

REFLECTION CENTRALIZERS IN COXETER GROUPS

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ABSTRACT. We give a new proof of Brink’s theorem that the non-reflection part of a reflection centralizer in a Coxeter group is free, and make several refinements. In particular we give an explicit finite set of generators for the centralizer and a method for computing the Coxeter diagram for its reflection part. In many cases, our method allows one to compute centralizers quickly in one’s head. We also define “Vinberg representations” of Coxeter groups, in order to isolate some of the key properties of the Tits cone.

Brink has proved the elegant result that the centralizer of a reflection in a Coxeter group is the semidirect product of a Coxeter group by a free group [5]. In fact this free group is the fundamental group of the component of the “odd Coxeter diagram” distinguished by the conjugacy class of the reflection. We give a new proof of her result, together with several refinements.

The first refinement is a method of computing the Coxeter diagram of the Coxeter-group part of the centralizer. With a little effort we develop this method to the point that many centralizer computations are very easy. For example, the fact that the reflection centralizer in $W(E_8)$ is $W(E_7) \times 2$ becomes an almost-instant mental computation. We offer many other examples, including the reflection centralizer in the Coxeter group of Bugaenko that acts cocompactly on 8-dimensional hyperbolic space [7]. Our method shares the same foundation as that of Brink and Howlett [6], which is a special case of an algorithm for understanding normalizers of parabolic subgroups. (See also [1] and [3] for related work.) However, in use it feels quite different.

The second refinement is an explicit finite set of generators for the reflection centralizer; Brink only gave explicit generators for the free part. This generating set plays a key role in the author’s work [2] with Lisa Carbone on Kac-Moody groups.

Our proof of Brink’s theorem is quite different from hers, using covering spaces and topology in place of induction on word lengths. We hope this alternate proof will be helpful to some people. In order to

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present the argument as clearly as possible we introduce what we call a Vinberg representation of a Coxeter group, in honor of Vinberg's [15]. This is not strictly needed for the rest of the paper. However, this notion isolates the relevant properties of the Tits cone and is more flexible than it.

In section 1 we give the definition of a Vinberg representation, in section 2 we describe the non-reflection part Γ_Ω of the centralizer, in section 3 we describe the Coxeter-group part $W_\Omega \times 2$, in section 4 we give many examples, and then in section 5 we give our explicit finite generating set for the centralizer.

We follow standard conventions regarding Coxeter systems and diagrams [4], [12]. We also use the semi-standard term "spherical" for a Coxeter system or diagram when the corresponding group is finite. This reflects the fact that the group acts naturally on a sphere, rather than say hyperbolic space. In the many places where we refer to the parity of an edge label in a diagram, we use the convention that ∞ is neither even nor odd.

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1. VINBERG REPRESENTATIONS

Let W be a group, which we will later see is a Coxeter group. A Vinberg representation of it means a pair (V, U) where V is a faithful finite-dimensional real representation of W and U is a nonempty W -invariant open convex subset, satisfying (V1)–(V3) below. We name these representations after Vinberg because they are essentially the representations he studied in [15]. An involution $s \in W$ is called a reflection if it fixes a hyperplane V^s of V pointwise; then it exchanges the two half-spaces V^s bounds. Since U is nonempty, W -invariant and convex, $U^s = V^s \cap U$ is nonempty. We call this the mirror of s .

- (V1) The mirrors of W 's reflections are locally finite in U .
- (V2) Each mirror is the mirror of only one reflection.
- (V3) W is generated by its reflections.

It follows from (V1) that $U - (\text{all mirrors})$ is open in V , and we call the closure in U of any one of its components a (Weyl) chamber. Also by (V1), it makes sense to speak of a chamber's faces, and in particular its codimension 1 faces, called facets. If C is a chamber then the reflections across (the mirrors containing) its facets are uniquely determined by (V2), and we write Π_C for the set of them. We call two chambers

adjacent (or neighbors) if their intersection is a facet of each. Then they are exchanged by the reflection across that facet.

The next theorem shows that in this situation, all the usual properties of Coxeter groups hold. We omit the proof because it is not strictly needed for our main results and is a minor variation on standard arguments [15, thm. 2][4, ch. V§4].

Theorem 1.

- (i) W acts simply transitively on chambers.
- (ii) W acts properly discontinuously on U .
- (iii) The W -stabilizer of any point of U is generated by reflections.
- (iv) Each chamber maps bijectively to U/W .

Furthermore, for any chamber C , Π_C generates W , and defining relations for W with respect to this generating set are $(st)^{m(s,t)} = 1$ when $m(s,t) < \infty$, where s, t vary over Π_C and $m(s,t)$ is the order of their product. In particular, (W, Π_C) is a Coxeter system. \square

Conversely, if (W, Π) is a Coxeter system then there is a Vinberg representation (V, U) of W and a chamber C of it, such that $\Pi = \Pi_C$. Then we call (V, U, C) a (Vinberg) representation of the Coxeter system. The standard construction is the (open) Tits cone, and the conclusions of theorem 1 are all standard results for it. See [15], [4] or [12] for details.

