The Leech Lattice and Complex Hyperbolic Reflections

Daniel Allcock
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allcock@math.utah.edu
web page: *http://www.math.utah.edu/∼allcock*
Department of Mathematics
University of Utah
Salt Lake City, UT 84112.

Abstract.
We construct a natural sequence of finite-covolume reflection groups acting on the complex hyperbolic spaces $\mathbb{C}H^{13}$, $\mathbb{C}H^{9}$ and $\mathbb{C}H^{5}$, and show that the 9-dimensional example coincides with the largest of the groups of Mostow [10]. These reflection groups arise as automorphism groups of certain Lorentzian lattices over the Eisenstein integers, and we obtain our largest example by using the complex Leech lattice. We also construct finite-covolume reflection groups on the quaternionic hyperbolic spaces $\mathbb{H}H^{7}$, $\mathbb{H}H^{5}$ and $\mathbb{H}H^{3}$, again using the Leech lattice, and apply results of Borcherds [3] to obtain automorphic forms for our groups.

1 Introduction

In [2] we constructed a large number of reflection groups acting on complex and quaternionic hyperbolic spaces. For the most part, the groups appeared as symmetry groups of selfdual Lorentzian lattices (*i.e.,* those of signature $(1, n)$) over the rings of Eisenstein, Gaussian and Hurwitz integers. We were surprised to find examples in even higher dimensions by using certain non-selfdual lattices. These are the subject of the paper. Although similar in spirit to [2], we do not need results from there.

The basic idea, as in [2], is that one can build reflection groups acting on $\mathbb{C}H^{n+1}$ or $\mathbb{H}H^{n+1}$ from suitable negative-definite n-dimensional lattices; we will use lattices over the Eisenstein integers $\mathcal{E} = \mathbb{Z}[\omega]$, where ω is a primitive cube root of unity, and over the quaternionic ring $\mathcal{H}$ of Hurwitz integers. If Λ is such a lattice then one constructs a Lorentzian lattice L by taking the direct sum of Λ and a suitable 2-dimensional Lorentzian lattice H. In [2] we took H to be given by the inner product matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, whereas here we use

$$\begin{pmatrix} 0 & \bar\theta \\ \theta & 0 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} 0 & 1+i \\ 1-i & 0 \end{pmatrix} ,$$

where $\theta \in \mathcal{E}$ is a square root of -3. If the covering radius of Λ is small enough, then $\operatorname{Aut} L$ is generated up to finite index by reflections.

The small change in H has surprising effects, allowing us to use certain lattices Λ that did not work before. In particular, there are Eisenstein and Hurwitz versions of the Leech lattice, but their covering radii are too large for the technology of [2] to apply. Our main results are that the automorphism groups of the Eisenstein lattices

$$\Lambda_4^{\mathcal{E}} \oplus H, \quad \Lambda_4^{\mathcal{E}} \oplus \Lambda_4^{\mathcal{E}} \oplus H \quad \text{and} \quad \Lambda_{12}^{\mathcal{E}} \oplus H$$

are generated (up to finite index, in the last case) by reflections. Here, $\Lambda_4^{\mathcal{E}}$ and $\Lambda_{12}^{\mathcal{E}}$ are the Eisenstein versions of the E_8 and Leech lattices, and $H = \left(\begin{smallmatrix} 0 & \bar\theta \\ \theta & 0 \end{smallmatrix}\right)$. Alternately, these are the unique Lorentzian $\mathcal{E}$-lattices L of dimensions 6, 10 and 14 satisfying $L = \theta L'$, where L' is the dual of

L. In particular, $\Lambda_{12}^{\mathcal{E}} \oplus H$ may also be described as $(\Lambda_4^{\mathcal{E}})^3 \oplus H$, so that our three groups form a natural sequence. Furthermore, we identify our 10-dimensional $\mathcal{E}$-lattice with a lattice described by Thurston [12] in terms of combinatorial triangulations of the sphere S^2, and we identify the automorphism group of the lattice with the largest of the groups described by Mostow [10]. (See also Deligne and Mostow [8] and Thurston [12].)

We also carry out a quaternionic version of this program, proving that the automorphism groups of the lattices

$$\Lambda_2^{\mathcal{H}} \oplus H, \quad \Lambda_2^{\mathcal{H}} \oplus \Lambda_2^{\mathcal{H}} \oplus H \quad \text{and} \quad \Lambda_6^{\mathcal{H}} \oplus H$$

are virtually generated by reflections. Here, $\Lambda_2^{\mathcal{H}}$ and $\Lambda_6^{\mathcal{H}}$ are the Hurwitz versions of the E_8 and Leech lattices, and $H = \left(\begin{smallmatrix} 0 & 1+i \\ 1-i & 0 \end{smallmatrix}\right)$. The lattices above may be described as the unique Lorentzian $\mathcal{H}$-lattices L in dimensions 4, 6 and 8 satisfying $L = L' \cdot (1+i)$. By the isomorphism $\Lambda_6^{\mathcal{H}} \oplus H \cong (\Lambda_2^{\mathcal{H}})^3 \oplus H$, we see that these three lattices also form a natural sequence.

These two sequences are obviously similar to each other, and they are also very similar to a sequence of real hyperbolic reflection groups, namely those of the lattices

$$E_8 \oplus H, \quad E_8 \oplus E_8 \oplus H \quad \text{and} \quad \Lambda_{24} \oplus H\ ,$$

where Λ_{24} is the Leech lattice and $H = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. These lattices are the unique even selfdual Lorentzian $\mathbb{Z}$-lattices in dimensions 10, 18 and 26. As before, $\Lambda_{24} \oplus H \cong E_8^3 \oplus H$, so that these lattices form a natural sequence. The subgroup of $\operatorname{Aut} L$ generated by reflections has index 2, 4 or ∞ in $\operatorname{Aut} L$ for these three lattices L. In the last case, although the reflection group has infinite index, it is still "almost all" of $\operatorname{Aut} L$, in the sense that the quotient by it is virtually $\mathbb{Z}^{24}$. See [4] for details.

Finally, we construct automorphic forms on complex and quaternionic hyperbolic space that are automorphic with respect to the symmetry groups of the various lattices listed above. This construction relies on the work of Borcherds [3], and allows us to explicitly describe the zeros and singularities of these forms. In the complex case, the forms are holomorphic with simple zeros along the mirrors of the reflection groups, and in the quaternionic case the forms are real analytic except at their singularities, which lie along the mirrors of the groups. The analogy with the real hyperbolic case extends to the construction of these automorphic forms.

2. Background on lattices

We write ω for a primitive cube root of unity and define the Eisenstein integers $\mathcal{E}$ to be $\mathbb{Z}[\omega]$. It will be useful to denote $\omega - \bar{\omega}$, a square root of -3, by θ. For $x \in \mathcal{E}$ we define $\operatorname{Im} x = (x - \bar{x})/2$. A lattice over $\mathcal{E}$ is a free (right) module over $\mathcal{E}$ equipped with a Hermitian form $\langle | \rangle$ taking values in $\mathcal{E}$. If x lies in a lattice then the norm x^2 of x is defined to be $\langle x | x \rangle$; some authors call this the squared norm. We will consider only nondegenerate forms—in fact only those of signature $(0, n)$ or $(1, n)$. The former are negative-definite and the latter are called Lorentzian. We use the convention that $\langle | \rangle$ be linear in its second argument rather than in its first. This is because it is convenient to regard lattice elements as column vectors, with linear transformations acting on the left. This means that in the quaternionic case (section 6), scalars should act on the right and this makes it natural for $\langle | \rangle$ to be linear in its second argument. Until section 6 we will allow ourselves to write the action of scalars on either side.

The dual L' of a lattice L is the set of all $x \in L \otimes \mathbb{Q}$ satisfying $\langle x | \lambda \rangle \in \mathcal{E}$ for all $\lambda \in L$. For the most part we will consider lattices L satsifying $L \subseteq \theta L'$, so that all inner products of lattice vectors are divisible by θ. We call $r \in L$ a root of L if $r^2 = -3$ and $r \in \theta L'$. Then the complex ξ-reflection in r,

$$w \mapsto w - r(1 - \xi) \frac{\langle r | w \rangle}{r^2}\ , \tag{2.1}$$

is an isometry of L if ξ is a cube root of unity. If $\xi \neq 1$ then the reflection has order 3 and is called a triflection. We define $R(L)$ to be the subgroup of $\operatorname{Aut} L$ generated by the triflections in the roots of L.

