Asphericity of moduli spaces via curvature

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Abstract.
We show that under suitable conditions a branched cover satisfies the same upper curvature bounds as its base space. First we do this when the base space is a metric space satisfying Alexandrov's curvature condition CAT(κ) and the branch locus is complete and convex. Then we treat branched covers of a Riemannian manifold over suitable mutually orthogonal submanifolds. In neither setting do we require that the branching be locally finite. We apply our results to hyperplane complements in several Hermitian symmetric spaces of nonpositive sectional curvature in order to prove that two moduli spaces arising in algebraic geometry are aspherical. These are the moduli spaces of the smooth cubic surfaces in $\mathbb{C}P^3$ and of the smooth complex Enriques surfaces.

§1. Introduction

It is well-known that taking branched covers usually introduces negative curvature. One can see this phenomenon in elementary examples using Riemann surfaces, and the idea also plays a role in the construction [8] of exotic manifolds with negative sectional curvature. In this paper we work in the setting of Alexandrov's comparison geometry; for background see [3]. In this setting we will establish the persistence of upper curvature bounds in branched covers. A simple way to build a cover $\widehat{Y}$ of a space $\widehat{X}$ branched over $\Delta \subseteq \widehat{X}$ is to take any covering space Y of $\widehat{X} - \Delta$ and define $\widehat{Y} = Y \cup \Delta$. We call $\widehat{Y}$ a simple branched cover of $\widehat{X}$ over Δ. Our first result, theorem 2.1, states that if $\widehat{X}$ satisfies Alexandrov's CAT(κ) condition and Δ is complete and satisfies a convexity condition then the natural metric on $\widehat{Y}$ also satisfies CAT(κ).

The question which motivated this investigation is whether the moduli space of smooth cubic surfaces in $\mathbb{C}P^3$ is aspherical (i.e., has contractible universal cover). It is, and our argument also establishes the analogous result for the moduli space of smooth complex Enriques surfaces. To prove these claims, we use the fact that each of these moduli spaces is known to be covered by a Hermitian symmetric space with nonpositive sectional curvature, minus an arrangement of complex hyperplanes. In each case the hyperplanes have the property that any two of them are orthogonal wherever they meet. In section 3 we show that such a hyperplane complement is aspherical. The result (theorem 3.1) is more general because the symmetric space structure is not needed. Theorem 5.3 of [4] morally contains this result and also suggests a substantial generalization of it. However, there are some difficulties with the proof.

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If $\hat{M}$ is the symmetric space and $\mathcal{H}$ is the union of the hyperplanes, then the idea is to apply standard nonpositive curvature techniques like the Cartan-Hadamard theorem to the universal cover N of $M = \hat{M} - \mathcal{H}$. The problem is that N is not metrically complete. One can pass to its metric completion $\hat{N}$, but this introduces problems of its own. First there is the issue of how N and $\hat{N}$ are related. We resolve this by a simple trick that shows that the inclusion $N \to \hat{N}$ is a homotopy equivalence. The second problem is that $\hat{N}$ is not a manifold and not even locally compact, so that one cannot use the techniques of Riemannian geometry. But it is still a metric space and it turns out to satisfy Alexandrov's CAT(0) condition locally. It is then easy to show that N and $\hat{N}$ are contractible.

In order to understand this curvature bound for $\hat{N}$, the reader should imagine a closed ball B in $\mathbb{C}^n$, equipped with some Riemannian metric, minus the coordinate hyperplanes. The metric completion of the universal cover of the hyperplane complement can be obtained by first taking a simple branched cover of B over one hyperplane, then taking a simple branched cover of this branched cover over (the preimage of) the second hyperplane, and so on. If the hyperplanes are mutually orthogonal and totally geodesic then theorem 2.1 may be used inductively to bound the curvature of the iterated branched cover. Observe that the base space of each branched cover fails to be locally compact except in the first step. This means that the inductive argument requires a theorem treating branched covers of spaces more general than manifolds. We also note that the condition of mutual orthogonality in branched covers has appeared before, for example for the modified Deligne complexes of [5], which are certain metric polyhedral complexes of piecewise constant curvature. In fact, in this polyhedral setting our results are already well-established. We need to go beyond the piecewise constant curvature case for the applications to algebraic geometry.