We have formulated these ideas because the same Coxeter system may have several different interesting representations. They all work equally well for many purposes, but sometimes the presence of additional structure makes a particular representation better than others. For example, the construction of the Tits cone starts with an inner product on the vector space \mathbb{R}^Π , and there is a choice involved for each edge labeled ∞ in the Coxeter diagram. This choice can affect the signature of the inner product, and in particular can make it degenerate. This can lead to natural representations of (W, Π) other than on the Tits cone. For example V could be Minkowski space, U one of the two cones of timelike vectors, and C the cone on a hyperbolic Coxeter polytope (meaning that its dihedral angles are integral submultiples of π).

Another example is that two Kac-Moody algebras may be very different from each other yet have isomorphic Weyl groups. The difference is the choice of generalized Cartan matrix, which is essentially a special choice of Vinberg representation.

2. THE NON-REFLECTION PART OF THE CENTRALIZER

Now suppose (W, Π) is a Coxeter system and $s \in W$ is conjugate into Π . Our goal is to understand $C_W(s)$. Choose a Vinberg representation (V, U, C) of (W, Π) and define W_Ω as the subgroup of W generated by the reflections that preserve V^s and each of the two half-spaces it bounds (call one of them $\frac{1}{2}V$). Then (V^s, U^s) is a Vinberg representation of W_Ω , since the assumptions on W restrict well. (W_Ω 's faithfulness on V^s follows from the preservation-of-half-spaces condition). In particular, W_Ω is a Coxeter group (c.f. [9],[10]).

Now choose a chamber C_Ω for W_Ω in U^s and define Γ_Ω as the subgroup of W that preserves C_Ω (hence V^s) and $\frac{1}{2}V$. Note that C_Ω has one dimension less than C . To avoid confusion we specify: when we speak of chambers without mentioning W_Ω explicitly, we always mean chambers of W . The following theorem, essentially due to Howlett [11, corollaries 3 and 7] (but see [6] for the non-spherical case) decomposes $C_W(s)$ into 3 pieces. It reduces our analysis of the centralizer to a study of W_Ω and Γ_Ω , which are called the reflection and nonreflection parts of $C_W(s)$.

Theorem 2. $C_W(s) = \langle s \rangle \times \langle W_\Omega, \Gamma_\Omega \rangle$, and the latter factor splits as the semidirect product of W_Ω by Γ_Ω . \square

As usual, the Coxeter diagram Δ_C of a chamber C means the graph with vertex set Π_C and an edge between $s, t \in \Pi_C$ labeled by the order $m(s, t)$ of st , when $m(s, t) > 2$. In this paper we will avoid fixing a chamber of W , and instead think of $\mathcal{C} := U/W$ as the “fundamental chamber”. By theorem 1(iv) it may be canonically identified with any chosen chamber. This allows us to speak of \mathcal{C} 's faces, facets and Coxeter diagram Δ . The proper faces of \mathcal{C} correspond to the spherical subdiagrams of Δ .

Our approach to understanding Γ_Ω is that the interior C_Ω° of C_Ω is the universal cover of part of the boundary of \mathcal{C} . We define X as \mathcal{C} minus its interior and those codimension 2 faces that correspond to even or absent edges in Δ . Our first step is to describe X ; then we establish the universal covering property.

Lemma 3. *The dual complex of X is the “odd Coxeter diagram” Δ^{odd} , i.e., the graph with the same vertices as Δ , two being joined just if their edge label in Δ is odd.*

Since X is a chamber minus some faces, we clarify our meaning: its dual complex means the simplicial complex with a vertex for each facet of \mathcal{C} , and a simplex with any given set of vertices just if X contains the

interior of the intersection of the corresponding facets. (The interior of a face means that face minus its lower-dimensional faces.) Standard arguments show that X and its dual complex are homotopy-equivalent. This uses the fact that the link in \mathcal{C} of any face is a simplex, which is a rephrasing of the standard fact that chambers of finite Coxeter groups are simplicial. By this and lemma 3, each component of X has free fundamental group.

Proof. X contains no codimension 3 faces of \mathcal{C} at all. This is because such a face corresponds to a spherical 3-vertex diagram in Δ , and every such diagram has a pair of unjoined vertices. So every codimension 3 face of \mathcal{C} lies in a codimension 2 face that we discarded in the definition of X . To finish the proof we observe that the codimension 2 faces of \mathcal{C} whose interiors lie in X are exactly the ones corresponding to odd edges in Δ , by definition. \square

Lemma 4. *The natural map $U \rightarrow U/W = \mathcal{C}$ induces a universal covering map from C_Ω° to a component of X , with deck group Γ_Ω .*

A nice mental image of $C_\Omega^\circ \rightarrow X$ is of folding C_Ω° along its intersections with mirrors of W other than U^s , and then wrapping it around the fundamental chamber \mathcal{C} as one might wrap a Weyl-chamber-shaped gift.