An element x of a lattice L is called primitive if it is not of the form $y\alpha$ with $y \in L$ and α a non-unit of $\mathcal{E}$. A sublattice M of L is called primitive if $L \cap (M \otimes \mathbb{Q}) = M$. A deep hole of a positive-definite lattice L is a point of $L \otimes \mathbb{R}$ at maximal distance from L; this distance is called the covering radius of L. It is awkward to directly adapt this definition for negative-definite lattices, so we make the following definition instead. The covering norm N of a negative-definite lattice L is the largest (negative) number N such that for each $\ell \in L \otimes \mathbb{R}$ there is a lattice vector λ satisfying $(\ell - \lambda)^2 \in [N, 0]$. The points ℓ of $L \otimes \mathbb{R}$ for which there are no $\lambda \in L$ satisfying $(\ell - \lambda)^2 > N$ are called the deep holes of L.

If L is a Lorentzian $\mathcal{E}$-lattice then $\operatorname{Aut} L$ acts on the set of positive-definite subspaces of $L \otimes \mathbb{R}$; this space is called complex hyperbolic space and denoted $\mathbb{C}H^n$, where $n = \dim L - 1$.

3. The stabilizer of a null vector

We define H to be the 2-dimensional lattice over $\mathcal{E}$ with inner product matrix $H = \left(\begin{smallmatrix} 0 & \bar\theta \\ \theta & 0 \end{smallmatrix}\right)$. That is, elements of H are column vectors with entries in $\mathcal{E}$, so that the inner product $\langle v|w\rangle$ of lattice vectors v and w is v^*Hw, where v^* denotes the conjugate-transpose of v. Let Λ be any $\mathcal{E}$-lattice and let $L = \Lambda \oplus H$. We write elements of L in the form $(\lambda; \mu, \nu)$ with $\lambda \in \Lambda$ and $\mu, \nu \in \mathcal{E}$. We give the vector $(0; 0, 1)$ the name ρ and define the height $\operatorname{ht} v$ of $v \in L \otimes \mathbb{R}$ to be the inner product $\langle \rho | v\rangle$. For $v = (\lambda; \mu, \nu)$, the height of v is just $\theta\mu$. If Λ is negative-definite then since H is Lorentzian, L is also Lorentzian.

Let $\lambda \in \Lambda \otimes \mathbb{R}$ and suppose $z \in \operatorname{Im} \mathbb{C}$. Then the map

$$T_{\lambda;\,z} : \begin{array}{l} (\ell; 0, 0) \mapsto (\ell; 0, \theta^{-1}\langle \lambda|\ell\rangle) \\ (0; 1, 0) \mapsto (\lambda; 1, \bar\theta^{-1}(z - \lambda^2/2)) \\ (0; 0, 1) \mapsto (0; 0, 1) \end{array}$$

is an isometry of $L \otimes \mathbb{R}$ that preserves ρ. Every isometry preserving ρ and acting trivially on $\rho^\perp/\langle\rho\rangle$ has this form, where $\langle\rho\rangle$ denotes the complex span of ρ. We call the maps $T_{\lambda;\,z}$ the translations by λ. Regarding elements of $L \otimes \mathbb{R}$ as column vectors, $T_{\lambda;\,z}$ acts by left multiplication by the matrix

$$T_{\lambda;\,z} = \begin{pmatrix} I & \lambda & 0 \\ 0 & 1 & 0 \\ \theta^{-1}\lambda^* & \bar\theta^{-1}(z - \lambda^2/2) & 1 \end{pmatrix} \tag{3.1}$$

where I represents the identity map of Λ and λ^* is the linear map $\ell \mapsto \langle\lambda|\ell\rangle$ on Λ induced by λ.

One verifies the relations

$$T_{\lambda;\,z} T_{\lambda';\,z'} = T_{\lambda+\lambda';\,z+z'+\operatorname{Im}\langle\lambda'|\lambda\rangle} \tag{3.2}$$

$$T_{\lambda;z}^{-1} = T_{-\lambda;\,-z} \tag{3.3}$$

$$T_{\lambda;z}^{-1} T_{\lambda';z'}^{-1} T_{\lambda;\,z} T_{\lambda';\,z'} = T_{0;\,2\operatorname{Im}\langle\lambda'|\lambda\rangle}\ . \tag{3.4}$$

These show that the translations form a group whose center and commutator subgroup coincide and consist of those $T_{\lambda;\,z}$ with $\lambda = 0$. We call these the central translations. Furthermore, if U is an isometry of Λ then we may regard it as acting on L, fixing H pointwise. Then we have

$$U T_{\lambda;\,z} U^{-1} = T_{U\lambda;\,z}\ . \tag{3.5}$$

In order for a translation to preserve L, it is obviously necessary that $\lambda \in \Lambda$. Furthermore, by considering the lower-left entry of (3.1), we see that the inner product $\langle \lambda | \ell \rangle$ must be divisible by θ, for all $\ell \in \Lambda$. Finally, by considering the bottom middle entry of (3.1) we see that $z \in \operatorname{Im} \mathbb{C}$ must be chosen so that $\bar{\theta}^{-1}(z - \lambda^2/2) \in \mathcal{E}$. We can choose such a z just if $\bar{\theta}^{-1}\lambda^2/2$ is an integer multiple of $\theta/2$. This condition holds because by our previous condition we know that $\lambda^2 = \langle \lambda | \lambda \rangle$ is real and divisible by θ, hence divisible by $3 = -\theta^2$. We conclude that $\operatorname{Aut} L$ contains a translation by λ just if $\lambda \in \Lambda \cap \theta\Lambda'$.

Theorem 3.1. *Let Λ be an $\mathcal{E}$-lattice and let $L = \Lambda \oplus H$. Then for each $\lambda \in \Lambda \cap \theta\Lambda'$, $R(L)$ contains a translation by λ.*

Proof: We begin by working in H, so we suppress all but the last pair of vectors' coordinates. Let $r_1 = (1, \bar{\omega})$ and $r_2 = (1, -\omega)$, each a root of L, and let R_1 and R_2 be the ω-reflections in r_1 and r_2, respectively. Computation reveals that matrices for the action of the R_i on H are

$$R_1 = \begin{pmatrix} 0 & \bar{\omega} \\ -\bar{\omega} & -\bar{\omega} \end{pmatrix} \qquad \text{and} \qquad R_2 = \begin{pmatrix} -\bar{\omega} & \bar{\omega} \\ -\bar{\omega} & 0 \end{pmatrix} .$$

Then the action of $R_1 R_2$ on H is

$$R_1 R_2 = \begin{pmatrix} -\omega & 0 \\ 2\omega & -\omega \end{pmatrix} = -\omega T_{0;\, 2\theta} \,. \tag{3.6}$$

Of course, $R_1 R_2$ acts trivially on $H^\perp = \Lambda$. We write the action of $R_1 R_2$ on L as $-\omega K T_{0;\, 2\theta}$, where $-\omega$ indicates scalar multiplication by $-\omega$ on all of L and K indicates scalar multiplication by $-\bar{\omega}$ on Λ.

