_3, \lambda x_4) .$$

Finally, if Δ degenerates to some other threefold T, then T must be semistable and also have a singularity of nullity ≥ 2. Therefore T degenerates to Δ. Since T and Δ degenerate to each other they are projectively equivalent. □

The threefolds $T_{A,B}$ defined in (1.1) play a similar central role for the threefolds with a singularity of nullity 1. As mentioned in the introduction, if $k \neq 0$ then $T_{A,B}$ and $T_{k^2A,kB}$ are projectively equivalent, and we write T_β, $\beta = 4A/B^2$ for their projective equivalence class. When $\beta = 1$, T_β is a chordal cubic; in fact, $T_{1,-2}$ is the secant variety of the standard rational normal curve of degree 4, which is the curve C of points $[x_0, \ldots, x_4] \in \mathbb{C}P^4$ where the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix}$$

has rank one. One can check that C is the entire singular locus of $T_{1,-2}$. The singular points of a chordal cubic are equivalent to each other under symmetries, each has rank one, and one can show that the threefold does not contain any planes at all. It follows from theorem 3.3 that chordal cubics are semistable. When $\beta \neq 1$, T_β has two A_5 singularities, and when $\beta = 0$, T_β also has an A_1 singularity. (The A_5 singularities of the $T_{A,B}$ lie at $[1,0,0,0,0]$ and $[0,0,0,0,1]$ with null lines L_{01} and L_{34}, and the A_1 singularity of $T_{0,B}$ lies at $[0,0,1,0,0]$.) There are no other singularities, and it follows from theorem 3.3 that all the T_β are semistable. They are strictly semistable because each $T_{A,B}$ is preserved by the 1-parameter group

$$\sigma_\lambda : (x_0, \ldots, x_4) \mapsto (\lambda^2 x_0, \lambda x_1, x_2, \lambda^{-1}x_3, \lambda^{-2}x_4) . \tag{4.1}$$

Theorem 4.2. *Suppose T is a cubic threefold with a singularity at P of nullity 1 and milnor number $\mu \geq 5$. If the singularity is a double line then T is unstable. Otherwise,*

(i) if P has type A_5 then T is strictly semistable and degenerates to some T_β, $\beta \neq 1$;
(ii) if $\mu \geq 6$ and T contains no plane containing the null line, then T is strictly semistable and degenerates to a chordal cubic;
(iii) if $\mu \geq 6$ and T contains a plane containing the null line, then T is unstable.

Proof: A double line makes T unstable, because in suitable coordinates T has type 3.1(e). So suppose henceforth that the singularity is not a double line. By choice of coordinates we may take

$$F = x_0(x_3^2 - x_2x_4) + C(x_1, \ldots, x_4) \ . \tag{4.2}$$

Probably the best way to follow the rest of this argument, and similar ones later, is to prepare a large picture of the 4-simplex and place and remove coins to indicate coefficients known to be zero, or known to be nonzero, or unknown. This method makes it easy to tell what the result of one of our linear substitutions is, and is how we discovered these arguments.

The x_1^3 term must vanish, or else P is only an A_2 singularity. We write the terms divisible by x_1^2 as $x_1^2(ax_2 + bx_3 + cx_4)$. At least one of a, b and c is nonzero, or else T would contain L_{01} as a double line. In fact, one of a and c must be nonzero because otherwise we would have $b \neq 0$ and then P would be only an A_3 singularity. By exchanging x_2 and x_4 if necessary we may take $c \neq 0$, and by rescaling the variables we may take $c = 1$. By a substitution $x_4 \mapsto x_4 + \lambda x_2 + \lambda' x_3$ we use the $x_1^2x_4$ term to kill the $x_1^2x_2$ and $x_1^2x_3$ terms. That is, $a = b = 0$. This has the side-effect of possibly reintroducing the $x_0x_2^2$ and $x_0x_2x_3$ terms, which we took to vanish in (4.2). By $x_3 \mapsto x_3 + \lambda x_2$ we use the $x_0x_3^2$ term to kill the reintroduced $x_0x_2x_3$ term. Then the other reintroduced term vanishes automatically, for otherwise P is only an A_3 singularity. For more convenient singularity analysis, by a substitution $x_1 \mapsto x_1 + \lambda x_2$ we use the $x_1^2x_4$ term to kill the $x_1x_2x_4$ term. Now the $x_1x_2^2$ term vanishes, for else P is only an A_4 singularity.