For the reader's convenience we recall some definitions from [3]. D_κ is the diameter of the simply-connected complete surface of constant sectional curvature κ. A metric space X is D_κ-geodesic if any two points at distance $< D_\kappa$ are joined by a geodesic. A subset Δ of X is called D_κ-convex if any two points of Δ at distance $< D_\kappa$ are joined by a geodesic of X, and every geodesic joining them lies in Δ. We say that X is CAT(κ) if it is D_κ-geodesic and any two points on any geodesic triangle in X with perimeter $< 2D_\kappa$ satisfy Alexandrov's inequality. We write $\ell(\gamma)$ for the length of a path γ.

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§2. Branched covers

We begin by showing that a branched cover satisfies the same upper bounds on curvature as its base space. We treat what we call simple branched covers. The idea is that one removes a closed subset Δ from a length space $\widehat{X}$, takes a cover of what is left, and then attaches a copy of Δ in the obvious way. More precisely, if Y is any covering of $X = \widehat{X} - \Delta$, then each component of Y carries a unique length metric under which projection to X is a local isometry. We take $\widehat{Y} = Y \cup \Delta$ and write $\pi : \widehat{Y} \to \widehat{X}$ for the obvious projection map. If $x, z \in \widehat{Y}$ then we define

$$d(x,z) = \inf\Big(\{\, d(\pi x, \pi y) + d(\pi y, \pi z) \mid y \in \Delta \,\} \;\cup\; \{\, \ell(\gamma) \mid \gamma \text{ is a path in } Y \text{ joining } x \text{ and } z \,\}\Big)\ .$$

One checks that d is a length metric, and we call $\widehat{Y}$ a simple branched covering of $\widehat{X}$ over Δ.

Theorem 2.1. *If $\widehat{X}$ is CAT(κ) and Δ is complete and D_κ-convex, then $\widehat{Y}$ is also CAT(κ).*

This theorem was stated by Gromov [7, section 4.4], without the completeness hypothesis. Without this, $\widehat{Y}$ may fail to be D_κ-geodesic. It may even happen that $\widehat{Y}$ contains points having no geodesic neighborhoods. On the other hand, every geodesic triangle in $\widehat{Y}$ of perimeter $< 2D_\kappa$ still satisfies CAT(κ). The same concerns arise for Reshetnyak's gluing lemma (see for example [3, p. 347]), to which this result is very similar.

Proof: We have obtained a proof in full generality, but here we make the additional assumptions that $\kappa \leq 0$ and $\widehat{X}$ is complete. This is sufficient for our applications. The idea is to realize $\widehat{Y}$ as a Gromov-Hausdorff limit of spaces which are obviously CAT(κ). For $\varepsilon > 0$ let Δ_ε be the closed ε-neighborhood of Δ, and let $\widehat{Y}_\varepsilon$ be obtained by gluing together a copy of Δ_ε and a copy of Y, so that all preimages in Y of any given point of $\Delta_\varepsilon - \Delta$ are identified. This space has a natural path metric—in fact it is a simple branched cover of $\widehat{X}$ over Δ_ε. It is obvious that $\widehat{Y}$ is a Gromov-Hausdorff limit of the $\widehat{Y}_\varepsilon$. Since $\widehat{Y}$ is complete it suffices by [3, Cor. 3.10] to show that each $\widehat{Y}_\varepsilon$ is CAT(κ). Since $\widehat{Y}_\varepsilon$ is complete and simply connected, it suffices to show that $\widehat{Y}_\varepsilon$ is locally CAT(κ). If $x \in \widehat{Y}_\varepsilon$ lies at distance $\neq \varepsilon$ from Δ then x admits a CAT(κ) neighborhood because it has a neighborhood isometric to an open subset of $\widehat{X}$. Now suppose $d(x, \Delta) = \varepsilon$. We let U be the closed $\varepsilon/2$-ball about the image of x in $\widehat{X}$. Then x admits a neighborhood which is the union of some number of copies of U, glued together along $U \cap \Delta_\varepsilon$. This neighborhood of x is CAT(κ) by Reshetnyak's lemma, since $U \cap \Delta_\varepsilon$ is complete and is convex in U. $\square$

Next we define precisely what we mean by a branched cover which is locally an iterated branched cover of a manifold over a family of mutually orthogonal totally geodesic submanifolds. Then we show that such a branched cover satisfies the same upper bounds on local curvature as the base manifold. We prove this only in the case of nonpositive curvature, and indicate what else is needed in the general case.

