CONGRUENCE SUBGROUPS AND ENRIQUES SURFACE AUTOMORPHISMS

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Abstract. We give conceptual proofs of some results on the automorphism group of an Enriques surface X, for which only computational proofs have been available. Namely, there is an obvious upper bound on the image of $\operatorname{Aut} X$ in the isometry group of X's numerical lattice, and we establish a lower bound for the image that is quite close to this upper bound. These results apply over any algebraically closed field, provided that X lacks nodal curves, or that all its nodal curves are (numerically) congruent to each other mod 2. In this generality these results were originally proven by Looijenga and Cossec–Dolgachev, developing earlier work of Coble.

1. Introduction

Our goal in this paper is to give conceptual proofs of some known computer-based results on the group of automorphisms of an Enriques surface X. These results are valid over any algebraically closed field. Of course, $\operatorname{Aut} X$ acts on $\operatorname{Pic} X$, hence on the quotient Λ of $\operatorname{Pic} X$ by its torsion subgroup $\mathbb{Z}/2$. This quotient Λ is called the numerical lattice, and is a copy of the famous E_{10} lattice. One can describe it as $E_8 \oplus U$ where we take E_8 to be negative definite and $U = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

The main object of interest in this paper is the image Γ of $\operatorname{Aut} X$ in $\mathrm{O}(\Lambda)$. This is "most" of $\operatorname{Aut} X$, because the kernel of $\operatorname{Aut} X \to \Gamma$ is finite, and in fact very tightly constrained [7, §7.2]. All of our arguments concern Λ and various Coxeter groups acting on it. For the underlying algebraic geometry we refer to [6], [7] and [9].

Because Λ has signature $(1, 9)$, the positive norm vectors in $\Lambda \otimes \mathbb{R}$ fall into two components. Just one of these contains ample classes; we call it the future cone and write $\mathrm{O}^{\uparrow}(\Lambda)$ for the subgroup of $\mathrm{O}(\Lambda)$ preserving it. Vinberg showed (theorem 2.2 below) that $\mathrm{O}^{\uparrow}(\Lambda)$ is the

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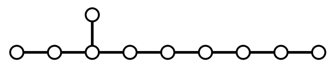
Coxeter group W_{237} with diagram

(1.1) 

Besides preserving Λ, the main constraint on Γ is that it must preserve the ample cone, hence its closure, the numerically effective cone $\mathrm{nef}(X)$. The nef cone is described in terms of X's nodal curves (i.e., smooth rational curves), which have self-intersection -2 by the adjunction formula. If X has nodal curves then $\mathrm{nef}(X)$ consists of the vectors in $\Lambda \otimes \mathbb{R}$ having nonnegative inner product with all of them. In the special case that X lacks nodal curves, $\mathrm{nef}(X)$ is the closure of the future cone.

The remaining constraint on Γ concerns the $\mathbb{F}_2$ vector space $V := \Lambda/2\Lambda$. Dividing lattice vectors' norms by 2 and then reducing modulo 2 defines on V an $\mathbb{F}_2$ quadratic form of plus type. "Plus type" means that V has totally isotropic spaces of largest possible dimension, in this case 5. Although we won't use this property, it does explain the presence of some superscripts $+$. We write $\mathrm{O}(V)$ for the isometry group of this quadratic form. We will use ATLAS notation for group structures and finite groups throughout the paper; see [1], especially §5.2. In this notation, $\mathrm{O}(V)$ has structure $\mathrm{O}_{10}^{+}(2) : 2$. (Caution: the ATLAS uses "O" for the simple composition factor of an orthogonal group—in this case an index 2 subgroup. Some authors write $\mathrm{O}_{10}^{+}(2)$ for $\mathrm{O}(V)$ itself.)

Theorem 1.1 (The unnodal case)**.** *Suppose an Enriques surface X has no nodal curves. Then Γ contains the level two congruence subgroup $W_{237}(2)$, meaning the kernel of the natural map $\mathrm{O}^{\uparrow}(\Lambda) \to \mathrm{O}(V)$.*

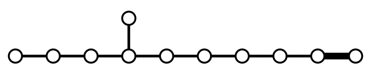
So Γ must be one of the finitely many groups between $W_{237}(2)$ and W_{237}. Because $\mathrm{O}^{\uparrow}(\Lambda) \to \mathrm{O}(V)$ is a surjection, the possibilities correspond to subgroups of $\mathrm{O}_{10}^{+}(2) : 2$. Different X can lead to different Γ, so one cannot say much more without specifying X more closely. If X is unnodal then $\mathrm{Aut}\, X$ acts faithfully on Λ, by [7, Thm. 7.3.6]. So one can identify Γ with $\mathrm{Aut}\, X$.

