A HOMOLOGICAL CHARACTERIZATION OF HYPERBOLIC GROUPS

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Abstract. A finitely presented group G is hyperbolic iff $H_1^{(1)}(G,\mathbb{R}) = 0 = \bar{H}_2^{(1)}(G,\mathbb{R})$, where $H_*^{(1)}$ (resp. $\bar{H}_*^{(1)}$) denotes the ℓ_1-homology (resp. reduced ℓ_1-homology). If Γ is a graph, then every ℓ_1 1-cycle in Γ with real coefficients can be approximated by 1-cycles of compact support. A 1-relator group G is hyperbolic iff $H_1^{(1)}(G,\mathbb{R}) = 0$.

§1. Introduction.

In [Ge1] the second author showed that a finitely presented group G is word hyperbolic iff a certain cohomology group vanishes, namely $H^2_{(\infty)}(G,\ell_\infty) = 0$. G. A. Swarup asked whether there are analogous vanishing theorems in homology. Our main result, Corollary 4.8, is that a finitely presented group G is word hyperbolic (henceforth called hyperbolic) iff $H_1^{(1)}(G,\mathbb{R}) = 0 = \bar{H}_2^{(1)}(G,\mathbb{R})$, where $H_*^{(1)}$ denotes the (unreduced) ℓ_1-homology and where $\bar{H}_*^{(1)}$ denotes the reduced ℓ_1-homology; their definitions are recalled in §2.

It is not clear whether there is any relation between vanishing theorems for ℓ_∞-cohomology and those for ℓ_1-homology, for while it is true that ℓ_∞ is the dual of ℓ_1, the Banach space ℓ_1 is not reflexive. For example, $H^1_{(\infty)}(G,\mathbb{R})$ and the reduced group $\bar{H}^1_{(\infty)}(G,\mathbb{R})$ are both nonzero for every infinite finitely generated group G.

Our main technical tool, Theorem 3.3, states that every ℓ_1 1-cycle with real coefficients on a graph can be approximated by 1-cycles of compact support. We give in 6.1 below an example due to E. Formanek, which shows that there is no analog of Theorem 3.3 in general for higher dimensional cycles on a complex.

In §5 we show that if G is either a 1-relator group or the fundamental group of a finite piecewise Euclidean 2-complex of nonpositive curvature, then $H_2^{(1)}(G,\mathbb{R}) = 0$. It follows from this and from Theorem 4.5 that such a group G is hyperbolic iff $H_1^{(1)}(G,\mathbb{R}) = 0$.

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§2. The ℓ_1-homology and ℓ_∞-cohomology of a group.

A norm on an abelian group A is a function $|\ \ |: A \to \mathbb{R}$ satisfying $|-a| = |a|$, $|a+a'| \le |a| + |a'|$, and $|a| \ge 0$ with $|a| = 0$ iff $a = 0$, for all $a, a' \in A$.

We recall that a group G is said to be of type $\mathcal{F}_n$ if there is a CW-complex X' of type $K(G,1)$ with finite n-skeleton. For example, G is of type $\mathcal{F}_1$ iff it is finitely generated and of type $\mathcal{F}_2$ iff it is finitely presented.

If G is a group of type $\mathcal{F}_{n+1}$, let X' be a CW-complex of type $K(G,1)$ with finite $(n+1)$-skeleton and let X be the universal cover of X'. A summable i-chain f on X with values in A is a skew-symmetric function from the oriented i-cells of X with values in A (so $f(\bar{e}) = -f(e)$ where e is an i-cell and $\bar{e}$ is the same geometric i-cell with the opposite orientation[1]) such that $\sum_e |f(e)| < \infty$, where the sum is over all oriented i-cells e. It is convenient to think of the chain f as an infinite sum $\sum_{e\in\mathcal{O}} f(e)e$ where $\mathcal{O}$ is an orientation on the i-cells, that is $\mathcal{O}$ contains precisely one of the pair e, $\bar{e}$ for each oriented i-cell e. Then we define the ℓ_1-norm $|f|$ of f by

$$|f| = \sum_{e\in\mathcal{O}} |f(e)|. \tag{2.1}$$

If $i \le n+1$ then because X' has only finitely many i-cells, there is an upper bound on the ℓ_1-norms of the boundaries ∂e of i-cells e. This means that if $C_i^{(1)}(X,A)$ denotes the set of summable i-chains with values in A, then the function ∂ extends to a continuous homomorphism $\hat{\partial} : C_i^{(1)}(X,A) \to C_{i-1}^{(1)}(X,A)$. One checks that $\hat{\partial}^2 = 0$, so we have a chain complex defined in a range of degrees $i \le n+1$. The ℓ_1-homology $H_i^{(1)}(X,A)$ is defined in the usual way as $Z_i^{(1)}(X,A)/B_i^{(1)}(X,A)$, where $Z_i^{(1)}(X,A)$ is the subgroup of summable i-cycles and where $B_i^{(1)}$ is the image of $\hat{\partial} : C_{i+1}^{(1)}(X,A) \to C_i^{(1)}(X,A)$. This makes sense for $i \le n$.

Now we consider the ℓ_∞ cohomology groups. We define $C^i_{(\infty)}(X,A)$ to be the subgroup of cellular i-cochains h such that there is a number $M_h > 0$ with $|h(\sigma)| \le M_h$ for all i-cells σ. It follows from the finiteness of ${X'}^{(n+1)}$ that the coboundary δh lies in $C^{i+1}_{(\infty)}(X,A)$ if $i \le n$. One defines cocycles $Z^i_{(\infty)}$, coboundaries $B^i_{(\infty)} = \delta(C^{i-1}_{(\infty)})$, and cohomology groups $H^i_{(\infty)} = Z^i_{(\infty)}/B^i_{(\infty)}$ in the usual way. This definition makes sense for all $i \le n+1$, since one does not require the finiteness conditions on the $(n+2)$-cells to formulate the condition $\delta h = 0$ for $h \in C^{n+1}_{(\infty)}(X,A)$.

The value of the ℓ_1-homology and ℓ_∞-cohomology groups arises from their quasi-isometry invariance.[2] It is known that the condition that a group G have type $\mathcal{F}_n$ is a quasi-isometry invariant [Al][Gr2]. It is also known that if X' and Y' are a $K(G,1)$ and a $K(G',1)$, respectively, both with finite $(n+1)$-skeleta, and if the groups G and G' are quasi-isometric, then there are isomorphisms $H_i^{(1)}(X,A) \cong H_i^{(1)}(Y,A)$ for

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[1] Precisely, $\bar{e}$ is obtained from e by precomposing the characteristic mapping of e with some fixed orientation reversing involution of the i-cell.

[2] For the notion of quasi-isometry of metric spaces and of finitely generated groups see [GH].