We say that a finite set $\{S_1, \ldots, S_n\}$ of codimension-2 subspaces of an even-dimensional real vector space A is normal if it is equivalent to some n of the m coordinate hyperplanes in $\mathbb{C}^m$ under an $\mathbb{R}$-linear isomorphism $A \to \mathbb{C}^m$. If A is odd-dimensional then we call $\{S_1, \ldots, S_n\}$ normal if it is equivalent to the products with $\mathbb{R}$ of some n of the m coordinate hyperplanes of $\mathbb{C}^m$, in $\mathbb{C}^m \times \mathbb{R}$. We write $\mathcal{S}$ for $\cup_i S_i$. Now suppose $\mathcal{H}_0$ is a family of immersed submanifolds of a Riemannian manifold $\hat{M}$ with union $\mathcal{H}$. We say that $\mathcal{H}_0$ is normal at $x \in \hat{M}$ if there is a set $\{S_1, \ldots, S_n\}$ of mutually orthogonal subspaces of $T_x\hat{M}$ that are normal in the sense above and have the following property. We require that there be an open ball U about 0 in $T_x\hat{M}$ which the exponential map carries diffeomorphically onto its image V, such that $V \cap \mathcal{H} = \exp_x(U \cap \mathcal{S})$, and such that each $\exp_x(S_i \cap U)$ is a convex subset of V. We say that $\mathcal{H}_0$ is normal if it is normal at each $x \in \hat{M}$. In this case, each element of $\mathcal{H}_0$ is totally geodesic and intersections of elements of $\mathcal{H}_0$ are orthogonal.

If $x \in \hat{M}$ then $\pi_1(V - \mathcal{H}) \cong \pi_1(U - \mathcal{S}) \cong \pi_1(T_x\hat{M} - \mathcal{S}) \cong \mathbb{Z}^n$. The first two isomorphisms are obvious and the last follows from the fact that $T_x\hat{M} - \mathcal{S}$ is a product of n punctured planes and a Euclidean space. We choose generators $\sigma_1, \ldots, \sigma_n$ for $\pi_1(T_x\hat{M} - \mathcal{S})$ by taking a representative for σ_i to be a simple circular loop that links S_i but none of the other S_j. We say that a connected covering space of $T_x\hat{M} - \mathcal{S}$ is standard if the subgroup of $\mathbb{Z}^n$ to which it corresponds is generated by $\sigma_1^{d_1}, \ldots, \sigma_n^{d_n}$ for some $d_1, \ldots, d_n \in \mathbb{Z}$. We apply the same terminology to the corresponding cover of $V - \mathcal{H}$. In particular, the universal cover is standard. An arbitrary covering space of $V - \mathcal{H}$ is called standard if each of its components is.

We write M for $\hat{M} - \mathcal{H}$. If $\pi : N \to M$ is a covering space then we say that N is a standard cover of M if for each $x \in \hat{M}$ with V as above, $\pi : \pi^{-1}(V - \mathcal{H}) \to V - \mathcal{H}$ is a standard covering in the sense above. In this case, we take $\hat{N}$ to be a certain subset of the metric completion of N, namely those points which map into $\hat{M}$ under the extension of π. In particular, if $\hat{M}$ is complete then $\hat{N}$ is the completion of N. We denote the natural extension $\hat{N} \to \hat{M}$ of π again by π, and call $\hat{N}$ a standard branched covering of $\hat{M}$ over $\mathcal{H}_0$. The simplest example of a standard branched cover is $\pi : \mathbb{C}^n \to \mathbb{C}^n$, carrying $(z_1, \ldots, z_n)$ to $(z_1^{d_1}, \ldots, z_n^{d_n})$. We have just extended this by making the definition local and allowing infinite branching.