We have killed so many terms that F has type 3.2(c). In particular,

$$F = Ax_2^3 + x_0x_3^2 + x_1^2x_4 - x_0x_2x_4 + Bx_1x_2x_3 + \text{terms marked by dots in figure 3.2(c)}$$

for some constants A and B. The five terms given explicitly define $F_{A,B}$ and are indicated by solid squares in figure 3.2(c). This means that F degenerates to $F_{A,B}$ under the 1-parameter group (4.1). By the analysis leading to (3.2), with $K = A$ and $L(x_3, x_4) = Bx_3$, there are local coordinates about P in which T is defined by

$$f(x_1, \ldots, x_4) = x_3^3 - x_2x_4 - Ax_1^6 - Bx_1^3x_3 + (\text{terms of degree} > 1) \tag{4.3}$$

with respect to the weights $\left(\frac{1}{6}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. If P has type A_5 then the degree 1 terms must define an isolated singularity, so that $4A \neq B^2$ and T degenerates to T_β for $\beta = 4A/B^2 \neq 1$. Since T_β is strictly semistable, so is T. On the other hand, if $\mu \geq 6$ then the degree 1 terms must not define an isolated singularity, so $4A = B^2$. Suppose that T contains no plane containing the null line at P, which is L_{01}. Then $A \neq 0$, for otherwise T contains the plane $x_3 = x_4 = 0$. Therefore $B \neq 0$, and by rescaling the variables we may take $B = -2$. Then $A = 1$, T degenerates to the chordal cubic $T_{1,-2}$, and T is strictly semistable. Finally, suppose T contains a plane containing the null line. Taking limits, we see that $T_{A,B}$ does also, and a calculation shows that this only occurs if $A = 0$. This forces $B = 0$, so T has type 3.1(d) and is unstable by lemma 3.1. (Note: by the proof of theorem 3.3, the milnor number at P is at least 7, so there is no cubic threefold containing an A_6 singularity and also a plane containing the null line. Our argument also shows that if T contains an A_5 singularity and a plane containing its null line, then $A = 0$ and T degenerates to T_0, the T_β with the extra singularity.) □

§5. Uniqueness theorems

In this section we characterize the special cubic threefolds Δ and T_β by their singularities. The arguments are independent of the rest of the paper, and the only results given here that are used later are the computations of the symmetry groups and projective equivalence classes of the T_β. Throughout this section, T denotes a cubic threefold with defining form F. We will begin with the uniqueness of Δ.

Lemma 5.1. *If one singularity of T lies in the null space of another, then T is singular along the line joining them.*

Proof: With no loss of generality we may suppose that one singularity lies at $P = [1,0,0,0,0]$, that its null space contains the line L_{04}, and that a second singularity lies at $P' = [0,0,0,0,1]$. Then all terms of F divisible by x_0^2, x_0x_4 or x_4^2 vanish, so T contains L_{04} as a double line. □

Lemma 5.2. *Suppose T has two isolated singularities of nullity 2. Then their null planes meet along a line.*

Proof: By the previous lemma, neither singularity lies on the null plane of the other. We need only exclude the possibility that the null planes meet in a point. Suppose without loss of generality that one singularity lies at P with null plane $x_3 = x_4 = 0$ and the other lies at P' with null plane $x_0 = x_1 = 0$. Then all terms of F divisible by x_0^2, x_0x_1, x_0x_2, x_4^2, x_4x_2 or x_4x_3 vanish. Each remaining term is divisible by a quadratic in x_1, x_2 and x_3, so that T contains L_{04} as a double line, contrary to hypothesis. □

Lemma 5.3. *Suppose T has two isolated singularities of nullity 2. Then the hyperplane spanned by their null planes contains no other singularities of T.*