$i \le n$ and $H^i_{(\infty)}(X,A) \cong H^i_{(\infty)}(Y,A)$ for $i \le n+1$; here X and Y are the universal covers of X' and Y' respectively. A proof of these facts can be constructed along the lines of [Ge2] §11. We may thus unambiguously define $H_i^{(1)}(G,A)$ as $H_i^{(1)}(X,A)$ and $H^i_{(\infty)}(G,A)$ as $H^i_{(\infty)}(X,A)$, and we note that the vanishing of either of these groups is a *geometric property* in the sense that it is an invariant of quasi-isometry type.

In this paper we shall be interested mainly in the case $A = \mathbb{R}$. For $i \le n+1$, $Z_i^{(1)}(X,\mathbb{R})$ is a closed subspace of $C_i^{(1)}(X,\mathbb{R})$ since it is the kernel of the bounded linear operator $\hat\partial : C_i^{(1)}(X,\mathbb{R}) \to C_{i-1}^{(1)}(X,\mathbb{R})$. However, the image $B_i^{(1)}(X,\mathbb{R})$ need not be a closed subspace. It is usual to define the reduced ℓ_1-homology $\bar H_i^{(1)}(X,\mathbb{R})$ to be $Z_i^{(1)}(X,\mathbb{R})/\bar B_i(X,\mathbb{R})$, where $\bar B_i(X,\mathbb{R})$ is the closure of $B_i(X,\mathbb{R})$ in the normed linear space $C_i^{(1)}(X,\mathbb{R})$ under the ℓ_1-norm, defined in (2.1) above. $\bar H_i^{(1)}(X,\mathbb{R})$ is defined for $i \le n+1$ since the closure operation $\bar B_{n+1}(X,\mathbb{R})$ does not depend on the continuity of ∂_{n+2}, and it is quasi-isometry invariant in the range where it is defined.

§3. The approximation theorem for summable 1-cycles.

We will use Serre's formulation of a graph Γ as consisting of two sets, its vertices $V(\Gamma)$ and edges $E(\Gamma)$. The set $E(\Gamma)$ is equipped with a fixed-point-free involution $e \mapsto \bar e$ as well as the initial-vertex map $\iota : E(\Gamma) \to V(\Gamma)$. One defines the terminal vertex $\tau(e)$ of an edge to be $\iota(\bar e)$. Serre's notion of an edge corresponds to what is usually called a directed edge, since one of its vertices is distinguished as its initial vertex and the other as its terminal vertex. A path is a sequence (e_i) of edges satisfying $\tau e_i = \iota e_{i+i}$ for all i. A path is simple if the vertices ιe_i are all distinct. A circuit is a path $(e_1, \dots, e_n)$ with $\tau e_n = \iota e_1$, up to the equivalence relation that two paths represent the same circuit if one is obtained by the other by cyclic permutation of its edges. Since our notion of edge includes a direction, these concepts might also be called "directed paths" and "directed circuits".

A real-valued 1-cochain is a function $f : E(\Gamma) \to \mathbb{R}$ such that $f(e) = -f(\bar e)$ (skew-symmetry) for all $e \in E(\Gamma)$. For each vertex v of Γ we define the divergence of f at v to be

$$\mathrm{Div}_v(f) = \sum_{\iota e = v} f(e),$$

provided that the sum converges absolutely. The 1-cochain f is called a 1-chain if f has finite support. If e is an edge of Γ then e is identified with the 1-chain which takes the value 1 on e, -1 on $\bar e$, and is zero on all other edges of Γ.

A 1-cochain f is called, by a slight abuse of terminology, a summable (or ℓ_1) 1-chain if it is summable as a function on $E(\Gamma)$. (This definition of a summable chain agrees with that given in §2.) In this case, $\mathrm{Div}_v(f)$ is defined for all vertices v, and we observe that f is a summable 1-cycle iff $\mathrm{Div}_v(f) = 0$ for all vertices v. We will write $\Gamma_+(f)$ for the set of edges on which f takes positive values.

Here is a geometric interpretation of these concepts. A cochain may be regarded as a "flow" along the edges of $\Gamma_+(f)$, with the magnitude of the flow along an edge

e given by $f(e)$. An edge of $\Gamma_+(f)$ is an edge of Γ which points in the direction of the flow. $\mathrm{Div}_v(f)$ is the net flow out of the vertex v, and f is a cycle just if the substance flowing is conserved. The following lemma is a combinatorial analog of Stokes' theorem.

Lemma 3.1. *If Γ is a graph and f is a summable 1-chain on Γ, then*

$$\sum_v \mathrm{Div}_v(f) = 0.$$

Proof. For each $e \in E(\Gamma)$ the term $f(e)$ occurs in the sum $\mathrm{Div}_{\iota e}(f)$ and the term $f(\bar{e}) = -f(e)$ occurs in the sum $\mathrm{Div}_{\tau e}(f)$. By absolute convergence we may rearrange terms freely; after doing so, all terms cancel.

Lemma 3.2. *If Γ is a graph and f is a summable 1-cycle on Γ such that $\Gamma_+(f)$ contains no nontrivial circuits, then $f = 0$.*

Proof. Suppose that $f(e_0) = A > 0$ for some edge e_0. Let Γ' be the subgraph of Γ defined by the condition that $e, \bar{e} \in \Gamma'$ if either e or $\bar{e}$ lies in a path of $\Gamma_+(f)$ that begins at ιe_0; that is, Γ' is the smallest subgraph of Γ that contains all edges of all paths in $\Gamma_+(f)$ which begin at e_0. Let f' be the summable 1-chain on Γ which coincides with f on Γ' and vanishes elsewhere. If v is a vertex of Γ' then an edge of $\Gamma_+(f)$ whose initial (resp. terminal) vertex is v does (resp. might) lie in $\Gamma_+(f')$. This shows that $\mathrm{Div}_v(f') \geq \mathrm{Div}_v(f) = 0$ for all vertices v. Lemma 3.1 shows that $\sum_v \mathrm{Div}_v(f') = 0$, so we conclude that $\mathrm{Div}_v(f') = 0$ for all v. Taking $v = \iota e_0$ we see that ιe_0 must be the terminal vertex of some edge of $\Gamma_+(f')$ and hence of $\Gamma_+(f)$. This contradicts the hypothesis that $\Gamma_+(f)$ contains no circuits, completing the proof.

We turn now to the main result of this section. We call a family $\mathcal{F}$ of 1-cocycles *coherent* if for any two 1-cocycles f and g in $\mathcal{F}$ and for each edge e we have $f(e)g(e) \geq 0$. That is, if both $f(e)$ and $g(e)$ are nonzero then they have the same sign. If $c = (e_1, \ldots, e_n)$ is a simple circuit in Γ then we construct the 1-cycle $\sum_{i=1}^n e_i$, which we will also call c and refer to as a simple circuit. This duplicate definition of c should cause no confusion, since the (simple) circuit may be reconstructed from the 1-cycle.