Theorem 2.2. *If a Riemannian manifold $\hat{M}$ has sectional curvature bounded above by $\kappa \leq 0$ and $\pi : \hat{N} \to \hat{M}$ is a standard branched cover over a normal family $\mathcal{H}_0$ of immersed submanifolds of $\hat{M}$, then $\hat{N}$ is locally CAT(κ).*

Lemma 2.3. *Let X be a length space with metric completion $\widehat{X}$. Then every open ball in $\widehat{X}$ meets X in a path-connected set.*

Proof: Suppose given an open ball U about $x \in \widehat{X}$, and $y, z \in U \cap X$. Choose $x' \in X$ near x and join y and z to x' by paths in X that are short enough that they are forced to lie in U. $\square$

Proof of theorem 2.2: We will write $\widetilde{\mathcal{H}}$ for $\pi^{-1}(\mathcal{H})$. Suppose $\tilde{x} \in \hat{N}$ lies over $x \in \hat{M}$ and let $S_1, \ldots, S_n$, U and V be as in the definition of the normality of $\mathcal{H}_0$ at x. Let r be the common radius of U and V. We write T_i for $\exp_x(U \cap S_i) \subseteq V$. It is clear that geodesics from $\tilde{x}$ to nearby points are lifts of radial geodesics from x. By choosing r small enough we may suppose that $\pi^{-1}(V)$ is the disjoint union of the r-balls about the points of $\pi^{-1}(x)$. We also choose r small enough so that V and all smaller balls centered at x are convex. We write $\widetilde{V}$ for the open r-ball about $\tilde{x}$; lemma 2.3 assures us that $\widetilde{V} - \widetilde{\mathcal{H}}$ is a connected covering space of $V - \mathcal{H}$. Taking generators $\sigma_1, \ldots, \sigma_n$ for $\pi_1(V - \mathcal{H})$ as above, the standardness of the cover assures us that the covering $\widetilde{V} - \widetilde{\mathcal{H}} \to V - \mathcal{H}$ corresponds to the subgroup generated by $\sigma_1^{d_1}, \ldots, \sigma_n^{d_n}$ for some $d_1, \ldots, d_n$. We take B (resp. $\widetilde{B}$) to be the closed r'-ball about x (resp. $\tilde{x}$), where we will choose $r' < r$ later. To show that x admits a CAT(κ) neighborhood, it suffices to show that $\widetilde{B}$ is CAT(κ) under the metric induced by lengths of paths in $\widetilde{B}$. We will prove this by realizing $\widetilde{B}$ as an iterated simple branched cover of B.

For each $k = 0, \ldots, n$, let G_k be the subgroup of $G = \pi_1(B - \mathcal{H})$ generated by $\sigma_1^{d_1}, \ldots, \sigma_k^{d_k}$, $\sigma_{k+1}, \ldots, \sigma_n$. We let B_k be the metric completion of the cover of $B - \mathcal{H}$ associated to G_k, equipped with the natural path metric. Then B_k is the standard branched cover of B, branched over the $T_i \cap B$, with branching indices $d_1, \ldots, d_k, 1, \ldots, 1$. In particular, $B_0 = B$ and $B_n = \widetilde{B}$. We write p_k for the natural projection $B_k \to B$ obtained by extending the covering map to a map of metric completions. Because $G_{k+1} \subseteq G_k$, there is a covering map $B_{k+1} - p_{k+1}^{-1}(\mathcal{H}) \to B_k - p_k^{-1}(\mathcal{H})$ whose completion $q_{k+1} : B_{k+1} \to B_k$ satisfies $p_k \circ q_{k+1} = p_{k+1}$. For each $k = 0, \ldots, n-1$ we let $\Delta_k = p_k^{-1}(T_{k+1})$. It is easy to see that q_{k+1} is a simple branched covering with branch locus $\Delta_k \subseteq B_k$.

In order to use theorem 2.1 inductively, we will need to know that Δ_k is a convex subset of B_k. This requires us to choose r' small enough so that the orthogonal projection maps from B to the T_i are well-behaved. By this we mean that for each i, there is fiberwise starshaped (about 0) set in the restriction to $T_i \cap B$ of the normal bundle of T_i, which is carried diffeomorphically onto B by the exponential map. Then each projection $B \to T_i$ has image in $B \cap T_i$, and the projection may be realized by a deformation retraction along geodesics. The retraction is distance non-increasing since T_i is totally geodesic and $\hat{M}$ has sectional curvature ≤ 0. Because the T_j are orthogonal to T_i for $j \neq i$, the track of the deformation retraction to T_i starting at a point outside $\cup_{j \neq i} T_j$ misses $\cup_{j \neq i} T_j$ entirely. Therefore the deformation lifts to a deformation retraction from $B_k - p_k^{-1}(\cup_{j \neq i} T_j)$ to $\Delta_k - p_k^{-1}(\cup_{j \neq i} T_j)$. This extends to a distance nonincreasing retraction $B_k \to \Delta_k$, which we will also call orthogonal projection.