Proof. First we show that C_Ω° contains no codimension 3 face ϕ of any chamber C . If it did then we could apply an element in ϕ 's stabilizer to suppose that C has a facet in C_Ω . Then $s \in \Pi_C$ and ϕ corresponds to a rank 3 spherical subdiagram of Δ_C containing s . Every reflection in every rank 3 spherical Coxeter group is centralized by some other reflection in it. (Just inspect the possible Coxeter diagrams and recall that oddly-joined generators are conjugate.) It follows that W_Ω contains a reflection fixing ϕ , so ϕ lies in a mirror of W_Ω . Since C_Ω° is a component of $U^s - (W_\Omega$'s mirrors), it can't contain ϕ .

The same argument shows that C_Ω° contains no codimension 2 face of any chamber C corresponding to an even edge label in Δ_C . This shows that $U \rightarrow U/W = \mathcal{C}$ carries C_Ω° into X . Finally, if C has a facet F in C_Ω and $t \in \Pi_C$ is oddly joined to s , then (the interior of) the corresponding codimension-2 face ϕ of C *does* lie in C_Ω° . This is because ϕ 's W -stabilizer is a dihedral group of twice odd order. In such a group, no reflection centralizes any other. Furthermore, the facet (of some other chamber) in U^s on the other side of ϕ is equivalent under this dihedral group to the other facet of C containing ϕ (i.e., not F). This is what we referred to when comparing $C_\Omega^\circ \rightarrow \mathcal{C}$ to wrapping a gift. It follows that $C_\Omega^\circ \rightarrow X$ is a local homeomorphism, and then it is

easy to see that $C_\Omega^\circ \rightarrow X$ is a covering map. It is a universal covering of a component of X because C_Ω° is convex, hence connected and simply connected.

That the deck group is Γ_Ω is the fact that if $x, y \in C_\Omega^\circ$ are W -equivalent, say $g(x) = y$, then they are also Γ_Ω -equivalent. If x, y are generic (they lie on no mirrors except U^s) then the stabilizer of each is $\langle s \rangle$, so g must centralize s . Multiplying by s if necessary, we may suppose g preserves $\frac{1}{2}V$, hence lies in Γ_Ω . In the non-generic case their stabilizers are dihedral of twice odd order. In such a group all reflections are conjugate, so after following g by an element of W stabilizing y , we may suppose g centralizes s . Then we may proceed as before. \square

All that remains is to determine the component of X in terms of s . Choose a chamber C with a facet F in C_Ω , so $s \in \Pi_C$. Then $C_\Omega^\circ/\Gamma_\Omega \subseteq X$ contains (the interior of) the facet of \mathcal{C} corresponding to F . To phrase this without reference to a specific chamber, we refer to the well-known fact that the conjugacy classes of reflections in W are in bijection with the components of Δ^{odd} . So s distinguishes a component Δ_s^{odd} of Δ^{odd} , hence of X , and $C_\Omega^\circ/\Gamma_\Omega$ is this component.

Having proven the lemmas we will switch our focus from C_Ω° to C_Ω to avoid fussing over the missing faces. By a *tile* we mean a facet (of some chamber) that lies in C_Ω . Its *type* means its image in \mathcal{C} , or the corresponding vertex of Δ . We summarize this section by phrasing Brink's theorem in our language. It follows from the lemmas and remarks above.

Theorem 5. *Suppose (V, U) is a Vinberg representation of W , $s \in W$ is a reflection, and C_Ω , W_Ω and Γ_Ω are as above. Then the tiles comprising C_Ω are in bijection with the vertices of the universal cover $\tilde{\Delta}_s^{\text{odd}}$ of Δ_s^{odd} , with tiles meeting in codimension one if and only if the corresponding vertices are joined. Under this correspondence, Γ_Ω acts by the deck transformations of $\tilde{\Delta}_s^{\text{odd}} \rightarrow \Delta_s^{\text{odd}}$. In particular, Γ_Ω is the free group $\pi_1(\Delta_s^{\text{odd}}, s)$. \square*

3. THE COXETER DIAGRAM OF W_Ω

By theorem 1 and the definition of C_Ω as a chamber for W_Ω in the Vinberg representation (V^s, U^s) , the set of reflections across its facets is a Coxeter system for W_Ω . We write Δ_Ω for the corresponding Coxeter diagram. Our goal in this section is to describe it concretely, in a manner making obvious the action of $\Gamma_\Omega \cong \pi_1(\Delta_s^{\text{odd}})$.

Recall the definition above of a tile and its type. An *arrow* means a facet of a tile, which is not a facet of any other tile—i.e., it lies in the

boundary of C_Ω . This peculiar terminology helps organize the calculations in examples. The arrows fall into equivalence classes according to which facet of C_Ω they lie in, which we call the arrow classes. To describe Δ_Ω we must find the arrow classes and understand how the corresponding facets of C_Ω meet.