Theorem 3.3. *If Γ is a graph and f is a summable real valued 1-cycle on Γ, then f can be approximated arbitrarily closely in the ℓ_1-sense by finite sums of simple circuits with real coefficients. More precisely, there is a countable coherent family C of simple circuits and a function $g : C \to [0, \infty)$ so that $f = \sum_c g(c)c$, where the convergence of the sum is monotone.*

Proof. The support of f is countable, since f is summable; this shows that the family of simple directed circuits in $\Gamma_+(f)$ is countable. We take C to be this family, which is obviously coherent. The space of functions from C to $[0, \infty)$ is partially ordered in the obvious way: $g \geq h$ if $g(c) \geq h(c)$ for all $c \in C$. To each summable $g : C \to [0, \infty)$ there is associated the cocycle z_g (not necessarily summable) on Γ defined by

$$z_g = \sum_{c \in C} g(c)c.$$

The convergence is monotone because $g(c) \geq 0$ for all c and C is a coherent family. Consider the set $\mathcal{S}$ of functions $g : C \to [0, \infty)$ satisfying

$$0 \leq z_g(e) \leq f(e) \text{ for all } e \in \Gamma_+(f). \tag{3.3.1}$$

A standard Zorn's lemma argument shows that $\mathcal{S}$ has a maximal element, say g. We will complete the proof by showing $z_g = f$.

By (3.3.1), $\Gamma_+(f - z_g)$ is a subset of $\Gamma_+(f)$. If $\Gamma_+(f - z_g)$ contained a simple circuit, say c_0, then letting a be the smallest value taken by $f - z_g$ on any edge of c_0 we would find that the function $g' : C \to [0, \infty)$ defined by

$$g'(c) = \begin{cases} g(c) + a, & \text{if } c = c_0; \\ g(c) & \text{otherwise,} \end{cases}$$

lies in $\mathcal{S}$ and strictly dominates g, contradicting the maximality of g. Therefore $\Gamma_+(f - z_g)$ contains no simple circuits and thus no circuits. By lemma 3.2 we have $f - z_g = 0$, completing the proof.

Remark. It is natural to ask whether theorem 3.3 admits generalizations, allowing approximations of summable n-chains in CW complexes. Example 6.1 below, due to E. Formanek, shows that this is not possible in general.

Corollary 3.4. *If T is a tree, then every summable 1-cycle on T is zero.*

If Γ is an arbitrary graph then $Z_1(\Gamma, \mathbb{R})$ is a closed subspace of $C_1(\Gamma, \mathbb{R})$ for the ℓ_1-norm topology, so we can consider its completion $\hat{Z}_1(\Gamma, \mathbb{R}) \subset C^{(1)}(\Gamma, \mathbb{R})$. The continuity of the boundary map $\partial : C_1(\Gamma, \mathbb{R}) \to C_0(\Gamma, \mathbb{R})$ shows that $\hat{Z}_1(\Gamma, \mathbb{R}) \subseteq Z^{(1)}(\Gamma, \mathbb{R})$. We have

Corollary 3.5. *For every graph Γ, $\hat{Z}_1(\Gamma, \mathbb{R}) = Z^{(1)}(\Gamma, \mathbb{R})$.*

Proof. If $f \in Z^{(1)}(\Gamma, \mathbb{R})$ then it follows from Theorem 3.3 that f can be approximated by 1-cycles of compact support. Since $\hat{Z}_1(\Gamma, \mathbb{R})$ is closed in $C^{(1)}(\Gamma, \mathbb{R})$, it follows that $f \in \hat{Z}_1(\Gamma, \mathbb{R})$.

§4. Hyperbolic groups.

A hyperbolic group G is a finitely presented group satisfying the linear isoperimetric inequality for fillings of edge-circuits in its Cayley graph [GH][Ge1]. If X' is a space of type $K(G, 1)$ with finite 2-skeleton and X is the universal cover of X', then G is hyperbolic iff there exists $K > 0$ (the isoperimetric constant) so that for all $z \in Z_1(X, \mathbb{Z})$ there exists $c \in C_2(X, \mathbb{Z})$ with $\partial c = z$ and $|c| \leq K|z|$ [Ge5]. The norms here are both ℓ_1-norms, for a basis of oriented i-cells for $C_i(X, \mathbb{Z})$, for $i = 1, 2$.

For the remainder of this section we abbreviate $H_i(X, \mathbb{R})$, $H_i^{(1)}(X, \mathbb{R})$, $B_i(X, \mathbb{R})$, etc., by H_i, $H_i^{(1)}$, B_i, etc.

Proposition 4.1. *If G is a hyperbolic group, then $H_1^{(1)} = 0$.*

Proof. Given $f \in Z_1^{(1)}$ we must show that $f = \partial c$ for some $c \in C_2^{(1)}$. By Theorem 3.3 we can write $f = \sum_{i\geq 1} a_i z_i$ where $\{z_i \mid i \geq 1\}$ is a coherent family of compactly supported integral 1-cycles on X and where $a_i \geq 0$ for all i.

By hyperbolicity there are $c_i \in C_2(X, \mathbb{Z})$ with $\partial c_i = z_i$ and $|c_i| \leq K|z_i|$ for all $i \geq 1$, where K is the isoperimetric constant. Let $c = \sum_i a_i c_i$. We calculate $|c| \leq \sum_i a_i |c_i| \leq K \sum_i a_i |z_i| = K|\sum_i a_i z_i| = K|f|$, where the second-to-last equality follows from the coherence of the z_i. In particular this shows that c is a summable 2-cochain and justifies the interchange of summation and boundary map in the calculation $\partial c = \sum_i a_i \partial c_i = \sum_i a_i z_i = f$. This completes the proof that $H_1^{(1)} = 0$.

The boundary map ∂ provides a surjection $j : C_2 \to B_1$, which induces a norm, called the filling norm, on B_1. The completion of B_1 with respect to this norm is denoted $\hat{B}_1$, and j extends to a continuous map (in fact a surjection; see below) $\hat{\jmath} : C_2^{(1)} \to \hat{B}_1$. The identity map $i : B_1 \to Z_1$ is continuous, where B_1 (resp. Z_1) is equipped with the filling (resp. ℓ_1) norm, and therefore extends to a continuous map $\hat{\imath} : \hat{B}_1 \to Z_1^{(1)}$.

Lemma 4.2. *The sequence*

$$0 \to Z_2^{(1)}/\hat{Z}_2 \to \hat{B}_1 \xrightarrow{\hat{\imath}} Z_1^{(1)} \to H_1^{(1)} \to 0 \tag{4.2.1}$$

of linear maps of vector spaces is exact. (The second map is induced by the restriction to $Z_2^{(1)}$ of the completion $\hat{\jmath}$ of the bounday map $j : C_2 \to B_1$.)

Proof. We will use several times the fact that if C is a normed vector space and Z is a closed subspace, then the completion of C/Z coincides with the quotient of the completion of C by the closure therein of Z, *i.e.* $\widehat{C/Z} = \hat{C}/\hat{Z}$, where $\hat{}$'s denote completions. The last paragraph of the proof of theorem 1.5.3 in [KR] is essentially a proof of this assertion.