Now we prove by simultaneous induction that B_k is CAT(κ) and that Δ_k is convex in B_k. The fact that $B_0 = B$ is CAT(κ) follows from its convexity in $\hat{M}$ and the fact that $\hat{M}$ has sectional curvature $\leq \kappa$. The convexity of $\Delta_0 = T_1 \cap B$ in B follows from the convexity of T_1 in V. Now the inductive step is easy. If B_k is CAT(κ) and Δ_k is convex in B_k then B_{k+1} is CAT(κ) by theorem 2.1. In particular, geodesics in B_{k+1} are unique. Then if γ is a geodesic of B_{k+1} with endpoints in Δ_{k+1}, the orthogonal projection to Δ_{k+1} carries γ to a path of length $\leq \ell(\gamma)$ with the same endpoints. By the uniqueness of geodesics, γ lies in Δ_{k+1}, so we have proven that Δ_{k+1} is convex in B_{k+1}. The theorem follows by induction. □

Remark: We indicate here the additional work required to prove the theorem when $\kappa > 0$. The projection maps $B \to B \cap T_j$ may increase distances in the presence of positive curvature. All that is important for us is that the length of a path in B *with endpoints in* T_k does not increase under projection to T_k. Even this is not true, but we only need the result for paths of length $< 2r'$. One should choose r' small enough so that any path in B of length $< 2r'$, with endpoints in T_k, grows no longer under the projection to T_k. Presumably this can be done but I have not checked the details.

Theorem 2.2 has been widely believed, but this seems to be the first proof. As mentioned before, it is morally contained in theorem 5.3 of Charney and Davis [4], who consider locally finite branched covers of Riemannian manifolds over subsets more complicated than mutually orthogonal submanifolds. Unfortunately there are gaps in their proof which I do not know how to bridge. (Lemma 5.7 does not seem to follow from lemma 5.6. Also, the techniques of [6] referred to in passing to finish the proof of theorem 5.3 use properties of Riemannian manifolds, like continuous dependence of sufficiently short geodesics on their endpoints, that are not established for branched covers.) Nevertheless their infinitesimal CAT(κ) condition (condition 3 of theorem 5.3) is very natural, and their theorem surely holds and extends to the case of locally infinite branching.

§3. Asphericity of Moduli Spaces

In this section we solve the problems which motivated our investigation, concerning the asphericity of certain moduli spaces. By using known models for the moduli spaces of cubic surfaces in $\mathbb{C}P^3$ and of Enriques surfaces we will show that these spaces have contractible universal covers. In both cases the main ingredient is the following theorem, which is a sort of global version of theorem 2.2.

Theorem 3.1. *Let $\hat{M}$ be a complete simply connected Riemannian manifold with sectional curvature bounded above by $\kappa \leq 0$. Let $\mathcal{H}$ be the union of a family of complete submanifolds which is normal in the sense of section 2. Then the metric completion $\hat{N}$ of the universal cover N of $\hat{M} - \mathcal{H}$ is CAT(κ), and N and $\hat{N}$ are contractible.*

We will conform to the notation of section 2 by writing M for $\hat{M} - \mathcal{H}$, π for the covering map $N \to M$ and its completion, and $\widetilde{\mathcal{H}}$ for $\pi^{-1}(\mathcal{H}) \subseteq \hat{N}$.