In characteristic 0 one can describe Γ in terms of the period of the K3 surface which covers X. In this way one can show that for a generic Enriques surface without nodal curves, Γ is exactly $W_{237}(2)$; see [2]. The positive characteristic analogue of this seems to be open.

Given a nodal curve, regarded as an element of $\mathrm{Pic}\, X$, the corresponding *nodal root* means its image under $\mathrm{Pic}\, X \to \Lambda$. It is called a root because it has norm -2 and so the reflection in it is an isometry of Λ. Distinct nodal curves have intersection number ≥ 0, hence different images in Λ. So the nodal curves and nodal roots are in natural bijection.

Given a nodal root, its corresponding *nodal class* means its image in V, always an anisotropic vector. Theorem 1.2 below is the analogue of theorem 1.1 in the "1-nodal case": when X has at least one nodal curve, and all nodal curves represent a single nodal class. We will use lowercase letters with bars to indicate elements of V, whether or not we have in mind particular lifts of them to Λ. By definition of the quadratic form on V, every nodal class $\bar{\nu}$ is anisotropic. So its transvection $\bar{x} \mapsto \bar{x}+(\bar{x}\cdot\bar{\nu})\bar{\nu}$ is an isometry of V. We indicate stabilizers using subscripts, for example $\mathrm{O}^{\uparrow}(\Lambda)_{\bar{\nu}}$ in the next theorem.

Theorem 1.2 (The 1-nodal case)**.** *Suppose an Enriques surface X has a single nodal class $\bar{\nu} \in V$. Then the $\mathrm{O}^{\uparrow}(\Lambda)$-stabilizer $\mathrm{O}^{\uparrow}(\Lambda)_{\bar{\nu}}$ of $\bar{\nu}$ is the Coxeter group*

(1.2) 

Write W_{246} for the subgroup generated by the reflections corresponding to the leftmost 10 *nodes. Then*

1. $\mathrm{nef}(X)$ *is the union of the W_{246}-translates of the fundamental chamber of the Coxeter group* (1.2)*.*
2. Γ *lies in W_{246}, which is the full $\mathrm{O}^{\uparrow}(\Lambda)$-stabilizer of* $\mathrm{nef}(X)$*.*
3. Γ *contains the subgroup $\overline{W}_{246}(2)$ defined as the subgroup of W_{246} that acts trivially on $\bar{\nu}^{\perp} \subseteq V$.*
4. $\overline{W}_{246}(2)$ *acts transitively on the facets of* $\mathrm{nef}(X)$*.*
5. $\mathrm{Aut}\, X$ *acts transitively on the nodal curves of X.*

Remarks. (a) The heavy edge in the diagram indicates parallelism of the corresponding hyperplanes in H^9, or equivalently that the last pair of roots has intersection number 2.

(b) Suppose X is a generic nodal Enriques surface. Then $\mathrm{Aut}\, X$ acts faithfully on Λ, so it can be identified with Γ; see [7, Prop. 7.4.1]. Furthermore, in characteristic $\neq 2, 3, 5, 7$ or 17, Γ coincides with $\overline{W}_{246}(2)$, by [8, Thm. 1].

(c) $\overline{W}_{246}(2)$ is the group called $\overline{W}(2)$ by Cossec and Dolgachev [8]. But, contrary to what the notation might suggest, the kernel of our $W_{246} \to \mathrm{O}(V)$ is not the same as their $W(2)$. This is because they define their congruence subgroups with respect to the Reye lattice rather than the Enriques lattice Λ. The Reye lattice has index 2 in Λ: it is the preimage of $\bar{\nu}^{\perp} \subseteq V$.

Theorems 1.1 and 1.2 are modern forms of results of Coble [4, Thms. (4) and (30)]. But Cossec–Dolgachev [6, p. 162] state that his proofs were incorrect. They credit Looijenga with the first proof of theorem 1.1, never published, and give proofs of both theorems, following Looijenga's ideas. See [6, Thms. 2.10.1 and 2.10.2]. Their proof of the first relied on a lengthy hand computation, and the second required computer assistance.