Since $B_1 = C_1/Z_1$, passing to completions shows that the natural map $C_2^{(1)}/\hat{Z}_2 \to \hat{B}_1$ is an isomorphism. Therefore the restriction of this map to $Z_2^{(1)}/\hat{Z}_2$ is injective, proving exactness at the term $Z_2^{(1)}/\hat{Z}_2$ of 4.2.1.

The map $\partial : C_2 \to C_1$ factors as the composition $i \circ j : C_2 \xrightarrow{j} B_1 \xrightarrow{i} Z_1 \subset C_1$. Therefore the completion $\hat{\partial}$ factors as $\hat{\imath} \circ \hat{\jmath}$. Since $\hat{\jmath}$ is surjective (by the fact above), we have the exact sequence

$$0 \to \mathrm{Ker}(\hat{\jmath}) \to \mathrm{Ker}(\hat{\partial}) \to \mathrm{Ker}(\hat{\imath}) \to 0.$$

Since $\mathrm{Ker}(\hat{\jmath})$ is the kernel of the completion $\hat{\jmath} : C_2^{(1)} \to \hat{B}_1$, namely $\hat{Z}_2$, and $\mathrm{Ker}(\hat{\partial}) = Z_1^{(1)}$ by definition, we have $\mathrm{Ker}(\hat{\imath}) = Z_2^{(1)}/\hat{Z}_2$, proving exactness of 4.2.1 at $\hat{B}_1$.

Because $\hat{\partial} = \hat{\imath} \circ \hat{\jmath}$ with $\hat{\jmath}$ surjective, we see that $\mathrm{Im}(\hat{\partial}) = \mathrm{Im}(\hat{\imath})$. Therefore $Z_1^{(1)}/\mathrm{Im}(\hat{\imath}) = Z_1^{(1)}/B_1^{(1)} = H_1^{(1)}$. This proves exactness at the terms $Z_1^{(1)}$ and $H_1^{(1)}$ of 4.2.1, completing the proof.

Lemma 4.3. *If G is of type $\mathcal{F}_3$, then there is an exact sequence*

$$0 \to \hat{Z}_2/B_2^{(1)} \to H_2^{(1)} \to Z_2^{(1)}/\hat{Z}_2 \to 0, \tag{4.3.1}$$

where $\hat{Z}_2$ denotes the closure of Z_2 in $C_2^{(1)}$.

Proof. This follows from the filtration $B_2^{(1)} \subseteq \hat{Z}_2 \subseteq Z_2^{(1)}$.

Remark. The term $Z_2^{(1)}/\hat{Z}_2$ in (4.3.1) represents the obstruction to approximating a summable 2-cycle by 2-cycles of compact support. The term $\hat{Z}_2/B_2^{(1)}$, as we shall see shortly, represents the obstruction to the linear isoperimetric inequality holding for 2-cycles.

Proposition 4.4. *If G is a hyperbolic group then $H_2^{(1)} = 0$.*

Proof. By a result of Rips [GH] hyperbolic groups are of type $\mathcal{F}_\infty$, so we can take X' a $K(G,1)$ with finite n-skeleton for all n and let $X = \widetilde{X'}$ as usual.

Now $B_1 = Z_1$ as normed linear spaces over $\mathbb{R}$, where B_1 is equipped with the filling norm, so $\hat{B}_1 = \hat{Z}_1 = Z_1^{(1)}$, where the last equality follows from Corollary 3.5. Thus the map $\hat{\imath}$ in (4.1.1) is an isomorphism, and it follows that $Z_2^{(1)}/\hat{Z}_2 = 0$. Thus every summable 2-cycle can be approximated by 2-cycles of compact support. By [AB] a hyperbolic group satisfies a linear isoperimetric inequality for fillings of compactly supported 2-cycles. By approximation it follows that the linear isoperimetric inequality holds for fillings of summable 2-cycles. It follows that the term $\hat{Z}_2/B_2^{(1)} = 0$. Thus $H_2^{(1)} = 0$ by Lemma 4.3, and the proof is complete.

We can now state our main result

Theorem 4.5. *A finitely presented group G is hyperbolic iff $H_1^{(1)} = 0$ and $Z_2^{(1)}/\hat{Z}_2 = 0$.*

Remark. The assertion that $Z_2^{(1)} = \hat{Z}_2$ is that every summable 2-cycle on X can be approximated by real 2-cycles of compact support. If G is of type $\mathcal{F}_2$ then $\bar{H}_2^{(1)}(G,\mathbb{R})$ is defined (see §2) even though $H_2^{(1)}$ may not be defined, and it is equal to $Z_2^{(1)}/\bar{Z}_2$, where $\bar{Z}_2 = \bar{B}_2 = \hat{Z}_2$ is the closure of Z_2 in $C_2^{(1)}$. If G is of type $\mathcal{F}_3$, then $H_2^{(1)}$ is defined, and one has the exact sequence (4.3.1) exhibiting $\bar{H}_2^{(1)} = Z_2^{(1)}/\bar{Z}_2$ as a quotient of $H_2^{(1)}$.

Proof of Theorem 4.5. The vanishing of $H_1^{(1)}$ and $Z_2^{(1)}/\hat{Z}_2$ for a hyperbolic group follow from the preceding lemmas.

We assume now that G is of type $\mathcal{F}_2$ with $H_1^{(1)} = 0$. The argument that follows is a reduction to the main result of [Ge1], that a finitely presented group is hyperbolic iff $H^2_{(\infty)}(X,\mathbb{R})$ "vanishes strongly"; this means that there is a constant $K > 0$ so that for every $f \in Z^2_{(\infty)}(X,\mathbb{R})$ there is a $c \in C^1_{(\infty)}(X,\mathbb{R})$ so that $f = \delta c$ and so that $|c|_\infty \leq K|f|_\infty$.

Lemma 4.6. *The vanishing of $H_1^{(1)}$ is equivalent to the linear isoperimetric inequality for summable 1-cycles. Formally, $H_1^{(1)} = 0$ iff $\exists K > 0\ \forall \epsilon > 0\ \forall z \in Z_1^{(1)}\ \exists c_\epsilon \in C_2^{(1)}$ such that*

(1) $\hat{\partial} c_\epsilon = z$, *and*
(2) $|c_\epsilon|_1 \leq K|z|_1 + \epsilon$.

Proof. One direction is clear, since the linear isoperimetric inequality for summable 1-cycles implies *a fortiori* that they admit summable fillings. For the converse, assume that $H_1^{(1)} = 0$. It is obvious that the sequence

$$C_2^{(1)} \xrightarrow{\hat{\partial}} Z_1^{(1)} \to H_1^{(1)} \to 0$$

is an exact sequence of linear maps between vector spaces. Because $H_1^{(1)} = 0$, we have an algebraic isomorphism $C_2^{(1)}/\mathrm{Ker}(\hat{\partial}) \cong Z_1^{(1)}$. Since $\hat{\partial}$ is continuous, the Banach inversion theorem implies that the inverse of this bijection is also continuous, so the ℓ_1 and filling norms on $Z_1^{(1)} = B_1^{(1)}$ are equivalent. This is just the linear isoperimetric inequality for summable 1-cycles. The assertions (1) and (2) in the proposition amount to interpreting this in terms of the definition of the filling norm as a quotient norm (i.e., in terms of infima), and the proof is complete.