Lemma 3.2. *Suppose $\tilde{x} \in \hat{N}$ lies over $x \in \hat{M}$ and V is an open ball about x that meets none of the submanifolds except those passing through x. If $\widetilde{V}$ is the ball of the same radius about $\tilde{x}$, then $\widetilde{V} - \widetilde{\mathcal{H}}$ is a copy of the universal cover of $V - \mathcal{H}$.*

Proof: Since $\widetilde{V} - \widetilde{\mathcal{H}}$ is connected (lemma 2.3), all we must show is that $\pi_1(V - \mathcal{H})$ injects into $\pi_1(\hat{M} - \mathcal{H})$. Writing $\mathcal{I}$ for the union in $\hat{M}$ of the submanifolds that pass through x, we have homomorphisms

$$\pi_1(V - \mathcal{I}) \to \pi_1(\hat{M} - \mathcal{H}) \to \pi_1(\hat{M} - \mathcal{I}) \to \pi_1(V - \mathcal{I}) ,$$

where the first two maps are induced by inclusions and the third by a retraction of $\hat{M} - \mathcal{I}$ into $V - \mathcal{I}$ along geodesics from x. The composition is obviously the identity map, so the first map is injective. $\square$

Lemma 3.3. *The inclusion $N \to \hat{N}$ is a weak homotopy equivalence.*

Proof: First we show that for each $\tilde{x} \in \hat{N}$ there is a homotopy of $\hat{N}$ to itself that (i) carries some neighborhood of x into N, (ii) carries N into itself, and (iii) fixes each point of $\hat{N} - N$ that doesn't get pushed into N. We write n for the number of submanifolds passing through $x = \pi(\tilde{x})$. By the previous lemma there is a closed neighborhood $\widetilde{V}$ of $\tilde{x}$ which is homeomorphic to $(\widetilde{A})^n \times D$, where $\widetilde{A}$ is the metric completion of the universal cover of a punctured disk and D is a closed Euclidean ball. It is easy to see that $\widetilde{A}$ is homeomorphic to a wedge in the plane, by which we mean

$$\widetilde{A} \cong \{(0,0)\} \cup \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; |y| < x \text{ and } x^2 + y^2 \leq 1 \right\} .$$

There is obviously a homotopy of $\widetilde{A}$ into $\widetilde{A} - \{(0,0)\}$ which is supported on a small neighborhood of $(0,0)$. Using this it is easy to construct a homotopy of $\hat{N}$ satisfying (i)–(iii).

Now, if $f : S^k \to \hat{N}$ represents any element of the homotopy group $\pi_k(\hat{N})$ then we may cover $f(S^k)$ with finitely many open sets, each of which is carried into N by some homotopy of $\hat{N}$ that satisfies (ii) and (iii). Applying these homotopies one after another shows that f is homotopic to a map $S^k \to N$. Therefore $\pi_k(N)$ surjects onto $\pi_k(\hat{N})$. The same argument applied to balls rather than spheres shows that $\pi_k(N)$ also injects. By elaborating this argument one can show that $N \to \hat{N}$ is actually a homotopy equivalence, but it is easier to prove theorem 3.1 first and then apply the contractibility of both spaces. □

Proof of theorem 3.1: By lemma 3.2, $\hat{N}$ is a standard branched cover of $\hat{M}$ over the normal family $\mathcal{H}_0$. Since $\hat{M}$ has sectional curvature $\leq \kappa \leq 0$, theorem 2.2 shows that $\hat{N}$ is locally CAT(κ). Since N is simply connected, lemma 3.3 implies that $\hat{N}$ is also. The Cartan-Hadamard theorem for Alexandrov spaces [3, p. 193] implies that $\hat{N}$ is CAT(κ) and hence contractible. In particular, all of its homotopy groups vanish, and by another application of lemma 3.3 the same is true of N. As a manifold all of whose homotopy groups vanish, N is contractible. □

Now we turn to moduli spaces. The set $\mathcal{C}$ of cubic surfaces in $\mathbb{C}P^3$ may be identified with $\mathbb{C}P^{19}$, because there are 20 cubic monomials in 4 variables. The smooth surfaces form an open subset $\mathcal{C}_0$, and it is known that $\mathrm{PGL}(4,\mathbb{C})$ acts properly on $\mathcal{C}_0$. Therefore the moduli space $\mathcal{M} = \mathcal{C}_0/\mathrm{PGL}(4,\mathbb{C})$ carries the natural structure of a complex analytic orbifold. The main result of [1] shows that $\mathcal{M}$ is orbifold-isomorphic to $(\mathbb{C}H^4 - \mathcal{H})/P\Gamma$, where $\mathbb{C}H^4$ is complex hyperbolic 4-space, $\mathcal{H}$ is the union of an infinite family of complex hyperplanes and Γ is a certain discrete group. To state this more precisely, let ω be a primitive cube root of unity and let $\mathcal{E}$ be the discrete subring $\mathbb{Z}[\omega]$ of $\mathbb{C}$. Let Λ be the lattice $\mathcal{E}^5$ equipped with the Hermitian inner product