We begin by finding the arrows. By theorem 5, the tiles are in 1-1 correspondence with the vertices of $\tilde{\Delta}_s^{\text{odd}}$. If \tilde{J} is a tile then we write J for its type. To \tilde{J} we associate the unique chamber C that lies in $\frac{1}{2}V$ and has \tilde{J} as a facet. A facet of \tilde{J} corresponds to a vertex of Δ_C that is not infinitely joined to s . By lemma 4, this facet of \tilde{J} lies in the boundary of C_Ω just if the edge-label is even. Therefore we have named the arrows:

Lemma 6. *The arrows are in bijection with the pairs $[\tilde{J}, K]$, where \tilde{J} and K are vertices of $\tilde{\Delta}_s^{\text{odd}}$ and Δ respectively, and the edge joining J and K in Δ is absent or evenly-labeled. \square*

The next step is to determine the dihedral angles among the arrows. (We say they have dihedral angle π/n as a shorthand for saying the product of the corresponding reflections has order n .)

Lemma 7. *If \tilde{J} is a vertex of $\tilde{\Delta}_s^{\text{odd}}$ and K and L are vertices of Δ , such that the subdiagram of Δ formed by J , K and L appears in table 1, then the indicated arrows intersect in codimension 1, with the stated dihedral angle. Conversely, if two arrows meet in codimension 1 then there exist such \tilde{J}, K, L , such that the arrows are the ones indicated in the table.*

Proof. Suppose $[\tilde{J}, K]$ and $[\tilde{J}', K']$ are arrows, whose intersection ϕ has codimension one in each. Write C for the chamber associated to \tilde{J} . Then ϕ is a codimension 3 face of C , so it corresponds to a spherical 3-vertex subdiagram of Δ_C , containing J and K . Write Y for the corresponding finite Coxeter group and L for the third vertex. We next verify the conclusions of the theorem if this subdiagram appears in table 1.

The calculation takes place entirely in the standard representation of Y , which we think of as transverse to ϕ . In this \mathbb{R}^3 , U^s appears as a hyperplane H , $\frac{1}{2}V$ as a half-space bounded by H , C as a chamber of Y with a facet in H , and \tilde{J} equal to this facet. The two other facets of this chamber correspond to K and L . We write F_J , F_K and F_L for these facets, and R_{JK} , R_{KL} and R_{LJ} for the rays where these facets meet.

The simplest case is when J and L are evenly joined. Then $[\tilde{J}, L]$ is also an arrow containing ϕ , so it is the only one other than $[\tilde{J}, K]$. Also,

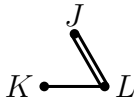



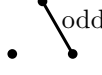


- (1)  $\implies [\tilde{J}, K]$ and $[\tilde{J}, L]$ make angle $\pi/4$.
- (2)  $\implies [\tilde{J}, K] \perp [\tilde{J}, L]$.
- (3)  $\implies [\tilde{J}, K]$ and $[\tilde{J}, L]$ make angle π/n .
- (4)  $\implies [\tilde{J}, K] \perp [\tilde{L}, K]$.
- (5)  $\implies [\tilde{J}, K]$ and $[\tilde{L}, K]$ make angle π .
- (6)  $\implies [\tilde{J}, K]$ and $[\tilde{K}, J]$ make angle π .
- (7)  $\implies [\tilde{J}, K] \perp [\tilde{K}, J]$.

TABLE 1. Dihedral angles between facets of tiles; see lemma 7. We implicitly label the vertices of all the diagrams J , K , L as in the first one. In the last two lines, \tilde{K} means the tile of type K adjacent to \tilde{L} , where (in the last four lines) \tilde{L} is the tile of type L adjacent to \tilde{J} .

$[\tilde{J}, K]$ and $[\tilde{J}, L]$ correspond to R_{JK} and R_{JL} , so the angle between them is the angle between these rays in \mathbb{R}^3 .

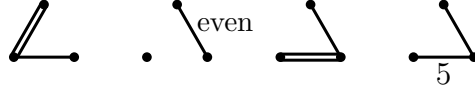
The next case is when J and L are oddly joined and L and K are unjoined or evenly joined. Let Θ be the rotation around R_{LJ} with $\Theta(F_L)$ in H but not overlapping F_J . Then \tilde{L} corresponds to $\Theta(F_L)$, and $[\tilde{L}, K]$ is the arrow meeting $[\tilde{J}, K]$ in ϕ ; it corresponds to $\Theta(R_{KL})$. So the angle between these arrows is the angle in \mathbb{R}^3 between R_{JK} and $\Theta(R_{KL})$. (This rotation process is the reverse of the gift-wrapping process of section 2.)

In the final case, J and L are oddly joined and so are L and K . We will apply a second rotation. Namely, let Θ' be the rotation around $\Theta(R_{KL})$ with $\Theta' \circ \Theta(F_K)$ in H but not overlapping $\Theta(F_L)$. Then \tilde{K} corresponds to $\Theta' \circ \Theta(F_K)$, and $[\tilde{K}, J]$ is the arrow meeting $[\tilde{J}, K]$ in ϕ ; it corresponds to $\Theta' \circ \Theta(R_{JK})$. The angle between these arrows is the angle in \mathbb{R}^3 between R_{JK} and $\Theta' \circ \Theta(R_{JK})$.