Suppose now that $f \in Z^2_{(\infty)}$. If $z \in Z_1$, we set $\langle F, z\rangle = \langle f, c\rangle$, where $c \in C_2^{(1)}$ is such that $\partial c = z$. Note that X is contractible, so such c certainly exist. The main step of the argument is contained in the next result.

Lemma 4.7. *The map $F : Z_1 \to \mathbb{R}$ is well-defined, linear, and $|F|_\infty \leq K|f|_\infty$, where K is the constant in Lemma 4.6.*

Proof. Let $c, c' \in C_2^{(1)}$ be such that $\partial c = \partial c' = z$. Then $c - c' \in Z_2^{(1)}$. Since $Z_2^{(1)} = \hat{Z}_2$ it follows that there is a sequence of elements $z_n \in Z_2$, $n \geq 1$, so that $z_n \to c - c'$, where convergence is in the sense of the ℓ_1-norm. Since $Z_2 = B_2$, there are elements $y_n \in C_3$ so that $z_n = \partial y_n$ for all n. Now calculate $\langle f, c - c'\rangle = \langle f, \lim z_n\rangle = \lim\langle f, z_n\rangle = \lim\langle f, \partial y_n\rangle = \lim\langle \delta f, y_n\rangle = 0$; the second equality holds here since $f \in Z^2_{(\infty)}$, the convergence $z_n \to c - c'$ is in the ℓ_1-sense, and ℓ_∞ is the dual of ℓ_1. Thus $\langle f, c\rangle = \langle f, c'\rangle$, and it follows that F is well-defined. A familiar argument we omit shows that F is linear.

Now with $z \in Z_1$, let $\epsilon > 0$ and let $c_\epsilon \in C_2^{(1)}$ be such that $\partial c_\epsilon = z$ and $|c_\epsilon|_1 \leq K|z|_1 + \epsilon$; the existence of c_ϵ is guaranteed by Lemma 4.6. Now calculate $|\langle F, z\rangle| = |\langle f, c_\epsilon\rangle| \leq |f|_\infty |c_\epsilon|_1 \leq |f|_\infty(K|z|_1 + \epsilon)$. Since this holds for all $\epsilon > 0$, it follows that $|\langle F, z\rangle| \leq K|f|_\infty|z|_1$. It follows that $|F|_\infty \leq K|f|_\infty$, and the proof of the lemma is complete.

Since $F : Z_1 \to \mathbb{R}$ is a bounded linear functional, it follows from the Hahn-Banach theorem that F admits a bounded linear extension H to C_1 with the same norm. Hence $|H|_\infty \leq K|f|_\infty$. Now calculate for $c \in C_2$, $\langle \delta H, c\rangle = \langle H, \partial c\rangle = \langle F, \partial c\rangle = \langle f, c\rangle$, and hence $\delta H = f$. This establishes strong vanishing of $H^2_{(\infty)}(X, \mathbb{R})$, and it follows that G is hyperbolic. This completes the proof of Theorem 4.5.

Corollary 4.8. *A finitely presented group G is hyperbolic iff $H_1^{(1)} = \bar{H}_2^{(1)} = 0$.* □

Remark. S. Weinberger has conjectured that a finitely presented group is hyperbolic iff $H_1^{(1)} = 0$. This assertion remains open at the time of writing. To establish it, in view of Theorem 4.5, would be equivalent to showing that summable 2-cycles can be approximated by 2-cycles of compact support if $H_1^{(1)} = 0$.

Remark. The exact sequence (4.2.1) has a higher dimensional analog proved in the same way when G is of type $\mathcal{F}_{n+1}$, namely, the exact sequence

$$0 \longrightarrow \bar{H}_{n+1}^{(1)} \longrightarrow \hat{B}_n \xrightarrow{\hat{\imath}} Z_n^{(1)} \longrightarrow H_n^{(1)} \longrightarrow 0.$$

To understand this better, one introduces the group $I_n^{(1)} = \hat{Z}_n / B_n^{(1)}$ which is defined for groups G of type $\mathcal{F}_{n+1}$. Then one has from the filtration $B_n^{(1)} \subset \hat{Z}_n \subset Z_n^{(1)}$ the short exact sequence

$$0 \to I_n^{(1)} \to H_n^{(1)} \to \bar{H}_n^{(1)} \to 0,$$

which is the analog of (4.3.1), and the exact sequence

$$0 \longrightarrow \bar{H}_{n+1}^{(1)} \longrightarrow \hat{B}_n \xrightarrow{\hat{\imath}} \hat{Z}_n \longrightarrow I_n^{(1)} \longrightarrow 0.$$

$I_n^{(1)}$ should be thought of as the obstruction to linear filling of summable n-cycles while $\bar{H}_n^{(1)}$ is the obstruction to approximating summable n-cycles by n-cycles of compact support. Thus Corollary 4.8 says that the finitely presented group G is hyperbolic iff both obstructions $I_1^{(1)}$ and $\bar{H}_2^{(1)}$ vanish.

§5. Vanishing theorems for $H_2^{(1)}$.

Proposition 4.4 established that $H_2^{(1)}$ vanishes for a hyperbolic group. In this section we prove that for all 1-relator groups and for all fundamental groups of finite piecewise Euclidean 2-complexes of nonpositive curvature one has vanishing of $H_2^{(1)}$. For these groups the hyperbolicity criterion of Corollary 4.8 reduces to the vanishing of $H_1^{(1)}$.

First let G be a 1-relator group, so G is defined by a presentation consisting of a finite set of generators and a single defining relation. We shall prove

Theorem 5.1. *If G is a 1-relator group then $H_2^{(1)}(G, \mathbb{R}) = 0$.*

An immediate consequence of this result is

Corollary 5.2. *The 1-relator group G is hyperbolic iff $H_1^{(1)}(G, \mathbb{R}) = 0$.*

Proof. If G has nontrivial torsion, then it is known that G is hyperbolic; this is a consequence of the "spelling theorem" of B. B. Newman [LS] p. 205. Thus $H_1^{(1)}$ and $H_2^{(1)}$ are both zero in this case.

If G is torsion-free, then by Lyndon's identity theorem [LS] the 1-relator presentation for G is aspherical, so hyperbolicity of G is equivalent to the vanishing of $H_1^{(1)}$ and $H_2^{(1)}$ by Corollary 4.3. But $H_2^{(1)}$ vanishes by Theorem 5.1, so hyperbolicity is equivalent to the vanishing of $H_1^{(1)}$, and the corollary follows from the theorem.

Definition. Suppose that K is a subcomplex of the CW complex L such that $K = T \times I$, where T is a complex and $I = [0,1] \subset \mathbb{R}$. We say that K is *isolated* in L if $L - (T \times (0,1))$ is a subcomplex of L.