Proof: We suppose the singularities lie at P and P', and that their null planes are $x_2 = x_4 = 0$ and $x_0 = x_2 = 0$. These planes span the hyperplane $x_2 = 0$ and meet along the line L_{13}. All terms of F divisible by x_i^2, x_ix_1 or x_ix_3 vanish for $i = 0$ or 4, leaving

$$F = Kx_0x_2x_4 + x_2^2(K'x_0 + K''x_4) + C(x_1, x_2, x_3) .$$

Now, $K \neq 0$, or else T would contain L_{04} as a double line. By the substitution $x_4 \mapsto x_4 + \lambda x_2$ we use the $x_0x_2x_4$ term to kill the $x_0x_2^2$ term, and by $x_0 \mapsto x_0 + \lambda x_2$ we use the $x_0x_2x_4$ term to kill the $x_2^2x_4$ term. These substitutions preserve the singularities and their null planes. After rescaling to take $K = 1$, we have $F = x_0x_2x_4 + C(x_1, x_2, x_3)$. An extra singularity of T lying in the hyperplane $x_2 = 0$ must have nonzero x_0 and x_4 coordinates in order to avoid lying in either null plane. But then its orbit under the 1-parameter group

$$(x_0, \ldots, x_4) \mapsto (\lambda x_0, x_1, x_2, x_3, \lambda^{-1}x_4)$$

is a curve of singularities of T whose closure contains both the given singularities, contrary to the hypothesis that they are isolated. □

Theorem 5.4. *If T has three D_4 singularities then T is projectively equivalent to Δ.*

Proof: If the singularities are s_1, s_2 and s_3, with null planes N_1, N_2 and N_3, then we claim that $N_1 \cap N_2 \cap N_3$ is a line. Otherwise, $N_3 \cap N_1$ and $N_3 \cap N_2$ are distinct lines, which forces N_3 and hence s_3 to lie in the span of N_1 and N_2, contrary to the previous lemma. We take $s_1 = [1,0,0,0,0]$, $s_2 = [0,1,0,0,0]$ and $s_3 = [0,0,1,0,0]$, and suppose that the intersection of the null planes is L_{34}. Then all terms of F divisible by x_i^2, $x_i x_3$ or $x_i x_4$ vanish for $i = 0, 1, 2$, leaving $F = Kx_0x_1x_2 + C(x_3, x_4)$. In order for the singularities to have type D_4, K must be nonzero and C must have three distinct roots. After a linear transformation we may take $K = 1$ and $C = x_3^3 + x_4^3$. $\square$

Next we will characterize the T_β. We begin by describing their symmetry groups and classifying them up to projective equivalence. Each $T_{A,B}$ is preserved by the symmetries σ_λ of (4.1), and also by the coordinate reversal $\tau : x_i \mapsto x_{4-i}$. A tedious computation shows that if $B \neq 0$ and $4A \neq B^2$ then this is the full symmetry group. (One considers the general linear transformation that preserves each singularity and its null line and also acts trivially on the line spanned by the A_5 singularities.) If $B = 0$ then the group is twice as large, containing the extra transformations

$$(x_0, \ldots, x_4) \mapsto (x_0, \pm x_1, x_2, \mp x_3, x_4) .$$

If $4A = B^2 \neq 0$ then $T_{A,B}$ is a chordal cubic, with symmetry group $\mathrm{PGL}(2, \mathbb{C})$. Thus there are three special values β can take: T_0 has an extra singularity, T_1 is the chordal cubic, and T_∞ has an extra symmetry.

Lemma 5.5. *T_β is projectively equivalent to $T_{\beta'}$ if and only if $\beta = \beta'$.*

Proof: Suppose $T_\beta \cong T_{\beta'}$. If either T_β or $T_{\beta'}$ is a chordal cubic then so is the other, so we may suppose that β and β' are different from 1. We choose A, B, A', B' with $\beta = 4A/B^2$ and $\beta' = 4A'/B'^2$ and suppose that $g \in \mathrm{GL}(5, \mathbb{C})$ carries $F_{A,B}$ to $F_{A',B'}$. Since the connected components of the stabilizers of these forms are the same subgroup of $\mathrm{GL}(5, \mathbb{C})$ and are conjugate under g, g normalizes the 1-parameter group (4.1). After multiplying g by τ we may even suppose that g centralizes it. Then g is a diagonal matrix and it is easy to check that $4A/B^2 = 4A'/B'^2$. $\square$

Lemma 5.6. *Suppose T has a singularity of type A_n, $n \geq 4$, and another singularity of any type. Then the null spaces of the singularities do not meet.*