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2. The case of no nodal curves

A root means a lattice vector of norm -2. In this section our model for Λ is the span of the roots in figure 2.1. Two of them have inner product 1 or 0, according to whether they are joined or not. By the theory of reflection groups [3, V.4], these 10 vectors form a set of simple roots for the group W_{237} generated by their reflections. We will write W_{235} resp. W_{236} for the subgroup generated by the reflections corresponding to the top 8 resp. 9 nodes. Also, we will write Λ_0 for the span of the first 8 roots. This is a copy of the E_8 lattice in the "odd" coordinate system, namely

$$\big\{(x_1,\dots,x_8)\bigm| \text{all } x_i \text{ in } \mathbb{Z} \text{ or all in } \mathbb{Z}+\tfrac{1}{2}, \text{ and } \textstyle\sum x_i \equiv x_8 \bmod 2\big\}$$

from [5, §8.1 of Ch. 4]. Its isometry group is the E_8 Weyl group W_{235}, which has structure $2\cdot \mathrm{O}_8^+(2):2$. Sometimes we will write lattice vectors as $(x;y,z)$ with $x\in\Lambda_0$ and $y,z\in\mathbb{Z}$, and inner product $(x;y,z)\cdot(x';y',z') = x\cdot x' + yz' + y'z$.

Lemma 2.1 (The stabilizer of a null vector)**.** *W_{236} is the full stabilizer $\mathrm{O}(\Lambda)_\rho$ of the null vector $\rho = (0;1,0)$. It has structure $\Lambda_0 : W_{235}$, where Λ_0 indicates the group of "translations"*

$$T_{\lambda\in\Lambda_0} : \begin{aligned} (x;0,0) &\mapsto (x;-\lambda\cdot x,0)\\ (0;1,0) &\mapsto (0;1,0)\\ (0;0,1) &\mapsto (\lambda;-\lambda^2/2,1)\end{aligned} \tag{2.1}$$

Proof. The T_λ are called translations because of how they act on hyperbolic space when ρ is placed at infinity in the upper halfspace model. One checks that they are isometries, that $T_{\lambda+\mu} = T_\lambda T_\mu$, and that $W_{235} = \operatorname{Aut}\Lambda_0$ acts on them in the same way it acts on Λ_0. Next, W_{236} contains the reflection in $\lambda = (\frac{-}{2}\frac{-}{2}\frac{-}{2}\frac{-}{2}\frac{-}{2}\frac{-}{2}\frac{-}{2}\frac{+}{2};00)$, because this is a root of Λ_0. Also, W_{236} contains the reflection in $(\frac{-}{2}\frac{-}{2}\frac{-}{2}\frac{-}{2}\frac{-}{2}\frac{-}{2}\frac{-}{2}\frac{+}{2};10)$,

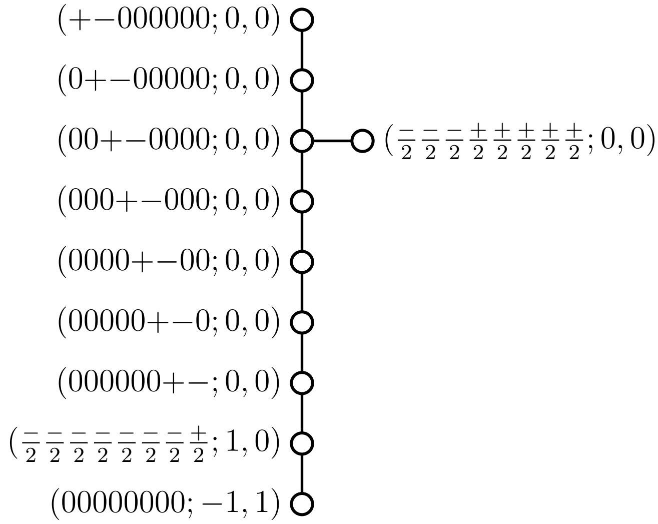

FIGURE 2.1. Simple roots for $W_{237} = \mathrm{O}^{\uparrow}(\Lambda)$, with respect to the norm $(x_1,\ldots,x_8;y,z)^2 = -x_1^2 - \cdots - x_8^2 + 2yz$. We have abbreviated ± 1 to $\pm$ and hidden some commas.

because this root is second from the bottom in figure 2.1. The product of these two reflections is $T_{\pm\lambda}$, the sign depending on the order of the factors. So W_{236} contains the translation by a root of Λ_0. Conjugating by W_{235} shows that W_{236} contains the translations by all the roots of Λ_0. Since Λ_0 is spanned by its roots, W_{236} contains all translations.