Lemma 5.3. *Suppose that the subcomplex $K = T \times I$ with T a tree is isolated in the 2-complex L. If f is an ℓ_1 2-cycle on L, then f vanishes on all 2-cells of L of the form $e \times (0,1)$, where e is an edge of T.*

Proof. If e is an edge of T, we define $F(e) = f(e \times (0,1))$. It is readily checked that F is a summable 1-cycle on T. But $Z_1^{(1)}(T,\mathbb{R}) = 0$ for every tree T, by Corollary 3.4. It follows that $F = 0$, and hence $f(e \times (0,1)) = 0$ for all edges of T, and the lemma is established.

Lemma 5.4. *Suppose that $G = H *_F K$ (resp. $G = H*_F$) where F is a finitely generated free group and H and K (resp. H) admit finite aspherical presentations. Then $H_2^{(1)}(G,\mathbb{R}) = 0$ iff $H_2^{(1)}(H,\mathbb{R}) = 0 = H_2^{(1)}(K,\mathbb{R})$ (resp. $H_2^{(1)}(H,\mathbb{R}) = 0$).*

Proof. We write out the proof in the HNN case, $G = H*_F$; the amalgam case involves only notational changes. Let X' be a finite 2-dimensional $K(H,1)$. Then we can construct a $K(G,1)$ by taking $Y' = X' \cup (\Gamma \times [-1,2])/\sim$, where Γ is a finite connected graph, and $\Gamma \times \{i\}$ is attached to X' by cellular maps inducing the given injective homomorphisms $F \to H$, $i = -1, 2$. We can subdivide the interval $[-1,2]$ at points $0, 1$ and take the induced product cell structure on $\Gamma \times [-1,2]$, so that $\Gamma \times [0,1]$ becomes a subcomplex of Y'. The proof that Y' is a $K(G,1)$ is standard and makes use of the Mayer-Vietoris exact sequence in homology for Y, the universal covering space of Y'.

Note that the preimage of $\Gamma \times I$ in Y is a disjoint union of isolated subcomplexes, each isomorphic to $T \times I$, where T is the universal covering of Γ. Furthermore, the preimage of X' in Y is a disjoint union of isomorphic copies of X, the universal covering space of X'.

We suppose first that $H_2^{(1)}(G,\mathbb{R}) = 0$, and we let f be a summable 2-cycle on X. Then f can be extended to a summable 2-cycle F on Y by defining F to be f on one connected component of the preimage of X' in Y and zero on all of the other such connected components, as well as setting $F = 0$ on all 2-cells of Y over those of $\Gamma \times (-1,2)$. But $H_2^{(1)}(Y,\mathbb{R}) = Z_2^{(1)}(Y,\mathbb{R}) = 0$, so it follows that $F = 0$. Thus $f = 0$ as well.

Next suppose that $H_2^{(1)}(H,\mathbb{R}) = 0$ and let F be a summable 2-cycle on Y. By Lemma 5.3 it follows that F vanishes on all 2-cells of Y which map to those of $\Gamma \times (-1,2)$. But this means that F is supported on the preimage of X' in Y. It follows from $H_2^{(1)}(H,\mathbb{R}) = Z_2^{(1)}(X,\mathbb{R}) = 0$ that F restricted to each connected component of the preimage of X' in Y is zero, and hence $F = 0$. This completes the proof.

Proof of Theorem 5.1. It suffices to consider torsion-free 1-relator groups, as the argument in the first paragraph of the proof of Corollary 5.2 shows. Given a 1-relator presentation $\mathcal{P}$ whose relator R is cyclically reduced and not a proper power in the free group of the generators, we shall prove that $H_2^{(1)}(G,\mathbb{R}) = 0$, where G is the group defined by $\mathcal{P}$, by induction on the length $\ell(R)$ of the word R. The induction starts when $\ell(R) = 1$, for in this case G is free.

For the inductive step, we need to recall the structure of 1-relator groups in the form presented in [MS] (*cf.* also [LS] pages 198–200). We assume that G has a 1-relator presentation $\mathcal{P}$ with $\ell(R) = n > 1$. There are two cases, depending on whether some generator of $\mathcal{P}$ actually occurring in R has exponent sum 0, or the contrary case when none of the generators appearing in R has exponent sum 0. In the first case $G = H*_F$ where H is a 1-relator group defined by a 1-relator presentation whose defining relator S has $\ell(S) < n$ and where F is a finitely generated free group. In the second case, there is an element $1 \neq g \in G$ and a number $d > 0$ so that $K = G *_{g=x^d} \langle x \rangle$, where $K = H*_F$; here H admits a 1-relator presentation whose defining relator S has $\ell(S) < n$, x is of infinite order, and F is a finitely generated free group.

In either case it follows from Lemma 5.4 that $H_2^{(1)}(H,\mathbb{R}) = 0$ iff $H_2^{(1)}(G,\mathbb{R}) = 0$. But $H_2^{(1)}(H,\mathbb{R}) = 0$ by the induction hypothesis, so it follows that $H_2^{(1)}(G,\mathbb{R}) = 0$. This completes the induction, and the proof that $H_2^{(1)}(G,\mathbb{R}) = 0$ is complete.

Example 5.5. It follows from Theorem 5.1 that $H_1^{(1)}(\mathbb{Z}^2,\mathbb{R}) \neq 0$. In fact, one can give explicit examples of nonzero classes in this homology group. If z_n denotes an edge-circuit which goes once in the positive direction around a square of side n, then it is easy to check that $z = \sum_{n\geq 1} \dfrac{z_n}{n^3}$ is a summable 1-cycle which admits no summable filling.

Remark. One of the open problems on 1-relator groups that has stimulated much interest since the second author proposed it nearly a decade ago asks whether a 1-relator group is hyperbolic iff it contains no Baumslag-Solitar subgroups $\langle x, y \mid yx^p = x^q y\rangle$, $p, q > 0$. One might ask whether the homological methods introduced here can be of use in attacking this previously unapproachable question.

Theorem 5.6. *If G is the fundamental group of a finite piecewise Euclidean simplicial 2-complex K of nonpositive curvature, then $H_2^{(1)}(G,\mathbb{R}) = 0$.*

Proof. For the definitions and properties of complexes of nonpositive curvature and CAT(0) spaces we refer the reader to W. Ballmann's article in [GH]. We need here the fact that the universal cover $\widetilde{K}$ of K is a simply connected nonpositively curved complex with finitely many isometry types of simplices, and hence, by the theorem of M. Bridson [Br], $\widetilde{K}$ possesses unique geodesic segments connecting any pair of points. We need

Lemma 5.7. *Let σ be an open 2-simplex of $\widetilde{K}$ and let $p \in \sigma$. Then there is a direction u in the tangent space of σ at p so that no geodesic segment in $\widetilde{K}$ passing through p with tangent directions $\pm u$ passes through a vertex of $\widetilde{K}$.*

Proof. There are only countably many vertices $v \in \widetilde{K}^{(0)}$ so there are only countably many tangent directions at p of the geodesic segments $[p, v]$. Choose u to avoid all of these directions and their negatives.