Suppose $\lambda \in \Lambda \cap \theta\Lambda'$, so that $\operatorname{Aut} L$ contains a translation $T_{\lambda;\, z}$ by λ. We will show that $R(L)$ contains a translation by $-\omega\lambda$. The commutator

$$T_{\lambda;\, z} R_1 R_2 T_{\lambda; z}^{-1} (R_1 R_2)^{-1}$$

is a member of $R(L)$, since $R(L)$ is a normal subgroup of $\operatorname{Aut} L$. By using our expression for $R_1 R_2$, together with (3.2)–(3.5), we discover that

$$\begin{aligned} T_{\lambda;\, z} R_1 R_2 T_{\lambda; z}^{-1} (R_1 R_2)^{-1} &= T_{\lambda;\, z} (-\omega K) T_{0;\, 2\theta} T_{\lambda; z}^{-1} T_{0;\, -2\theta} (-K^{-1} \bar{\omega}) \\ &= T_{\lambda;\, z} K T_{\lambda; z}^{-1} K^{-1} \\ &= T_{\lambda;\, z} T_{K\lambda; z}^{-1} \\ &= T_{\lambda;\, z} T_{-\bar{\omega}\lambda; z}^{-1} \\ &= T_{\lambda;\, z} T_{\bar{\omega}\lambda;\, -z} \\ &= T_{\lambda + \bar{\omega}\lambda;\, \operatorname{Im}\langle \bar{\omega}\lambda | \lambda \rangle} \\ &= T_{-\omega\lambda;\, \operatorname{Im}(\omega\lambda^2)} \\ &= T_{-\omega\lambda;\, \theta\lambda^2/2} \,. \end{aligned}$$

□

Theorem 3.2. *Let Λ be an $\mathcal{E}$-lattice and $L = \Lambda \oplus H$. If $\Lambda = \theta\Lambda'$ then $R(L)$ contains all the translations of L and all the scalars of H.*

Proof: By theorem 3.1, $R(L)$ contains a translation by λ, for each $\lambda \in \Lambda$. Thus, to show that $R(L)$ contains all the translations, it suffices to show that it contains the central translations. By the condition $\Lambda = \theta\Lambda'$, we may find $\lambda, \lambda' \in \Lambda$ such that $\langle\lambda'|\lambda\rangle = -\theta\omega$, so that $\mathrm{Im}\langle\lambda'|\lambda\rangle = \theta/2$. Choosing translations $T_{\lambda;\,z}$ and $T_{\lambda';\,z'}$ that lie in $R(L)$, we see by (3.4) that $R(L)$ contains

$$T^{-1}_{\lambda;z}T^{-1}_{\lambda';z'}T_{\lambda;\,z}T_{\lambda';\,z'} = T_{0;\,\theta}\ .$$

It remains to show that $R(L)$ contains the scalars of H. In the proof of theorem 3.1 we saw that $R(L)$ contains the product $-\omega T_{0;\,2\theta}$. Since $R(L)$ contains the central translations, it must also contain the scalar $-\omega$, which generates the group of all scalars of H. □

A consequence of these results is that if Λ is negative-definite then the stabilizer of ρ in $R(L)$ has finite index in the stabilizer in $\mathrm{Aut}\,L$. This holds even if there are no reflections of L stabilizing ρ—we will see an example of this in the next section. There are no analogues of theorems 3.1 and 3.2 for $\mathbb{Z}$-lattices. In particular, if L is a Lorentzian $\mathbb{Z}$-lattice, ρ is any null vector of L, and G is a subgroup of $\mathrm{Aut}\,L$ generated by reflections, then the stabilizer of ρ in G is also generated by reflections. The most notatble example of this phenomenon occurs with $L = \Lambda_{24} \oplus \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\rho = (0; 0, 1)$, where Λ_{24} is the Leech lattice. From the fact that Λ_{24} admits no reflections one can deduce that no reflections of L stabilize ρ, so the stabilizer of ρ in the reflection group of L is trivial, so the reflection group has infinite index in $\mathrm{Aut}\,L$. (However, the stabilizer of ρ is essentially the entire difference between the two groups. See [4] for details.)

The following theorem is not needed elsewhere in the paper but casts light on a conjecture made in [2].

Theorem 3.3. *Let Λ be a negative definite $\mathcal{E}$-lattice and let $L = \Lambda \oplus \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then the subgroup of $\mathrm{Aut}\,L$ generated by reflections in norm -1 vectors contains a finite-index subgroup of the stabilizer of $\rho = (0; 0, 1)$.*

Proof sketch: Take R_1 and R_2 to be the $(-\omega)$-reflections in the norm -1 vectors $r_1 = (1, \bar{\omega})$ and $r_2 = (1, \omega)$ of the second summand of L. Then argue as in the proofs of theorems 3.1 and 3.2. (Note that the matrices for translations of L are given by formula (4.2) of [2] rather than by (3.1), because we have replaced H by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.) □

Conjecture 9.1 of [2] asserts that the lattice $\mathcal{E}^{1,n+1}$ is reflective just if each negative-definite selfdual $\mathcal{E}$-lattice of dimension n is virtually spanned by its elements of norms -1 and -2. (A lattice L is called reflective if the subgoup of $\mathrm{Aut}\,L$ generated by reflections has finite index in $\mathrm{Aut}\,L$.) The purpose of the last condition was to assure that the reflection group of L contains a finite-index subgroup of the stabilizer of each null vector. Theorem 3.3 shows that this conclusion holds with no hypotheses at all! In light of this, the natural reformulation of the conjecture is simply that $\mathcal{E}^{1,n+1}$ is reflective for all n. This sounds too good to be true, considering Vinberg's result [13] that there are no reflective Lorentzian $\mathbb{Z}$-lattices in high dimensions.

4. A reflection group on $\mathbb{C}H^{13}$

The Leech lattice Λ_{24} is the unique even selfdual positive-definite lattice of dimension 24 with minimal norm 4. Its automorphism group contains an element of order 3 that fixes no vectors except the origin. We may regard this transformation as defining an action of $\mathcal{E}$ on Λ_{24}, making Λ_{24} into a 12-dimensional $\mathcal{E}$-lattice. Wilson [15] describes this lattice, the complex Leech lattice, in detail. By $\Lambda^{\mathcal{E}}_{12}$ (in this section, Λ) we denote his lattice with all inner products multiplied by -3. Then Λ is a negative-definite $\mathcal{E}$-lattice whose real form is the standard Leech lattice with inner products muliplied by $-3/2$. By the main result of [5], the covering norm of Λ is -3. The minimal vectors of Λ (those of norm -6) span the lattice, and Wilson's computation (p. 158) of the inner products of minimal vectors with each other shows that $\Lambda \subseteq \theta\Lambda'$. By [9, p. 248] the determinant of Λ is θ^{12}, so $\Lambda = \theta\Lambda'$. The automorphism group of Λ is the universal central extension $6{\cdot}Suz$ of Suzuki's sporadic simple group. For further background on Λ we refer the reader to [15].

We set $L = \Lambda \oplus H$ and note that because Λ has no roots, $\rho = (0; 0, 1)$ is orthogonal to no roots of L.

Theorem 4.1. *The group $R(L)$ acts transitively on the primitive null vectors of L that are orthogonal to no roots of L. The obvious subgroup $6{\cdot}Suz$ of $\operatorname{Aut} L$ maps onto the quotient $\operatorname{Aut} L/R(L)$. In particular, the index of $R(L)$ in $\operatorname{Aut} L$ is finite.*

Proof: Suppose that $v \in L$ has norm 0, is not a multiple of ρ (so that it has nonzero height), and is orthogonal to no roots of L. We claim that by applying a reflection of L we may reduce the height of v. By this we mean that we may carry v to a vector v' with $|\operatorname{ht} v'| < |\operatorname{ht} v|$.