One can find these angles without computation. Consider the edges E_J , E_K and E_L of the spherical triangle defined by F_J , F_K and F_L . In

the three cases the desired angle is $\ell(E_J)$, $\ell(E_J) + \ell(E_L)$ and $\ell(E_J) + \ell(E_K) + \ell(E_L)$, where ℓ indicates length. Drawing the tessellation of the sphere by Y 's chambers makes it easy to recognize which submultiple of π this is. This justifies the entries in table 1.

The table is complete because one can write down all possibilities for the spherical diagram on J, K and L , with J and K evenly joined or unjoined since $[\tilde{J}, K]$ is an arrow. The possibilities other than those in table 1 are



where we use the table's labeling of vertices. The first and second differ from (1) and (2) by $K \leftrightarrow L$, the third from (4) by $J \leftrightarrow L$ and the last from (7) by $J \leftrightarrow K$. In all cases the conclusion of the relevant line of the table is symmetric under the same interchange. So if the diagram of J, K, L doesn't appear in the table then we just swap $[\tilde{J}, K]$ and $[\tilde{J}', K']$. \square

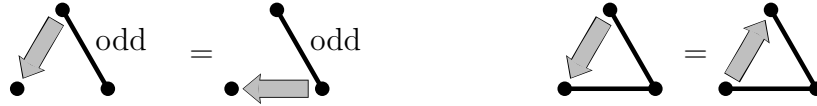
The lemma allows one to compute the Coxeter diagram Δ_Ω :

Theorem 8. *The equivalence relation of arrows lying in the same facet of C_Ω is generated by the equivalence of $[\tilde{J}, K]$ with $[\tilde{L}, K]$ in the situation of (5) and that of $[\tilde{J}, K]$ with $[\tilde{K}, J]$ in the situation of (6).*

If two arrow classes have representative arrows as in one of the other entries of the table, then the corresponding facets of C_Ω have the listed dihedral angle. If they have no such representatives then they do not meet, and the edge of Δ_Ω joining them is labeled ∞ .

Proof. If two arrows lie in the same facet of C_Ω then there is a chain of arrows joining them, each lying in that facet of C_Ω and meeting the next in codimension 1. This proves the first claim. The second follows immediately from lemma 7, and the third is obvious. \square

We close the section with a few remarks on the language. We visualize $[\tilde{J}, K]$ as an arrow pointing from the vertex \tilde{J} of Δ_s^{odd} to the vertex K of Δ . We define a tail class as an equivalence class of facets under the relation (5), i.e., $[\tilde{J}, K]$ and $[\tilde{L}, K]$ are equivalent when J, K, L form the configuration (5). The reason for the name is that the equivalence corresponds to the first of the following “moves” on arrows: the tail moves around.



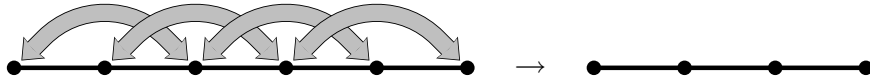
The second picture is a graphical interpretation of (6), although one must be careful keeping track of which vertices lie in Δ_s^{odd} and which lie in Δ . In our examples in the next section Δ_s^{odd} will be a tree, so $\tilde{\Delta}_s^{\text{odd}}$ may be regarded as a subdiagram of Δ . In that case we can take the figure literally.

4. EXAMPLES

In this section we give many examples illustrating theorem 8. If Δ_s^{odd} is not a tree then Δ_Ω is usually infinite, so we focus on the tree case. We generally proceed by working out the tail classes, fusing them into arrow classes, and then finding the angles.

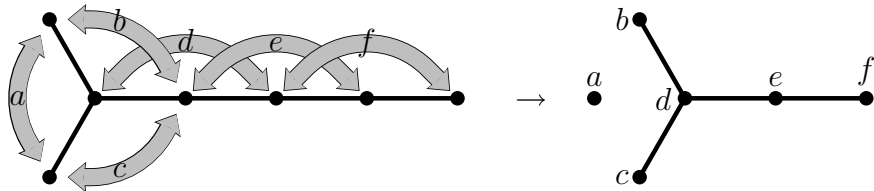
Suppose first that Δ is a tree of single edges (edge label 3). Then there is only one class of reflection, so we don't need to choose a component of Δ^{odd} . The tail classes are easy to work out: each contains a unique arrow $[J, K]$ where J and K have distance 2 in Δ . (Proof: move the tail toward the head.) The tail classes fuse in pairs, got by reversing these arrows. The end result is that the generators for W_Ω are in bijection with the A_3 diagrams in Δ . Almost all the angles can be worked out using (3). In our situation it reads: if two arrow classes have representatives with the same tail, then their edge label in Δ_Ω is the same as the one between their tips in Δ .

The first example is $A_{n \geq 3}$, which has $n - 2$ arrow classes:



We have drawn double-headed arrows because of the fusion of tail classes. Every arrow class has a representative with tail at the left-most vertex. So the joins between these arrow classes are the same as the joins between the right-hand tips of the arrows. So $W_\Omega = W(A_{n-2})$.