Returning to the proof of the theorem, let f be a summable 2-cycle on $\widetilde{K}$ and let σ be an open 2-simplex of $\widetilde{K}$. Let $p \in \sigma$ and choose the direction u in the tangent space to σ at p as in the lemma. Consider T, the union of all geodesic segments through p in $\widetilde{K}$ whose tangent directions at p are $\pm u$. Then T is a tree with a discrete set of vertices where T intersects the 1-skeleton of $\widetilde{K}$. The edges of T are the nonempty intersections of T with 2-simplices of $\widetilde{K}$. Such an edge e has a product neighborhood in the 2-simplex σ in which it lies, and these product neighborhoods fit together to give a product neighborhood $N = T \times I$ of T in $\widetilde{K}$ which does not meet $\widetilde{K}^{(0)}$ (note however that one cannot choose N of uniform thickness in the normal I-direction in general in the metric of $\widetilde{K}$). Having fixed an orientation on I, giving an orientation on $N \cap \sigma \cong e \times \mathring{I}$ is equivalent to giving an orientation on the edge e. In this way we define a 1-cochain F on T so that $F(e) = f(\sigma)$, where $e = T \cap \sigma$ and where the orientations are compatible in the sense just described. That F is a summable 1-chain on T follows from the summability of f and that F is summable 1-cycle follows from the facts that f is a summable 2-cycle and that T does not meet $\widetilde{K}^{(0)}$. It follows from Corollary 3.4 that $F = 0$, and hence in particular $f(\sigma) = 0$. Since σ was an arbitrary 2-simplex, it follows that $f = 0$, and the proof is complete.

Remark. For higher dimensional CAT(0) spaces it is certainly not the case that $H_2^{(1)}$ vanishes (although the question of approximating summable 2-cycles by 2-cycles of compact support remains open). For example, if X is the tesselation of $\mathbb{R}^3$ by the unit cube lattice and z_n is the 2-cycle which is the boundary of a cube of side n with the orientation determined by the outward pointing normal, then it is easy to see that $\sum_{n\geq 1} \dfrac{z_n}{n^4}$ is a summable 2-cycle which admits no summable filling; thus $H_2^{(1)}(\mathbb{Z}^3, \mathbb{R}) \neq 0$.

Corollary 5.7. *The fundamental group G of a finite piecewise Euclidean 2-complex of nonpositive curvature is hyperbolic iff $H_1^{(1)}(G, \mathbb{R}) = 0$.* $\square$

§6. Examples.

Theorem 3.3 says that if Γ is a graph, then elements of $\mathrm{Ker}(\hat{\partial})$ can be ℓ_1-approximated by elements of $\mathrm{Ker}(\partial)$, where $\partial : C_1(\Gamma, \mathbb{R}) \to C_0(\Gamma, \mathbb{R})$ is the boundary map and $\hat{\partial}$ is its ℓ_1-completion. Equivalently, $Z_1^{(1)}(X, \mathbb{R}) = \hat{Z}_1(X, \mathbb{R})$ for every CW-complex X. It is natural to ask whether there is an analogous approximation theorem for n-cycles, *i.e.* whether $Z_n^{(1)}(X, \mathbb{R}) = \hat{Z}_n(X, \mathbb{R})$ for $n \geq 2$. The next example shows that it is futile to look for general results of this type.

Example 6.1 (E. Formanek[3]). Let G be the free group with free basis $\{x, y\}$. Then the row vector $M = (\, 2 - x, y - 2 \,)$ defines a monomorphism of right $\mathbb{R}G$-modules

[3] private communication

$(\mathbb{R}G)^2 \to \mathbb{R}G$ by $\begin{pmatrix} a \\ b \end{pmatrix} \mapsto (2-x)a + (y-2)b$. However if $a = 1 + \frac{1}{2}x + \frac{1}{4}x^2 + \dots$ and $b = 1 + \frac{1}{2}y + \frac{1}{4}y^2 + \dots$, then $(2-x)a = (2-y)b = 2$, so the induced map of ℓ_1-completions $\widehat{\mathbb{R}G}^2 \to \widehat{\mathbb{R}G}$ is not injective.

For each $n \geq 2$ one may use this to build a CW complex X of dimension n and a summable n-chain thereon that cannot be ℓ_1-approximated by compactly supported n-chains. Furthermore, we may take X to admit a cocompact free action of G and to be simply connected (if $n > 2$) or aspherical (if $n = 2$). The topological construction is standard: if $n > 2$ then take X as the universal cover of X', where X' is the join of two circles and an $n-1$-sphere with a pair of n-cells attached in such a way that boundaries of their lifts to X are given by the entries of Formanek's matrix M. Then M gives the boundary map $C_n \to C_{n-1}$, so $Z_n(X) = 0$ but $Z_n^{(1)}(X) \neq 0$. The same basic construction yields a slightly different result for $n = 2$. Then X' may be taken to be the 2-complex associated to the presentation

$$\langle x, y, z \mid 1 = z^2xz^{-1}x^{-1} = z^2yz^{-1}y^{-1} \rangle$$

and X to be the covering space of X' associated to the normal closure of z; the free group G is the group of covering transformations. X' is aspherical because it is the presentation complex of an iterated HNN extension of $\langle z \rangle \cong \mathbb{Z}$. The boundary map $C_2(X) \to C_1(X)$ has image in a rank one G-submodule of $C_1(X)$, with matrix given by M. As before, this shows that $Z_2(X) = 0$ but $Z_2^{(1)}(X) \neq 0$.

Question. If X is a simply-connected 2-complex which admits a discrete cocompact group action by cellular homeomorphisms, can summable 2-cycles on X be approximated by 2-cycles of compact support? This is for our purposes the most interesting open case of the general question of approximating summable n-cycles by cycles of compact support that is not already answered positively by Theorem 3.3 for $n = 1$ and negatively by examples constructed from Formanek's example in the preceding paragraph.

In the introduction we noted that the lack of reflexivity of the Banach space ℓ_1 has the peculiar effect that the vanishing theorems for hyperbolic groups are in different degrees, $H^2_{(\infty)}$ and $H_1^{(1)}$. We complete this discussion by showing

Proposition 6.2. *If G is an infinite finitely generated group then $H^1_{(\infty)}(G, \mathbb{R})$ and the reduced group $\bar{H}^1_{(\infty)}(G, \mathbb{R})$ are both nonzero.*

Proof. Let X' be a $K(G, 1)$ with finite 1-skeleton and let X be the universal cover of X'. Choose a vertex v_0 of X as base point and give the one-skeleton $X^{(1)}$ the path metric d where each edge has length 1. For an oriented edge e of X set $\Delta(e) = d(v_0, \tau e) - d(v_0, \iota e)$. Then Δ is a 1-cocycle, and the triangle inequality shows that $|\Delta|_\infty = 1$ (where $|\Delta|_\infty$ denotes the sup (or ℓ_∞) norm of the cochain Δ). Furthermore $\Delta \notin B^1_{(\infty)}(X, \mathbb{R})$ since its potential function $D : X^{(0)} \to \mathbb{R}$, given by $D(v) = d(v_0, v)$, is unbounded, as follows since G is infinite.