To prove this it suffices to reduce the height of any scalar multiple w of v, say the one of the form $w = (\ell; 1, \alpha - \theta\ell^2/6)$, with $\ell \in \Lambda \otimes \mathbb{R}$ and $\alpha \in \mathbb{R}$. (The imaginary part of the last coordinate is determined by the condition $w^2 = 0$.) We may choose $\lambda \in \Lambda$ such that $(\ell - \lambda)^2 \in [-3, 0]$. Because $\lambda^2 \equiv 0(3)$ we may choose $\beta \in \frac{1}{2}\mathbb{Z}$ such that $\beta + \theta(-3 - \lambda^2)/6$ lies in $\mathcal{E}$, and then $r' = (\lambda; 1, \beta + \theta(-3 - \lambda^2)/6)$ is a root of L. For any $n \in \mathbb{Z}$ the vector

$$r = r' + (0; 0, n) = (\lambda; 1, n + \beta + \theta(-3 - \lambda^2)/6)$$

is also a root of L, and we will show that for suitable n, some triflection in r reduces the height of w. We will need to know $\langle r|w\rangle$:

$$\begin{aligned}
\langle r|w\rangle &= \langle\lambda|\ell\rangle + \bar{\theta}\alpha - \frac{\ell^2}{2} + n\theta + \beta\theta + \frac{-3 - \lambda^2}{2} \\
&= -\frac{3}{2} - \frac{1}{2}\left(\ell^2 - 2\langle\lambda|\ell\rangle + \lambda^2\right) + n\theta + \bar{\theta}\alpha + \beta\theta \\
&= -\frac{3}{2} - \frac{1}{2}(\ell - \lambda)^2 + \operatorname{Im}\langle\lambda|\ell\rangle + n\theta + \bar{\theta}\alpha + \beta\theta \\
&= -3\left[\frac{1}{2} + \frac{(\ell - \lambda)^2}{6} + n\theta^{-1} + \alpha\bar{\theta}^{-1} + \beta\theta^{-1} - \frac{\operatorname{Im}\langle\lambda|\ell\rangle}{3}\right] \\
&= -3[a + b]\ ,
\end{aligned}$$

where $a = \frac{1}{2} + \frac{1}{6}(\ell - \lambda)^2$ is the real part of the term in brackets and b is the imaginary part. By construction of λ we have $a \in [0, \frac{1}{2}]$. By choice of n we may suppose $b \in [\theta^{-1}/2, \bar{\theta}^{-1}/2] = [\bar{\theta}/6, \theta/6]$.

Let w' be the image of w under ξ-reflection in r, where ξ is a cube root of unity that we will choose later. Using (2.1) and the computation above, we can compute the height of w':

$$\begin{aligned}
\langle\rho|w'\rangle &= \langle\rho|w\rangle - \langle\rho|r\rangle(1 - \xi)\frac{-3(a + b)}{-3} \\
&= \theta - \theta(1 - \xi)(a + b)\ .
\end{aligned}$$

For the absolute value of this to be smaller than that of $\langle\rho|w\rangle = \theta$, we need to choose ξ so that $1 - (1 - \xi)(a + b)$ has norm less than 1. If $b \in [0, \theta/6]$ (resp. $b \in [\bar{\theta}/6, 0]$) then taking $\xi = \omega$ (resp. $\xi = \bar{\omega}$) accomplishes this unless $a = b = 0$. We have assumed that v is orthogonal to no roots, so w cannot be orthogonal to r, which rules out the case $a = b = 0$. This proves the claim.

We have shown that we may reduce the height of v with a reflection. Repeating this as necessary, we may suppose that v has height 0, so that it is a multiple of ρ. By theorem 3.2, $R(L)$ contains the scalars of H, so $R(L)$ acts transitively on the unit scalar multiples of ρ. This proves the first claim of the theorem. Also by theorem 3.2, $R(L)$ contains the translations. Since these act transitively on the the null vectors of height θ, we conclude that a complete set of coset representatives for $R(L)$ in $\operatorname{Aut} L$ may be taken from the simultaneous stabilizer of $(0; 0, 1)$ and $(0; 1, 0)$. Since this stabilizer is just $\operatorname{Aut}\Lambda = 6{\cdot}Suz$, the proof is complete. $\square$

5. Further examples

We can construct other reflection groups by following the arguments of section 4 with other lattices in place of $\Lambda_{12}^{\mathcal{E}}$. In this section we construct two further examples, one acting on $\mathbb{C}H^9$ and the other on $\mathbb{C}H^5$. The first of these has already been discovered in a different guise—it is the largest of the groups discovered by Mostow [10] (see also Deligne and Mostow [8], Mostow [11] and Thurston [12]). We conjecture below that the 5-dimensional group coincides with a certain one of the other groups found in [10].

The E_8 root lattice may be regarded as an Eisenstein lattice, and is most easily described in terms of the tetracode $\mathcal{C}_4$. This is the 2-dimensional subspace of $\mathbb{F}_3^4$ consisting of the scalar multiples of the images of the vectors $(0,1,1,1)$ and $(1,0,1,-1)$ under cyclic permuation of the last three coordinates. We define $\Lambda_4^{\mathcal{E}}$ to be the set of vectors in $\mathcal{E}^4$ whose coordinates are elements of $\mathcal{C}_4$ when reduced modulo θ. Here we regard $\mathcal{E}^4$ as being equipped with the standard (negative-definite) inner product

$$\langle (x_1,\ldots,x_4)|(y_1,\ldots,y_4)\rangle = -\sum \bar{x}_i y_i \ .$$

This is the smallest scale at which $\Lambda_4^{\mathcal{E}}$ is an integral $\mathcal{E}$-lattice, and we have $\Lambda_4^{\mathcal{E}} = \theta(\Lambda_4^{\mathcal{E}})'$.

The largest of Mostow's groups is his $\Gamma(\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3})$, for which we will simply write Γ. The only fact we will need about Γ is that it is an infinite image of the spherical braid group B on 12 strands, with the standard generators $b_1,\ldots,b_{11}$ mapping to triflections $S_1,\ldots,S_{11}$. We will first construct an explicit set of reflections that generate a copy of Γ.

It is easy to check that the S_i are all distinct—if any two coincide then all coincide. (This follows by repeated use of the fact that if $S_i = S_j$ then any S_k which braids with one of them and commute with the other coincides with both of them.) Since the b_i are conjugate in B, the S_i are either all ω-reflections or all $\bar{\omega}$-reflections. Which of these is the case is irrelevant to the question of determining Γ, so we suppose that they are all ω-reflections. (In any case, the two possibilities are exchanged by complex conjugation, or equally well by the automorphism of B exchanging each b_i with its inverse.) We may choose vectors s_i of norm -3 in $\mathbb{C}^{1,9}$ such that each S_i is the ω-reflection in s_i. In order for S_i and S_{i+1} to braid, matrix computations show that we must have $|\langle s_i|s_j\rangle| = \sqrt{3}$. If S_i and S_j commute then (since $S_i \neq S_j$) we must have $s_i \perp s_j$. Therefore we may successively replace each of $s_2,\ldots,s_{11}$ by a scalar multiple of itself norm so that $\langle s_i|s_j\rangle = 0$ unless $|i-j| = 1$, when $\langle s_i|s_j\rangle = \theta$ or $\bar{\theta}$ according to whether or not j is closer than i to 6. That is, we may take the vectors s_i to be those elements of $\Lambda_4^{\mathcal{E}} \oplus \Lambda_4^{\mathcal{E}} \oplus H$ given in figure 5.1. We chose our strange convention about $\langle s_i|s_j\rangle$ when $|i-j| = 1$ so that the "diagram automorphism" $s_i \mapsto s_{12-i}$ would be an isometry.

We write K_1 and K_2 for the first and second summands of $L = \Lambda_4^{\mathcal{E}} \oplus \Lambda_4^{\mathcal{E}} \oplus H$, and set $L_0 = K_2 \oplus H$. We define Γ_0 to be the subgroup of Γ generated by $S_6,\ldots,S_{11}$.

Theorem 5.1. *Γ (resp. Γ_0) coincides with $\operatorname{Aut} L$ (resp. $\operatorname{Aut} L_0$) and acts transitively on the primitive null vectors of L (resp. L_0).*

Proof: All the s_i are roots of L and $s_6,\ldots,s_{11}$ lie in L_0, so $\Gamma \subseteq \operatorname{Aut} L$ and $\Gamma_0 \subseteq \operatorname{Aut} L_0$. By theorem 5.2 below, $S_8,\ldots,S_{11}$ generate $\operatorname{Aut} K_2$, so Γ_0 contains $\operatorname{Aut} K_2$. Similarly, $S_1,\ldots,S_4$ generate $\operatorname{Aut} K_1$. Since the automorphism $b_i \mapsto b_{12-i}$ of B is inner, Γ contains an element sending each s_i to s_{12-i} (up to a scalar). Such a transformation exchanges K_1 with K_2, so Γ contains $\operatorname{Aut}(K_1 \oplus K_2)$.