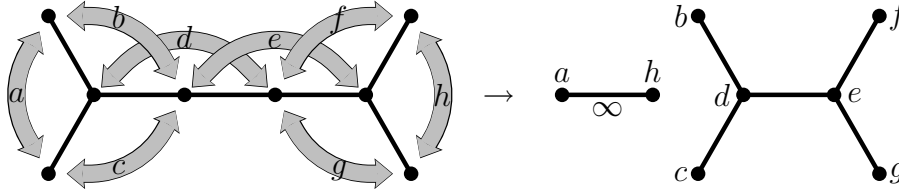
The second example is $D_{n \geq 6}$, which has $n - 1$ arrow classes:



Choosing representative arrows with tails at the top left shows that a is orthogonal to all the other generators except perhaps c . Repeating the argument with tails at the lower left shows that a is also orthogonal to c . Then taking tails at the rightmost vertex shows that b, \dots, f form

a D_{n-2} diagram. So $W_\Omega = W(A_1 D_{n-2})$. The D_4 and D_5 cases are the same provided one interprets D_2 and D_3 as A_1^2 and A_3 .

The third example is the affine diagram $\tilde{D}_{n \geq 6}$, which gives $W_\Omega = W(\tilde{A}_1 \tilde{D}_{n-2})$ in a similar way:

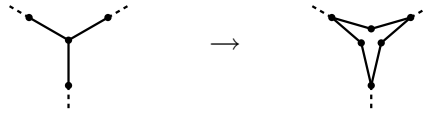


The new phenomenon is that the arrows a and h cannot be moved into a spherical 3-vertex diagram. So those facets of C_Ω don't meet, hence the edge label ∞ . The \tilde{D}_4 and \tilde{D}_5 cases are the same, provided one interprets \tilde{D}_2 and \tilde{D}_3 as \tilde{A}_1^2 and \tilde{A}_3 .

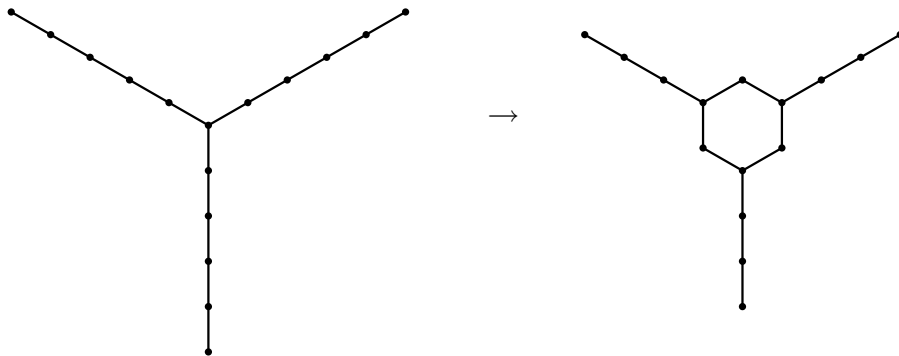
These examples are enough to treat the general case:

Theorem 9. *If Δ is a tree of single edges, then the vertices of Δ_Ω are the A_3 subdiagrams of Δ , with edge labels as follows. If the convex hull of two A_3 's in Δ has type D (resp. \tilde{D}), then their edge label in Δ_Ω is 2 (resp. ∞). Otherwise, it is the same as the one in Δ between their middle vertices. \square*

In the special case of trivalent branch points, no two adjacent, Δ_Ω can be got from Δ by the following operation: “blow up” each branch point



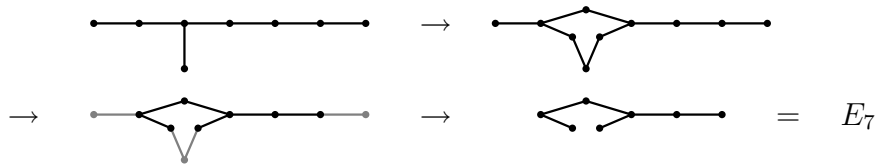
and then erase all the end vertices of Δ , and finally add some edges labeled ∞ . When there is only one branch point there are no ∞ 's. For example, the Y_{555} diagram gives



We chose this example because it explains the appearance of the latter figure in the ATLAS [13] entry for the monster simple group M ,

given the appearance of the former. Namely, the bimonster $(M \times M):2$ is described as a quotient of the Y_{555} Coxeter group, and $M \times 2$ as a quotient of the Coxeter group of the second figure. Given the first, one should expect the second, because a Y_{555} reflection maps to an involution in the bimonster with centralizer $M \times 2$. (One can repeat the process, so the reflection centralizer in the second diagram maps to the involution centralizer $2B \times 2 \subseteq M \times 2$ where B is the baby monster. Since the nonreflection part is now \mathbb{Z} , the Y_{555} approach to M distinguishes a conjugacy class in B , up to inversion. I don't know what class this is or whether it has any real meaning.)

Another example is $\Delta = E_8$, which we show in several steps to illustrate the interaction between blowing up branch points and erasing ends:



The first step blows up the branch point, the second shows what will be erased, and the third actually erases it.