$$h(x,y) = -x_0\bar{y}_0 + x_1\bar{y}_1 + \cdots + x_4\bar{y}_4 .$$

Then the complex hyperbolic space $\mathbb{C}H^4$ is the set of lines in $\mathbb{C}^5$ on which h is negative-definite, $\mathcal{H}$ is the union of the hyperplanes in $\mathbb{C}H^4$ which are the orthogonal complements of those $r \in \Lambda$ with $h(r,r) = 1$, and Γ is the unitary group of Λ, which is obviously discrete in $\mathrm{U}(4,1)$.

Corollary 3.4. *$\mathcal{M}$ has contractible orbifold universal cover.*

Proof: Since $\mathbb{C}H^4 - \mathcal{H}$ covers $(\mathbb{C}H^4 - \mathcal{H})/P\Gamma$ and $\mathbb{C}H^4$ has negative sectional curvature, all we have to prove is the normality of the family of hyperplanes. Suppose $r, r' \in \Lambda$ satisfy $h(r,r) = h(r',r') = 1$. If $r^\perp$ meets $r'^\perp$ then r and r' span a positive-definite sublattice of Λ. This requires $|h(r,r')| < 1$, and since $h(r,r') \in \mathcal{E}$ we must have $h(r,r') = 0$, so that $r^\perp$ and $r'^\perp$ meet orthogonally. The local finiteness of the family of hyperplanes follows from a standard argument: if $x \in \mathbb{C}^5$ represents a point of $\mathbb{C}H^4$ and N is given, then there are only finitely many $r \in \Lambda$ satisfying $h(r,r) = 1$ and $|h(r,x)| \leq N$. $\square$

Enriques surfaces are smooth compact complex surfaces that satisfy certain cohomological conditions; see for example Horikawa [9], [10]. Horikawa's global Torelli theorem identifies the set of isomorphism classes of these surfaces with $(\mathcal{D} - \mathcal{H})/\Gamma$, where $\mathcal{D}$ is the Hermitian symmetric space for $O(2,10)$, Γ is a certain discrete subgroup and $\mathcal{H}$ is the union of an infinite family of complex hyperplanes. Namikawa [11] refined Horikawa's work, and according to the rephrasing of these results in [2], Γ may be taken to be the isometry group of the lattice L which is $\mathbb{Z}^{12}$ equipped with the inner product

$$x \cdot y = x_1 y_1 + x_2 y_2 - x_3 y_3 - \cdots - x_{12} y_{12} \,.$$

A concrete model for $\mathcal{D}$ is the set of $v \in P(L \otimes \mathbb{C})$ satisfying $v \cdot v = 0$ and $v \cdot \bar{v} > 0$, and $\mathcal{H}$ may be taken to be the union of the (complex) hyperplanes in $\mathcal{D}$ which are the orthogonal complements of the norm -1 vectors of L. We regard the moduli space of Enriques surfaces as being the orbifold $(\mathcal{D} - \mathcal{H})/\Gamma$.

Corollary 3.5. *The moduli space* $(\mathcal{D} - \mathcal{H})/\Gamma$ *has contractible orbifold universal cover.*

Proof: The proof that hyperplanes that meet do so orthogonally is the same as before. Local finiteness follows by essentially the same standard argument: any $v \in \mathcal{D}$ defines a 2-dimensional positive-definite subspace V of $L \otimes \mathbb{R}$, namely the projection to $L \otimes \mathbb{R}$ of the complex line it represents. For each N there are only finitely many $r \in L$ with $r \cdot r = -1$ having projection to V of norm $\leq N$. This proves local finiteness and hence normality. Then we use the fact that $\mathcal{D}$ has nonpositive sectional curvature and appeal to theorem 3.1. $\square$

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