By theorem 5.2, $\operatorname{Aut} K_2$ is transitive on the roots of K_2, so Γ_0 contains the ω-reflection S' in $\lambda = (0,0,0,\bar{\theta}) \in K_2$. Computation reveals that $S'S_7^{-1}$ is the translation $T_{\lambda\bar{\omega};\,\theta/2}$ of L_0. The conjugates of this translation by $\operatorname{Aut} K_2$ are the translations $T_{x;\,\theta/2}$ where x varies over the roots of K_2. Since the roots of K_2 span K_2, Γ_0 contains a translation by λ for each $\lambda \in K_2$. By taking

$$
\begin{array}{l|cccccccccc}
s_1 & (\;\bar\theta, & , & , & ; & , & , & , & ; & , &) \\
s_2 & (-1, & -1, & 1, & ; & , & , & , & ; & , &) \\
s_3 & (& \theta, & , & ; & , & , & , & ; & , &) \\
s_4 & (& 1, & 1, & 1; & , & , & , & ; & , &) \\
s_5 & (& , & , & \bar\theta; & , & , & , & ; & , & 1) \\
s_6 & (& , & , & ; & , & , & , & ; & 1, & \bar\omega) \\
s_7 & (& , & , & ; & , & , & , & \bar\theta; & , & 1) \\
s_8 & (& , & , & ; & , & 1, & 1, & 1; & , &) \\
s_9 & (& , & , & ; & , & \theta, & , & ; & , &) \\
s_{10} & (& , & , & ; & -1, & -1, & 1, & ; & , &) \\
s_{11} & (& , & , & ; & \bar\theta, & , & , & ; & , &)
\end{array}
$$

Figure 5.1. For each $i = 1, \ldots, 11$, S_i is the ω-reflection in s_i. The reflections S_i and S_j braid (resp. commute) if the corresponding nodes of the diagram are joined (resp. not joined). All of the s_i lie in $\Lambda_4^{\mathcal{E}} \oplus \Lambda_4^{\mathcal{E}} \oplus H$, with coordinates given in an obvious notation; each blank entry represents the value 0. The diagram automorphism $s_i \mapsto s_{12-i}$ corresponds to the bodily exchange of the first two blocks of coordinates.

commutators of these translations, as in the proof of theorem 3.2, we see that Γ_0 contains the central translations, hence all the translations of L_0. By applying this result together with its image under the diagram automorphism, we see that Γ contains all the translations of L. Together with the previous paragraph, this implies that Γ (resp. Γ_0) contains the full stabilizer of ρ in $\operatorname{Aut} L$ (resp. $\operatorname{Aut} L_0$).

Observe that the roots r_1 and r_2 appearing in the proof of theorem 3.1 are the vectors s_6 and $T_{0;\,-\theta}(s_6)$. We conclude from (3.6) that Γ_0 (and hence Γ) contains the scalars of H, and therefore acts transitively on the unit scalar multiples of ρ.

Now we study the orbits under Γ of the primitive null vectors of L. Suppose $v \in L$ is such a vector, is not a multiple of ρ, and has minimal height in its Γ-orbit. Then for some multiple w of v we have $w = (\ell_1; \ell_2; 1, \alpha - \theta\ell^2/6)$, where $\alpha \in \mathbb{R}$ and $\ell = (\ell_1; \ell_2)$ with $\ell_1, \ell_2 \in \Lambda_4^{\mathcal{E}} \otimes \mathbb{R}$.

The argument for theorem 4.1 proves the following statement: if $\lambda = (\lambda_1; \lambda_2) \in K_1 \oplus K_2$ satisfies $(\ell - \lambda)^2 \in [-3, 0]$ then there is a root r of L of the form

$$
r = \left(\lambda_1;\, \lambda_2;\, 1, \beta + \frac{\theta(-3 - \lambda^2)}{6}\right) \tag{5.1}
$$

(with $\beta \in \mathbb{R}$) such that either a triflection in r reduces the height of w or else $(\ell - \lambda)^2 = -3$ and $r \perp w$. Since v has minimal height in its Γ-orbit, the latter possibility must apply. By [6, p. 121], the covering norm of $\Lambda_4^{\mathcal{E}}$ is $-3/2$ and its deep holes are the halves of the vectors with norm divisible by 6. Therefore the condition $(\ell - \lambda)^2 = -3$ for all $\lambda \in K_1 \oplus K_2$ nearest ℓ implies that after a translation we may suppose that each of ℓ_1 and ℓ_2 is one-half of a norm -6 vector of $\Lambda_4^{\mathcal{E}}$. Furthermore, by the transitivity (theorem 5.2) of $\operatorname{Aut} \Lambda_4^{\mathcal{E}}$ on such vectors, we may suppose that each is one-half of a specific one, say $(\theta, \theta, 0, 0)$.

The argument above, applied to L_0 and Γ_0 instead of L and Γ, rules out the existence of w, so any null vector of L_0 is Γ_0-equivalent to ρ. Since the stabilizers of ρ in Γ_0 and $\operatorname{Aut} L_0$ coincide, we have proven all our claims regarding Γ_0.

We now return to studying Γ and L. We have deduced that any null vector of L not Γ-equivalent to a multiple of ρ is instead Γ-equivalent to a multiple of

$$w = \left(\frac{\theta}{2}, \frac{\theta}{2}, 0, 0;\ \frac{\theta}{2}, \frac{\theta}{2}, 0, 0;\ 1, \alpha + \frac{\theta}{2}\right) \tag{5.2}$$

for some $\alpha \in \mathbb{R}$. (We have used the equality $\ell^2 = -3$.) Furthermore, we know that for each $\lambda \in K_1 \oplus K_2$ with $(\ell - \lambda)^2 = -3$, there is a root r of the form (5.1) orthogonal to w. Taking $\lambda = 0$, so that $r = (0;\ 0;\ 1, n + \bar{\omega})$ for some $n \in \mathbb{Z}$, the condition $r \perp w$ requires $\alpha \in \frac{1}{2} + \mathbb{Z}$. Taking $\lambda = (\omega, -\bar{\omega}, 0, 1;\ 0, 0, 0, 0)$, so that $r = (\lambda; 1, n)$ for some $n \in \mathbb{Z}$, the condition $r \perp w$ requires $\alpha \in \mathbb{Z}$. We conclude that w cannot exist, so that any primitive null vector of L is Γ-equivalent to ρ. The proof is completed by the equality of the stabilizers of ρ in Γ and $\operatorname{Aut} L$. $\square$

One can show that $\left(\Lambda_4^{\mathcal{E}}\right)^3 \oplus H \cong \Lambda_{12}^{\mathcal{E}} \oplus H$, so that the three Lorentzian lattices we have considered in this section and the previous one form a natural sequence. We have shown that Γ appears on the lists of [11] and [12], and we conjecture that Γ_0 also appears, as $\Gamma(\frac{5\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$. The quotients of complex hyperbolic space by the various discrete groups of [8], [10] and [11] are constructed in terms of the moduli spaces of point-sets in $\mathbb{C}P^1$; it would be very interesting to find a moduli-space interpretation for our quotient of $\mathbb{C}H^{13}$.

Thurston [12] states that there is an $\mathcal{E}$-lattice L in $\mathbb{C}^{1,9}$ invariant under Γ whose points of positive norm (up to Γ-equivalence) may be identified with the triangulations of the sphere S^2 in which each vertex meets at most 6 triangles. Furthermore, the number of triangles in a triangulation equals the norm of a corresponding element of L. One can show that up to scaling there is only one $\mathcal{E}$-lattice invariant under Γ. Since the fewest number of triangles possible is 2, Thurston's lattice must be a copy of $\Lambda_4^{\mathcal{E}} \oplus \Lambda_4^{\mathcal{E}} \oplus H$ with all norms multiplied by $2/3$.

The following theorem (used in the proof of theorem 5.1) is not new, but I have been unable to find a reference for it, especially for the transitivity on norm -6 vectors.

Theorem 5.2. $\operatorname{Aut} \Lambda_4^{\mathcal{E}}$ *is generated by the triflections* $S_8, \ldots, S_{11}$ *and acts transitively on lattice vectors of norms* -3 *and* -6.