Proof: We place the singularity known to have type A_n at P, with null line L_{01}. By lemma 5.1, the second singularity Q does not lie on this line, so we place it at $[0, 0, 1, 0, 0]$. Again by lemma 5.1, Q's null space misses P, so under the assumption that the null spaces meet we may suppose that they meet at $[0, 1, 0, 0, 0]$. Then L_{12} lies in the null space of Q. We will show that P is a non-isolated singularity, which is a contradiction. All terms of F divisible by x_0^2, x_0x_1, x_2^2 or x_1x_2 vanish. By mixing together x_3 and x_4 we may suppose that the $x_0x_2x_3$ term vanishes. Then the $x_0x_3^2$ and $x_0x_2x_4$ terms must be nonzero, because the nullity at P is only 1. By rescaling the variables we take the coefficients to be 1 and -1. By $x_3 \mapsto x_3 + \lambda x_4$ we use the $x_0x_3^2$ term to kill the $x_0x_3x_4$ term, and then by $x_2 \mapsto x_2 + \lambda x_4$ we use the $x_0x_2x_4$ term to kill the $x_0x_4^2$ term. This yields

$$F = x_0(x_3^2 - x_2x_4) + x_2Q(x_3, x_4) + C(x_1, x_3, x_4) .$$

The x_1^3 term must vanish, or else P is only an A_2 singularity. The $x_1^2x_3$ term must vanish, or else P is only an A_3 singularity. If the $x_1^2x_4$ term vanishes then T contains L_{01} as a double line and we are done. So suppose it does not vanish; by rescaling the variables we take the coefficient to be 1. By $x_1 \mapsto x_1 + \lambda x_4$ we use this term to kill the $x_1x_4^2$ term, and by $x_1 \mapsto x_1 + \lambda x_3$ we use it to kill the $x_1x_3x_4$ term. Finally, by $x_0 \mapsto x_0 + \lambda x_4$ we use the $x_0x_2x_4$ term to kill the $x_2x_4^2$ term, and by $x_0 \mapsto x_0 + \lambda x_3$ we use it to kill the $x_2x_3x_4$ term. We have reduced to the situation

$$F = x_0(x_3^2 - x_2x_4) + x_1^2x_4 + x_3^2(Kx_1 + K'x_2) + C(x_3, x_4) .$$

We obtain a local defining equation for T at P by taking $x_0 = 1$ and then substituting $x_2 + x_1^2$ for x_2. The result is

$$x_3^2 - x_2x_4 + x_3^2(Kx_1 + K'x_1^2 + K'x_2) + C(x_3, x_4) ,$$

which is singular along the curve $x_2 = x_3 = x_4 = 0$, so P is not isolated. $\square$

Theorem 5.7. *If T has two A_5 singularities then T is projectively equivalent to some T_β, $\beta \neq 1$.*

Proof: We place the singularities at P and P', and since their null lines are skew we may take them to be L_{01} and L_{34}, respectively. All terms of F divisible by x_0^2, x_0x_1, x_4^2 or x_4x_3 vanish. Since the nullity at P is only 1, the $x_0x_3^2$ and $x_0x_2x_4$ terms are nonzero, and the symmetric argument shows that the $x_4x_1^2$ and (again) $x_0x_2x_4$ terms are nonzero. By rescaling the variables we take the coefficients of the $x_0x_3^2$, $x_0x_2x_4$ and $x_1^2x_4$ terms to be 1, -1 and 1. Now we perform six linear substitutions. First, by $x_0 \mapsto x_0 + \lambda x_1$ we use the $x_0x_3^2$ term to kill the $x_1x_3^2$ term. Second, by $x_4 \mapsto x_4 + \lambda x_3$ we use the $x_4x_1^2$ term to kill the $x_3x_1^2$ term. Third, by $x_3 \mapsto x_3 + \lambda x_2$ we use the $x_0x_3^2$ term to kill the $x_0x_2x_3$ term. Fourth, by $x_1 \mapsto x_1 + \lambda x_2$ we use the $x_1^2x_4$ term to kill the $x_4x_2x_1$ term. Fifth, by $x_4 \mapsto x_4 + \lambda x_2$ we use the $x_0x_2x_4$ term to kill the $x_0x_2^2$ term. Finally, by $x_0 \mapsto x_0 + \lambda x_2$ we use the $x_4x_2x_0$ term to kill the $x_4x_2^2$ term. (Each even-numbered step differs from the previous step by coordinate reversal.)