As for the reduced cohomology group $\bar{H}^1_{(\infty)}(X,\mathbb{R})$, recall that this is the quotient of $Z^1_{(\infty)}(X,\mathbb{R})$ by the closure of $B^1_{(\infty)}(X,\mathbb{R})$. We claim that the cocycle Δ is not in the closure of $B^1_{(\infty)}(X,\mathbb{R})$. To see this we need the following

Lemma 6.3. *If $f \in Z^1_{(\infty)}(X,\mathbb{R})$ is in the closure of $B^1_{(\infty)}(X,\mathbb{R})$, then every potential function F for f satisfies*

$$\lim_{d(v_0,v)\to\infty} \frac{F(v)}{d(v_0,v)} = 0.$$

Proof. Let $\epsilon > 0$ and let $h \in C^0_{(\infty)}(X,\mathbb{R})$ be such that $|f - \delta h|_\infty \le \epsilon$. Let $\gamma = e_1 e_2 \dots e_n$ be a geodesic edge-path starting at v_0 and ending at v. Then

$$F(v) - F(v_0) - (h(v) - h(v_0)) = \sum_{i=1}^{n} (f(e_i) - \delta h(e_i)),$$

so $|F(v)| \le n\epsilon + |F(v_0)| + 2|h|_\infty$, and hence $\dfrac{|F(v)|}{n} \le 2\epsilon$ for n sufficiently large.

Returning to the proof of the Proposition we see that $\dfrac{D(v)}{d(v_0,v)} = 1$ for all vertices v, so it follows from Lemma 6.3 that Δ is not in the closure of $B^1_{(\infty)}(X,\mathbb{R})$. Hence $\bar{H}^1_{(\infty)}(X,\mathbb{R}) \ne 0$.

Appendix.

Let Γ be a graph and let A be a normed abelian group. It is an open question whether summable 1-cycles on Γ with coefficients in A can be approximated by cycles of compact support in general, although the case $A = \mathbb{R}$ was settled affirmatively by Theorem 3.3. Another special case will be established in this Appendix.

We recall that A is called *ultrametric* if $|a + b| \le \max(|a|, |b|)$ for all $a, b \in A$. We call the norm *discrete* if the set $\{|a|, a \in A\}$ has only 0 as a limit point. For example the p-adic completion of the rational numbers is ultrametric with discrete norm.

Theorem A1. *Let Γ be a graph and let A be a ultrametric normed abelian group with discrete norm. Then summable 1-cycles on Γ with coefficients in A can be approximated in the ℓ_1-sense by 1-cycles of compact support.*

Proof. Let f be a nonzero summable 1-cycle on Γ with coefficients in A. It follows from summability that for each $r > 0$ the set E_r of edges in Γ so that $|f(e)| \ge r$ is finite, and consequently f is nonzero on at most a countable set of edges. Let M be the maximum value of $|f(e)|$ as e runs over the edges of Γ and let $|f(e_1)| = M$. We need

Lemma A2. *There exists a simple circuit z_1 so that $|f(e)| = M$ for each edge e in z_1.*

Proof. If $v = \tau e_1$, then $\sum_{\tau e = v} f(e) = 0$ from the cycle condition, and the term $f(e_1)$ in the sum is of maximal norm. It follows from the ultrametric inequality that there exists an edge e_2 with $\iota e_2 = v$ and so that $|f(e_2)| = M$. Continuing in this way, we produce by induction a path $e_1 e_2 \dots e_n$ for each $n \geq 1$ so that $|f(e_i)| = M$. Since there are only finitely many edges e altogether with $|f(e)| = M$, there must be exist a number $k > 0$ so that $e_i = e_{i+k}$ for some $i \geq 1$. Taking k minimal produces a simple circuit z_1 satisfying the conclusion of the lemma.

Next let $f_1 = f - a_1 z_1$, where a_1 is the value of f on one of the edges of z_1 (it makes no difference for the argument which edge is chosen). It follows from the ultrametric inequality that $|f_1(e)| \leq |f(e)|$ for each edge e and consequently $|f_1|_1 \leq |f|_1$, where $|f|_1$ denotes the ℓ_1-norm of f, which, we recall from (2.1) above, is the sum of the norms of the values of f on an orientation $\mathcal{O}$ of the edges of Γ. Also note that f_1 has at least one fewer edge than f with value of maximal norm.

Now repeat the process, replacing f by f_1, finding a simple edge-circuit z_2 of edges on which f_1 assumes values of maximal norm, defining $f_3 = f_2 - a_2 z_2$, where a_2 is value of f_1 on one of the edges of z_2, and so forth, producing thereby the sequence of summable 1-cycles $f_2, f_3, \dots$. The relevant facts about this sequence are

(1) $|f_{n+1}(e)| \leq |f_n(e)|$ for every edge e of Γ, and $|f_{n+1}(e)| < |f_n(e)|$ for at least one edge e on which f_n assumes its value of maximal norm, and
(2) $|f_{n+1}|_1 \leq |f_n|_1 \leq |f|_1$ for all $n \geq 1$.

It follows from (1) and the discreteness of the norm on A that $\lim_{n\to\infty} |f_n(e)| = 0$ for each edge e while $|f_n|_1 \leq |f|_1$ for all $n \geq 1$. But the Lebesgue dominated convergence theorem (applied to the sequence of real valued functions g_n on the discrete measure space of edges of Γ, where g_n is given by $g_n(e) = |f_n(e)|$) implies that $\lim_{n\to\infty} |f_n|_1 = |\lim_{n\to\infty} g_n|_1 = 0$. If we set $s_n = a_1 z_1 + a_2 z_2 + \cdots + a_n z_n$, then $f = s_n + f_n$ for each $n \geq 1$, s_n is a 1-cycle of compact support, and $\lim_{n\to\infty} s_n = f$ where the limit is taken in the sense of the ℓ_1-norm. Thus f can be approximated by 1-cycles of compact support, and the proof of the theorem is complete.

Remark. It follows from arguments of [Mi] that if the graph Γ is combable in the sense of [EC], then approximation of summable 1-cycles by cycles of compact support is possible for all normed abelian coefficient groups. Furthermore, it is a result of [Fl] that a summable 1-cycle f with values in A on the connected graph Γ satisfying the stronger assumption that $\sum_e (d(v_0, \iota e) + 1)|f(e)| < \infty$, where v_0 is the base point and d is the path metric assigning length 1 to each edge, can be approximated by cycles of compact support. It is also not difficult to see that a summable 1-cycle on a tree must necessarily be zero. However the general question, of approximating summable 1-cycles on arbitrary graphs with values in an arbitrary normed abelian group by 1-cycles of compact support, remains open.

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