Proof: $\Lambda_4^{\mathcal{E}}/\theta\Lambda_4^{\mathcal{E}}$ may be regarded as a vector space V over $\mathbb{F}_3 = \mathcal{E}/\theta\mathcal{E}$, and the reduction modulo θ of the Hermitian form yields a symplectic form on V. No more than 3 roots may be congruent to each other modulo θ, because a tetrahedron of edge-length 3 cannot be inscribed in a sphere of radius $\sqrt{3}$. Therefore the 240 roots represent all $3^4 - 1 = 80$ nontrivial elements of V. A triflection in a root acts on V by a symplectic transvection in the corresponding element of V, and these generate the full symplecic group $\mathrm{Sp}_4(3)$. Since this group acts transitively on the 80 classes we obtain the transitivity on roots. Any element of $\operatorname{Aut} \Lambda_4^{\mathcal{E}}$ acting trivially on V must carry each root to a multiple of itself by a cube root of 1. Since $\Lambda_4^{\mathcal{E}}$ is not a direct sum of lower-dimensional lattices, any such transformation must be a scalar. This shows that $|\operatorname{Aut} \Lambda_4^{\mathcal{E}}| = |\mathrm{Sp}_4(3)| \cdot 3 = 155{,}520$. By the last entry of the table on p. 133 of [7], the triflections $S_8, \ldots, S_{11}$ generate a group of this order, so this group is $\operatorname{Aut} \Lambda_4^{\mathcal{E}}$.

None of the 2160 vectors of norm -6 lie in $\theta\Lambda_4^{\mathcal{E}}$, so each is congruent modulo θ to a root. By transitivity on roots, there are $2160/80 = 27$ vectors of norm -6 in each class, and to prove transitivity it suffices to prove transitivity on one such class of 27, say those congruent to $(\theta, 0, 0, 0)$. The 27 vectors are the images of $(0, 0, \theta, \bar{\theta})$ under the group generated by cyclic permutation of the last three coordinates and multiplication of the 4th coordinate by ω. The transitivity follows from this description. $\square$

6. The quaternionic case

All of the above constructions have analogues when the ring of Eisenstein integers is replaced by the quaternionic ring $\mathcal{H}$ of Hurwitz integers. In particular, the Leech and E_8 lattices may be regarded as $\mathcal{H}$-lattices, and we can use them to construct finite-covolume reflection groups acting on the quaternionic hyperbolic spaces $\mathbb{H}H^7$, $\mathbb{H}H^5$ and $\mathbb{H}H^3$. The Hurwitz integers are the elements of the skew field $\mathbb{H}$ of quaternions that have the form $(a+bi+cj+dk)/2$ with a,b,c and d being integers that are all congruent modulo 2. The left ideal $\mathfrak{p}$ generated by $p=1-i$ (or equally well by any other norm 2 element of $\mathcal{H}$) is two-sided.

The constructions may be described loosely as "those of sections 2–5 with θ replaced by p." In particular, if L is an $\mathcal{H}$-lattice, $r\in L$ has norm -2, and all inner products of r with lattice vectors lie in $\mathfrak{p}$, then the ξ-reflections (2.1) in r are isometries of L if $\xi\in\{\pm1,\pm i,\pm j,\pm k\}$. In this case we call r a root of L; we set $R(L)$ to be the group generated by these reflections in roots of L. We define H to be the 2-dimensional $\mathcal{H}$-lattice with inner product matrix $\left(\begin{smallmatrix}0&\bar{p}\\p&0\end{smallmatrix}\right)$. If Λ is an $\mathcal{H}$-lattice then as before we write elements of $L=\Lambda\oplus H$ in the form $(\lambda;\mu,\nu)$ and define $\rho=(0;0,1)$ and $\operatorname{ht}v=\langle\rho|v\rangle$. If $\lambda\in\Lambda\otimes\mathbb{R}$ and $z\in\operatorname{Im}\mathbb{H}$ then the translation

$$
\begin{aligned}
&(\ell;0,0)\mapsto(\ell;0,-\bar{p}^{-1}\langle\lambda|\ell\rangle)\\
T_{\lambda;\,z}:\ &(0;1,0)\mapsto(\lambda;1,\bar{p}^{-1}(z-\lambda^2/2))\\
&(0;0,1)\mapsto(0;0,1)
\end{aligned}
$$

is an isometry of $L\otimes\mathbb{R}$ preserving ρ. One may write these transformations in matrix form in a manner similar to (3.1) and check that the relations (3.2)–(3.5) hold. $\operatorname{Aut}L$ contains a translation by λ just if $\lambda\in\Lambda\cap\Lambda'\mathfrak{p}$.

We have an analogue of theorems 3.1 and 3.2, with slightly weaker conclusions:

Theorem 6.1. *Suppose Λ is an $\mathcal{H}$-lattice and $L=\Lambda\oplus H$. If $\operatorname{Aut}L$ contains a translation by $\lambda\in\Lambda$ then $R(L)$ contains a translation by $\lambda(i-1)$. In particular, if Λ is definite then the stabilizer of ρ in $R(L)$ has finite index in the stabilizer in $\operatorname{Aut}L$.*

Proof: This is similar to the proof of theorem 3.1. Let R_1 and R_2 be the i-reflections in the roots $r_1=(1,i)$ and $r_2=(1,-1)$ of H. Computation reveals that $R_1R_2=-iT_{0;\,4i}$, by which we mean the product of left scalar multiplication by $-i$ on H and the translation $T_{0;\,4i}$. For any $T_{\lambda;\,z}\in\operatorname{Aut}L$,

$$R_1R_2T_{\lambda;\,z}R_2^{-1}R_1^{-1}=T_{\lambda i;\,-izi}\,.$$

(One must verify this by multiplying the matrices together rather than following the proof of theorem 3.1, because there may be no concept of left-multiplication by scalars on Λ.) Then $R(L)$ contains

$$T_{\lambda;z}^{-1}R_1R_2T_{\lambda;\,z}R_2^{-1}R_1^{-1}=T_{\lambda(i-1);\,-z-izi+i\lambda^2}\,.$$

This proves the first claim; the second follows by taking commutators to obtain central translations, as in the proof of theorem 3.2. □

The Leech lattice admits an action of the binary tetrahedral group (the multiplicative group of the 24 units of $\mathcal{H}$) such that no nontrivial group element fixes any nontrivial lattice vector. This action allows us to regard Λ_{24} as a 6-dimensional $\mathcal{H}$-lattice. Wilson [14] describes this lattice in detail; the only facts we require about it are that suitably scaled it has no roots, all inner products are divisible by p, and that its minimal vectors have norm -4. In Wilson's description [14], the minimal norm of the lattice is 8 and all inner products are divisible by $2p$, as may be verified using the basis he gives on p. 453. By $\Lambda_6^{\mathcal{H}}$ we mean his lattice with all inner products divided by -2. There is also a quaternionic form of the E_8 lattice, denoted $\Lambda_2^{\mathcal{H}}$ and defined to be the set of pairs (x_1,x_2) in $\mathcal{H}^2$ satisfying $x_1\equiv x_2\,(\mathfrak{p})$, under the standard (negative-definite) Hermitian form $\langle(x_1,x_2)|(y_1,y_2)\rangle=-\bar{x}_1y_1-\bar{x}_2y_2$.

Theorem 6.2. *Let Λ be one of the lattices $\Lambda_6^{\mathcal{H}}$, $\Lambda_2^{\mathcal{H}} \oplus \Lambda_2^{\mathcal{H}}$ and $\Lambda_2^{\mathcal{H}}$, and let $L = \Lambda \oplus H$. Then $R(L)$ has finite index in $\operatorname{Aut} L$. If $\Lambda = \Lambda_2^{\mathcal{H}}$ (resp. $\Lambda_6^{\mathcal{H}}$) then $R(L)$ acts transitively on the (1-dimensional) primitive null lattices of L (resp. those orthogonal to no roots of L).*

Proof: This is very similar to the proof of theorem 4.1. By theorem 6.1, $R(L)$ contains a subgroup of finite index in the stabilizer of ρ in $\operatorname{Aut} L$. In light of this, it suffices to prove the claims regarding the transitivity on null lattices, and also that if $\Lambda = \Lambda_2^{\mathcal{H}} \oplus \Lambda_2^{\mathcal{H}}$ then $R(L)$ acts with only finitely many orbits of primitive null lattices.