An example with an edge label ∞ is the reflection group of the even unimodular Lorentzian lattice of dimension 18. By [16] (see also [8]), Δ is



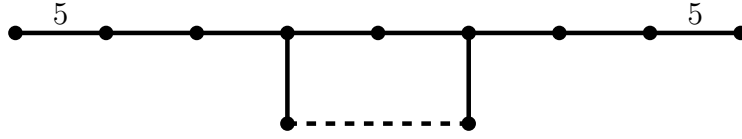
which after explosion and erasure yields



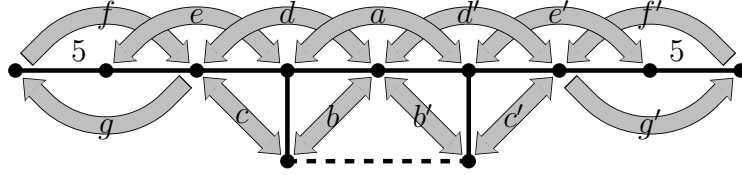
Since the vertices marked “!” correspond to A_3 's in Δ whose convex hull is a \tilde{D}_{16} , they should be joined by an edge labeled ∞ . Adjoining this edge completes the description of Δ_Ω . (Remark: this is the reflection group of the even Lorentzian lattice of dimension 17 and determinant ± 2 . Because it acts on hyperbolic space H^{16} , it makes sense to ask whether this ∞ represents parallelism or ultraparallelism. It represents parallelism, because the corresponding infinite dihedral group lies in the affine group $W(\tilde{D}_{16})$.)

As a meatier example we treat a Coxeter group found by Bugaenko [7]. It acts cocompactly on H^8 , and is the only known cocompact

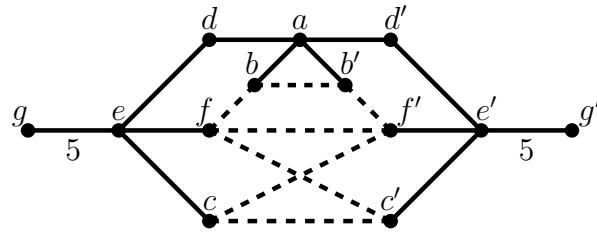
example on any $H^{n \geq 8}$. Here Δ is



where the dashed line means an edge label ∞ . The same argument as before shows that every tail class is represented by a unique arrow from one vertex to another at distance 2. So we can name the 13 generators of W_Ω :



Note that f and g represent distinct arrow classes, because only “ A_3 arrows” are reversible. Taking tails at the left end of Δ shows that $f, e, d, c, a, b', d', e', g'$ are joined the same way as their right endpoints are joined in Δ . Exchanging primed and unprimed letters gives all joins among $f', e', d', c', a, b, d, e, g$, so we know all the joins except those between a member of $\{g, b, c', f'\}$ and a member of $\{g', b', c, f\}$. By the priming symmetry there are only 10 cases left to work out. Taking tails based at the middle vertex gives $bb' = \infty$ and $gg' = 2$. Taking tails based at the lower left vertex gives $bc = bg' = cg = 2$. We have $fg = 2$ by (7). Finally, we have $bf = cc' = cf' = ff' = \infty$ because in none of these cases is there a 3-vertex spherical diagram containing representatives for both arrow classes. So Δ_Ω is



We have avoided the case of Δ_s^{odd} containing cycles, because then W_Ω is seldom finitely generated. But $\Delta = \tilde{A}_n$ gives $W_\Omega = W(\tilde{A}_{n-2})$ with Γ_Ω acting nontrivially. We recommend this to the reader as a pleasing exercise. Another interesting exercise is to work out what happens when one attaches a vertex using only even edges. This could be used to derive the diagrams for the reflection groups of the odd bimodular Lorentzian lattices [16, p. 34] from the diagrams for the odd unimodular Lorentzian lattices of one larger dimension [16, p. 32].

5. EXPLICIT GENERATORS FOR THE CENTRALIZER

In this section we give an explicit finite generating set for any reflection centralizer, which we wish to record for a forthcoming application to Kac-Moody groups [2].

Suppose (W, Π) is a Coxeter system with diagram Δ_Π and $\gamma = (t_0, \dots, t_n)$ is an edge-path in Δ_Π^{odd} , with $2l_i + 1$ being the label on the edge joining t_{i-1} and t_i . Then we set

$$p_\gamma := (t_1 t_0)^{l_1} (t_2 t_1)^{l_2} \cdots (t_n t_{n-1})^{l_n}$$

(or $p_\gamma = 1$ if γ has length 0). This word is Brink's $\pi(t_0, \dots, t_n)$. If u is a vertex of Δ with $m(t_n, u)$ even, say 2λ , then we define

$$r_{\gamma, u} := p_\gamma \cdot u(t_n u)^{\lambda-1} \cdot p_\gamma^{-1}.$$

Theorem 10. *Suppose $s \in \Pi$, Z is a set of edge-loops in Δ_Π^{odd} based at s that generate $\pi_1(\Delta_\Pi^{\text{odd}}, s)$, and δ_t is an edge-path in Δ_Π^{odd} from s to t , for each vertex t in s 's component of Δ_Π^{odd} . Suppose (V, U, C) is any Vinberg representation (V, U, C) of (W, Π) . Then the $p_{z \in Z}$ generate Γ_Ω . Together with the $r_{\delta_t, u}$ such that $m(t_n, u)$ is even, they generate $W_\Omega : \Gamma_\Omega$. Finally, $C_W(s) = \langle s, p_z, r_{\delta_t, u} \rangle$.*