We have reduced to the situation

$$
\begin{aligned}
F = {} & Ax_2^3 + x_0x_3^2 + x_1^2x_4 - x_0x_2x_4 + Bx_1x_2x_3 \\
& + \left(Cx_1^3 + C'x_3^3\right) + x_2\left(Dx_1^2 + D'x_3^2\right) + x_2^2\left(Ex_1 + E'x_3\right)
\end{aligned}
$$

for some constants $A, \ldots, E'$. We have $C = 0$, for otherwise P is only an A_2 singularity. Then $D = 0$, for otherwise P is only an A_3 singularity. Finally, $E = 0$, for otherwise P is only an A_4 singularity. The symmetric argument proves $C' = D' = E' = 0$, so $F = F_{A,B}$. We must have $4A \neq B^2$ in order for the singularities to be isolated, so that $\beta = 4A/B^2$ lies in $\mathbb{C} \cup \{\infty\}$ and $\beta \neq 1$. □

§6. The main results

In this section we prove the theorems stated in the introduction.

Lemma 6.1. *The chordal cubics are the only semistable cubic threefolds with non-isolated singularities.*

Proof: Suppose T is semistable with non-isolated singularities. It follows from theorem 4.1 that each positive-dimensional component of the singular locus is a curve. We fix such a component C and observe that C is not a line by theorem 4.2. Since T has degree 3, it contains the secant variety of C. C cannot lie in a plane, or else T would contain this plane and be unstable by theorem 4.2(iii). Since C does not lie in a plane, its secant variety is three-dimensional. If C lies in a 3-space then T contains this 3-space, and after a linear transformation we may take $F = x_0 Q(x_1, \ldots, x_4)$. Then T is unstable because of the 1-parameter group

$$(x_0, \ldots, x_4) \mapsto (\lambda^{-4}x_0, \lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4).$$

Therefore C does not lie in any hyperplane. Since a cubic surface can have at most 4 isolated singularities [5], a generic hyperplane meets C in ≤ 4 points, so that C has degree ≤ 4. It follows that C is a rational normal curve of degree 4 and that T is its secant variety. □

Theorems 1.1, 1.3 and 1.4 follow from theorems 3.3, 3.5, 4.1, and 4.2 and lemma 6.1. To prove theorem 1.2 we will need the following converse to theorem 4.2(ii).

Lemma 6.2. *A cubic threefold that degenerates to a chordal cubic contains a singularity of nullity one and milnor number ≥ 6, but no plane containing its null line.*

Proof: It follows from the previous lemma that the chordal cubics form a minimal orbit. Therefore Richardson's relative form of the Hilbert-Mumford criterion applies [4, theorem 4.2]. That is, if T degenerates to $T_{1,-2}$ then T is projectively equivalent to a threefold that degenerates to $T_{1,-2}$ under some diagonalizable 1-parameter subgroup of $\mathrm{SL}(5,\mathbb{C})$ that stabilizes $T_{1,-2}$. Since all such subgroups are conjugate in the automorphism group of $T_{1,-2}$, we may take the group to be (4.1). That is, we may suppose that T has type 3.2(c) and that

$$F = x_2^3 + x_0x_3^2 + x_1^2x_4 - x_0x_2x_4 - 2x_1x_2x_3 + \text{terms marked by dots in figure 3.2(c)}\ .$$

The nullity at P is clearly one, the milnor number is at least 6 by the argument concerning (4.3), and T cannot contain a plane because the chordal cubic does not. □

Proof of theorem 1.2: We showed in theorem 4.1 that the orbit of Δ is minimal. It follows from theorems 4.1 and 4.2 that the only other possibilities for minimal strictly semistable orbits are the T_β. We have just seen that the chordal cubics form a minimal orbit. The other T_β are minimal because they cannot degenerate to chordal cubics by lemma 6.2, cannot degenerate to Δ because they have singularities of milnor number > 4, and cannot degenerate to each other because their symmetry groups all have the same dimension. □

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