Take $\Lambda = \Lambda_6^{\mathcal{H}}$. Suppose v is a null vector of L not proportional to ρ and let w be its multiple of the form $w = (\ell; 1, \bar{p}^{-1}(\alpha - \ell^2/2))$ where $\ell \in \Lambda \otimes \mathbb{R}$ and $\alpha \in \operatorname{Im} \mathbb{H}$. Because the covering norm of Λ is -2, we may choose $\lambda \in \Lambda$ with $(\ell - \lambda)^2 \in [-2, 0]$. Then we may find a root r' of L of the form $r' = (\lambda; 1, \bar{p}^{-1}(\beta - 1 - \lambda^2/2))$ with $\beta \in \operatorname{Im} \mathbb{H}$. For any $n \in \operatorname{Im} \mathfrak{p}$, to be chosen later,

$$r = r' + (0; 0, n) = \big(\lambda; 1, \bar{p}^{-1}(n + \beta - 1 - \lambda^2/2)\big) \tag{6.1}$$

is also a root of L. Computation proves that

$$\begin{aligned} \langle r|w\rangle &= -2\left[\frac{1}{2} + \frac{1}{4}(\ell - \lambda)^2 - \frac{\operatorname{Im}\langle \lambda|\ell\rangle + \alpha + \bar{\beta}}{2} + \frac{n}{2}\right] \\ &= -2[a + b + n/2] \end{aligned}$$

where $a = \frac{1}{2} + \frac{1}{4}(\ell - \lambda)^2 \in [0, 1/2]$ and b is purely imaginary. Denoting by w' the image of w under the i-reflection in r, we have

$$\operatorname{ht} w' = p[1 - (1 - i)(a + b + n/2)] \; .$$

In order to have reduced the height of w we need to choose $n \in \operatorname{Im} \mathfrak{p}$ such that the term in brackets has norm less than 1.

That is, we desire

$$|1 - i|^2 \left| \frac{1}{1 - i} - (a + b + n/2) \right|^2 < 1 \; ,$$

which is equivalent to

$$\left| \frac{1 + i}{2} - (a + b + n/2) \right|^2 < \frac{1}{2}$$

and to

$$\left| \frac{1}{2} - a \right|^2 + \left| \frac{i}{2} - b - n/2 \right|^2 < \frac{1}{2} \; . \tag{6.2}$$

By our condition on a, the first term on the left lies in $[0, 1/4]$ and equals $1/4$ just if $a = 0$. Since $\operatorname{Im} \mathfrak{p}$ is a copy of the D_3 root lattice (it is spanned by $i \pm j$ and $j \pm k$), p. 112 of [6] shows that the covering radius of $\frac{1}{2} \operatorname{Im} \mathfrak{p}$ is $1/2$. Therefore by choice of $n/2$ we may suppose that the second term is bounded by $1/4$, with equality just if b is a deep hole of $i/2 + \frac{1}{2} \operatorname{Im} \mathfrak{p}$. All deep holes of $\operatorname{Im} \mathfrak{p}$ are equivalent by translations by elements of $\mathfrak{p}$, and 0 is such a deep hole. Therefore if b is a deep hole of $i/2 + \frac{1}{2} \operatorname{Im} \mathfrak{p}$ we may take $n/2 = -b \in \frac{1}{2} \operatorname{Im} \mathfrak{p}$. We have shown that for each $\lambda \in \Lambda$ with $(\lambda - \ell)^2 \in [-2, 0]$, there is a root r of L of the form (6.1) such that either the i-reflection in r reduces the height of w, or else $a = 0$ and $b = -n/2$, in which case $(\lambda - \ell)^2 = -2$ and $r \perp w$.

For $\Lambda = \Lambda_6^{\mathcal{H}}$, this shows that if v is a null vector of L orthogonal to no roots, then by repeated reflections its height may be reduced to 0, so that v is equivalent under $R(L)$ to a multiple of ρ.

For $\Lambda = \Lambda_2^{\mathcal{H}}$, the same argument proves that the height of *any* null vector may be reduced to 0, since the covering norm of $\Lambda_2^{\mathcal{H}}$ is -1. For $\Lambda = \Lambda_2^{\mathcal{H}} \oplus \Lambda_2^{\mathcal{H}}$, the argument shows that if the height of w cannot be reduced by an element of $R(L)$ then ℓ is a deep hole of Λ. Modulo translations there are only finitely many possibilites for ℓ. For each such ℓ the condition that w be orthogonal to a root of the form (6.1) for each λ nearest ℓ imposes an integrality condition on α; modulo central translations there are only finitely many possibilities for α. This proves that $R(L)$ acts with only finitely many orbits on the primitive null sublattices of L. $\square$

7. Automorphic forms

Borcherds [3] has developed machinery for constructing automorphic forms on the symmetric spaces for the orthogonal groups $O(m,n)$. That is, for an even $\mathbb{Z}$-lattice M of signature (m,n) and a suitable modular form F on the usual upper half-plane, he constructs an $(\operatorname{Aut} M)$-invariant function on the Grassmannian $G(M)$ of maximal-dimensional positive-definite subspaces of $M \otimes \mathbb{R}$. Furthermore, he explicitly describes the singularities of this function in terms of the lattice M and the Fourier coefficients of F. In this section we use his results to obtain automorphic forms on complex and quaternionic hyperbolic spaces for the symmetry groups of the various Lorentzian lattices we have considered.

We begin with the complex case. We define the real form of an $\mathcal{E}$-lattice to be the underlying $\mathbb{Z}$-module, equipped with the bilinear form given by the real part of the Hermitian form. The real forms of $\Lambda_{12}^{\mathcal{E}}$, $\Lambda_4^{\mathcal{E}}$ and H, with inner products multiplied by $2/3$, are even unimodular $\mathbb{Z}$-lattices. The evenness follows because all norms of lattice vectors are divisible by 3, and the unimodularity from that of the Leech and E_8 lattices (as $\mathbb{Z}$-lattices) together with a computation of the determinant of the inner product matrix of a $\mathbb{Z}$-basis for the real form of H. Borcherds' machinery simplifies dramatically in the unimodular case, and we obtain the following theorem.

Theorem 7.1. *Suppose $\Lambda = \Lambda_4^{\mathcal{E}}$ (resp. $\Lambda_4^{\mathcal{E}} \oplus \Lambda_4^{\mathcal{E}}$, $\Lambda_{12}^{\mathcal{E}}$) and $L = \Lambda \oplus H$. Then there is a holomorphic automorphic form Ψ_0 on $\mathbb{C}H^5$ (resp. $\mathbb{C}H^9$, $\mathbb{C}H^{13}$) of weight 84 (resp. 44, 4) for a one-dimensional representation of $\operatorname{Aut} L$ taking values among the cube roots of unity. Furthermore, Ψ_0 vanishes exactly on the subspaces orthogonal to roots of L, and these zeros have multiplicity one.*

Proof: Suppose $\Lambda = \Lambda_4^{\mathcal{E}}$. Then the real form $L^{\mathbb{R}}$ of L is a copy of $\mathrm{II}_{2,10}$ (the selfdual even $\mathbb{Z}$-lattice of signature $(2,10)$) with all norms multiplied by $3/2$. Let F be the modular form

$$F(\tau) = E_4^2(\tau)/\Delta(\tau) = \sum_{n \geq -1} c(n) q^n = q^{-1} + 504 + 73764q + \cdots \tag{7.1}$$

of weight -4 for $\mathrm{SL}_2\mathbb{Z}$, where $q = e^{2\pi i \tau}$ and τ lies in the upper half-plane. Then theorem 13.3 of [3] provides a meromorphic automorphic form Ψ on $G(\mathrm{II}_{2,10})$ of weight $c(0)/2 = 252$ for some one-dimensional character of $\operatorname{Aut} \mathrm{II}_{2,10}$. Furthermore, the only zeros and poles of Ψ lie along the divisors orthogonal to those $\lambda \in \mathrm{II}_{2,10}$ with $\lambda^2 < 0$, and are zeros of order

$$\sum_{\substack{x \in \mathbb{R}^+ \\ x\lambda \in \mathrm{II}_{2,10}}} c(x^2 \lambda^2 / 2) \ .$$

That is, Ψ has a simple zero along the divisor orthogonal to each norm -2 vector of $\mathrm{II}_{2,10}$ and no other zeros or poles; this implies that Ψ is holomorphic. By definition, $\mathbb{C}H^5$ is the space of all positive-definite complex lines in $L \otimes \mathbb{R}$, and is obviously a subspace of the Grassmannian $G(\mathrm{II}_{2,10})$. By restricting Ψ to $\mathbb{C}H^5$ we obtain an automorphic form for $\operatorname{Aut} L$. The orthogonal complement of a root of L meets $\mathbb{C}H^5$ in the obvious complex hyperplane. The orthogonal complements of $\lambda\omega$ and $\lambda\bar{\omega}$ meet $\mathbb{C}H^5$ in this same hyperplane, so the zeros of the restriction of Ψ to $\mathbb{C}H^5$ are precisely the hyperplanes orthogonal to roots of L, with multiplicity 3.