In the rest of the section we assume the hypotheses of the theorem, and identify C with $\mathcal{C} = U/W$, so that the elements of Π label the nodes of Δ . We will consider vertices t_0, \dots, t_n of Δ and chambers C_0, \dots, C_n , and write $t_{i,j}$ for the element of Π_{C_i} corresponding to t_j .

Lemma 11. *Suppose C_0 is a chamber in $\frac{1}{2}V$ containing a tile T_0 of type $t_0 \in \Pi$, and t_0 is joined to $t_1 \in \Pi$ by an edge labeled $2l_1 + 1$. Define $w_1 = (t_{0,1}s)^{l_1}$; then $C_1 := w_1(C_0)$ is the chamber in $\frac{1}{2}V$ containing the tile T_1 of type t_1 that is adjacent to T_0 in $\tilde{\Delta}_s^{\text{odd}}$.*

Proof. After unwinding the language this becomes a simple statement about a dihedral group of order $2(2l_1 + 1)$ acting on \mathbb{R}^2 . \square

Extending the notation of the lemma, suppose $\gamma = (t_0, \dots, t_n)$ is an edge-path in Δ_s^{odd} , with $m(t_{i-1}, t_i) = 2l_i + 1$. Then we use the lemma inductively to construct a sequence of chambers C_1, \dots, C_n that lie in $\frac{1}{2}V$ and contain tiles T_1, \dots, T_n of types t_1, \dots, t_n , with T_{i-1} and T_i adjacent in $\tilde{\Delta}_s^{\text{odd}}$ for each i . Namely, $C_i := w_i(C_{i-1})$ where $w_i = (t_{i-1,i}s)^{l_i}$. Obviously $w_n \cdots w_2 w_1$ sends C_0 to C_n . We defined p_γ above so that it would also do this:

Lemma 12. *If $C_0 = C$, so that $t_{0,j} = t_j$ for all j , then p_γ sends C to C_n .*

Proof. We use induction on n , the case $n = 0$ being trivial. For the inductive step we write β for the initial segment of γ of length $n - 1$. So $p_\beta(C_0) = C_{n-1}$ and therefore $t_{n-1,j} = p_\beta t_j p_\beta^{-1}$ for all j . Also, the meaning of “ T_{n-1} has type t_{n-1} ” is that $s = t_{n-1,n-1}$. So

$$w_n \cdots w_1 = (t_{n-1,n}s)^{l_n} p_\beta = (t_{n-1,n}t_{n-1,n-1})^{l_n} p_\beta = p_\beta (t_n t_{n-1})^{l_n} = p_\gamma.$$

□

Lemma 13. *Writing u' for the element of Π_{C_n} corresponding to some $u \in \Pi$ with $m(t_n, u) = 2\lambda$, $u'(su')^{\lambda-1}$ is a reflection in W_Ω , across the arrow $[T_n, u]$ of the tile T_n .*

Proof. Similar to lemma 11, this is a simple check in the dihedral group of order 4λ . □

Proof of theorem 10. First we show that the p_z generate Γ_Ω . If T is any tile of type s , then there is a path in $\tilde{\Delta}_s^{\text{odd}}$ from T_0 to T . After pushing this down to a path γ in Δ_s^{odd} , lemma 12 says that p_γ carries T_0 to T . The linear span of each of T_0 and T is V^s , so p_γ centralizes s . Also, by the inductive construction, p_β carries the chamber in $\frac{1}{2}V$ containing T_0 to the chamber in $\frac{1}{2}V$ containing T . So p_γ preserves $\frac{1}{2}V$, hence lies in Γ_Ω . Since the $z \in Z$ generate $\pi_1(\Delta_s^{\text{odd}}, s) = \Gamma_\Omega$, the $p_{z \in Z}$ generate Γ_Ω .

If $m(t_n, u) = 2\lambda$, then

$$r_{\gamma,u} = p_\gamma \cdot u(t_n u)^{\lambda-1} \cdot p_\gamma^{-1} = (p_\gamma u p_\gamma^{-1})(s \cdot p_\gamma u p_\gamma^{-1})^{\lambda-1},$$

which in the notation of lemma 13 is $u'(su')^{\lambda-1}$. So the $r_{\delta_t,u}$ are the reflections across the arrows of one tile of each type. Their conjugates by Γ_Ω are therefore the reflections across the facets of C_Ω . This proves $W_\Omega : \Gamma_\Omega = \langle p_z, r_{\delta_t,u} \rangle$. The final claim follows from theorem 2. □

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