If r is a root of L then the real form of the $\mathcal{E}$-span of r , with inner products multiplied by 2/3, is a copy of the A_2 lattice. The three real reflections in the 6 roots of A_2 generate the symmetric group S_3, and the 120° rotations are commutators in S_3. This proves that the triflections in r are commutators in $\operatorname{Aut}\mathrm{II}_{2,10}$. Since the triflections generate $\operatorname{Aut} L$, we find that Ψ transforms according to the trivial character of $\operatorname{Aut} L$. A cube root Ψ_0 of the restriction of Ψ is automorphic with respect to a character of $\operatorname{Aut}(L)$ taking values among the cube roots of unity. Finally, Ψ_0 obviously has simple zeros along each mirror, and weight $252/3 = 84$.

If $\Lambda = \Lambda_4^{\mathcal{E}} \oplus \Lambda_4^{\mathcal{E}}$ or $\Lambda_{12}^{\mathcal{E}}$ then our assertions follow from the argument above, with F replaced by

$$F(\tau) = E_4(\tau)/\Delta(\tau) = q^{-1} + 264 + 8244q + \cdots \qquad \text{or}$$
$$F(\tau) = 1/\Delta(\tau) = q^{-1} + 24 + 324q + \cdots \ ,$$

respectively. In the case $\Lambda = \Lambda_{12}^{\mathcal{E}}$, to prove that Ψ is invariant under $\operatorname{Aut} L$, one must use the fact that $\operatorname{Aut} L$ is generated by triflections together with the perfect group $\operatorname{Aut}\Lambda_{12}^{\mathcal{E}}$. ($\operatorname{Aut}\Lambda_{12}^{\mathcal{E}}$ is perfect because it is the universal central extension of a simple group.) □

Now we consider the quaternionic case. As before, the real forms of $\Lambda_6^{\mathcal{H}}$, $\Lambda_2^{\mathcal{H}}$ and H are even unimodular lattices (this time, no rescaling is required). Borcherds obtained the form Ψ used above by exponentiating another function Φ which has logarithmic singularities along the mirrors. In the quaternionic case, the relevant real lattices are $\mathrm{II}_{4,4n}$, and the analogue of Φ has poles rather than logarithmic singularities.

Theorem 7.2. *Let $\Lambda = \Lambda_2^{\mathcal{H}}$ (resp. $\Lambda_2^{\mathcal{H}} \oplus \Lambda_2^{\mathcal{H}}$, $\Lambda_6^{\mathcal{H}}$) and let $L = \Lambda \oplus H$. Then there is a function Φ on $\mathbb{H}H^3$ (resp. $\mathbb{H}H^5$, $\mathbb{H}H^7$) that is invariant under $\operatorname{Aut} L$ and real-analytic except at its singularities. The set on which Φ is singular is the union of the mirrors orthogonal to roots of L. Along the mirror orthogonal to a root r, the singularity has type*

$$\frac{12}{\pi}\frac{w^2}{|\langle w|r\rangle|^2} \ ,$$

where w is a vector in $L \otimes \mathbb{R}$ representing a point v of hyperbolic space near $r^\perp$.

Proof: Suppose $\Lambda = \Lambda_2^{\mathcal{H}}$, so that $L^{\mathbb{R}}$ is isometric to $\mathrm{II}_{4,12}$. We take $F(\tau) = E_4^2(\tau)/\Delta(\tau)$, as in (7.1). Then the function Φ on $G(L^{\mathbb{R}})$ defined in [3, §6] (with the polynomial p set to 1) is real analytic except at its singularities. If v_0 is a maximal-dimensional positive-definite subspace of $L^{\mathbb{R}} \otimes \mathbb{R}$ then for $v \in G(L^{\mathbb{R}})$ near v_0, Φ has a singularity of type

$$\sum_{\substack{\lambda \in L^{\mathbb{R}} \cap v_0^\perp \\ \lambda \neq 0}} c(\lambda^2/2)\left(2\pi\lambda_{v^+}^2\right)^{-1} \ ,$$

where λ_{v^+} is the projection of λ to the subspace v. Any λ appearing in the sum satisfies $\lambda^2 < 0$, so the only λ for which the term in the sum is nonzero have norm -2. We now restrict to $\mathbb{H}H^3$, with v_0 a generic point of $r^\perp$ (*i.e.,* orthogonal to no elements of L except multiples of r), represented by a vector w_0 of $L \otimes \mathbb{R}$ of positive norm. The sum extends over the 24 roots λ (the unit scalar multiples of r) orthogonal to w_0. For v a point of $\mathbb{H}H^3$ near v_0, represented by a vector w, λ_{v^+} is the projection λ to $w\mathbb{H}$ with respect to the Hermitian form, and $\lambda_{v+}^2 = |\langle w|\lambda\rangle|^2/w^2$. Therefore the singularity near v_0 has type

$$\sum_u c(-1)\frac{w^2}{2\pi|\langle w|ru\rangle|^2}$$

where the sum extends over the units u of $\mathcal{H}$. Since $c(-1) = 1$ and the 24 terms in the sum all coincide, we have proven our claims in the case $\Lambda = \Lambda_2^{\mathcal{H}}$.

If $\Lambda = \Lambda_2^{\mathcal{H}} \oplus \Lambda_2^{\mathcal{H}}$ or $\Lambda_6^{\mathcal{H}}$ then one can repeat the proof above, with $F = E_4/\Delta$ or $F = 1/\Delta$ respectively. □

In the introduction we mentioned the analogy between the two series of groups we have studied here and the series of real hyperbolic reflection groups discussed by Conway [4], namely the groups of the lattices

$$\mathrm{II}_{1,9}, \quad \mathrm{II}_{1,17} \quad \text{and} \quad \mathrm{II}_{1,25}\ .$$

The analogy extends even to the construction of automorphic forms. In each of the real, complex and quaternionic cases, there are automorphic forms constructed from the modular forms E_4^2/Δ, E_4/Δ and $1/\Delta$. (See examples 10.7 and 12.2 of [3] for the real case.) Borcherds has observed (private communication and [3, §12]) that there seems to be a very close (maybe one-to-one?) correspondence between nice reflection groups on real, complex and quaternionic hyperbolic spaces and automorphic forms with singularities exactly along reflection hyperplanes. The examples of this paper strengthen this correspondence.

In an earlier paper [1] we constructed two reflection groups acting on the octave hyperbolic plane $\mathbb{O}H^2$, and it is natural to wonder whether the analogies extend to this case. That they might is suggested by the fact that one of the groups may be regarded as the automorphism group of the "lattice" $\Lambda_1^{\mathcal{K}} \oplus \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, where $\Lambda_1^{\mathcal{K}}$ is the E_8 lattice regarded as a "1-dimensional lattice" over a suitable discrete subring $\mathcal{K}$ of the skew field $\mathbb{O}$ of octaves. This is obviously analogous to the first term in each of the series of lattices over $\mathbb{Z}$, $\mathcal{E}$ and $\mathcal{H}$.

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