The translations act transitively on $\{(x; -x^2/2, 1) \mid x \in \Lambda_0\}$, which is the set of null vectors having inner product 1 with ρ. The simultaneous stabilizer of ρ and $(0;0,1)$ is the orthogonal group of Λ_0, which is $W_{235} \subseteq W_{236}$. Since $\mathrm{O}^{\uparrow}(\Lambda)_\rho$ and its subgroup W_{236} act transitively on the same set, with the same stabilizer, they are the same group. $\square$

The proof of the following theorem of Vinberg illustrates the technique of cusp-counting, which we will use several times. To avoid repetition we take "null vector" to mean a future-directed primitive lattice vector of norm 0. "Cusp counting" means: when a Coxeter group acts on an integer quadratic form of signature $(1, n)$ and has finite volume fundamental chamber in hyperbolic space, then its orbits on null vectors are in bijection with the ideal vertices of the chamber. And these in turn are in bijection with the maximal affine subdiagrams of the Coxeter diagram.

Theorem 2.2 (Vinberg [10])**.** $\mathrm{O}^{\uparrow}(\Lambda) = W_{237}$.

Proof. The image in hyperbolic 9-space of the fundamental chamber has finite volume, with all vertices in H^9 except for one on its boundary. This follows from the general theory of hyperbolic reflection groups: the vertices in H^9 correspond to the rank 9 spherical subdiagrams of figure 2.1, and the last vertex corresponds to the affine subdiagram $\widetilde{E}_8$. It follows that there is only one W_{237}-orbit of null vectors, i.e., W_{237} acts transitively on them. Since W_{237} contains the full $\mathrm{O}^{\uparrow}(\Lambda)$-stabilizer of one of them (lemma 2.1), it is all of $\mathrm{O}^{\uparrow}(\Lambda)$. □

The most important ingredient in the proof of theorem 1.1 is the construction of automorphisms of X, for which we refer to the proof of theorem 3 in [9, §6]. Λ has many direct sum decompositions as a copy of E_8 plus a copy of U. For every such decomposition, the transformation which negates the E_8 summand is called a Bertini involution, and arises from an automorphism of X. (Very briefly: consider the linear system $|2E_1 + 2E_2|$, where E_1 and E_2 are the effective classes corresponding to the null vectors in the U summand. This is a 2-to-1 map onto a 4-nodal quartic del Pezzo surface in $\mathbb{P}^4$, and the Bertini involution is the deck transformation of this covering.) Bertini involutions obviously lie in the level 2 congruence subgroup of $\mathrm{O}^{\uparrow}(\Lambda)$, hence in $W_{237}(2)$. Also, every conjugate of a Bertini involution is again an Bertini involution. So the group they generate is normal in $\mathrm{O}(\Lambda)$.

Proof of theorem 1.1. The proof amounts to showing that the Bertini involutions generate $W_{237}(2)$. We write S ("small") for the group they generate, and think of $\mathrm{O}^{\uparrow}(\Lambda)$ as the "large" group. To understand the relation between small and large, we will introduce a "medium" group M. Its relationships with S and $\mathrm{O}^{\uparrow}(\Lambda)$ are easy to work out. Then the relationship between S and $\mathrm{O}^{\uparrow}(\Lambda)$ will be visible.

We define M as the group generated by S and W_{236}. The central involution B of $W_{235} \subseteq M$ is a Bertini involution. Also, its conjugacy action on $\Lambda_0 \subseteq W_{236}$ is inversion. By the normality of S, M contains

$$T_\lambda B T_\lambda^{-1} \circ B^{-1} = T_\lambda \circ B T_\lambda^{-1} B^{-1} = T_\lambda \circ T_{-\lambda}^{-1} = T_{2\lambda}$$

for all $\lambda \in \Lambda_0$. It follows that M/S is a quotient of $W_{236}/\langle B, \text{all } T_{2\lambda}\rangle = 2^8 : \mathrm{O}_8^+(2) : 2$. On the other hand, it is easy to see that W_{236} acts on V as the full $\mathrm{O}(V)$-stabilizer of $\bar{\rho}$, which has structure $2^8 : \mathrm{O}_8^+(2) : 2$. (Repeat the proof of lemma 2.1, reduced mod 2.) This shows how S is related to M: since it has index $\leq |2^8 : \mathrm{O}_8^+(2) \cdot 2|$ in M, and lies in the kernel of the surjection $M \to \mathrm{O}(V)_{\bar{\rho}} \cong 2^8 : \mathrm{O}_8^+(2) \cdot 2$, S coincides with the kernel. That is, $M \to \mathrm{O}(V)$ induces an isomorphism $M/S \to \mathrm{O}(V)_{\bar{\rho}}$.

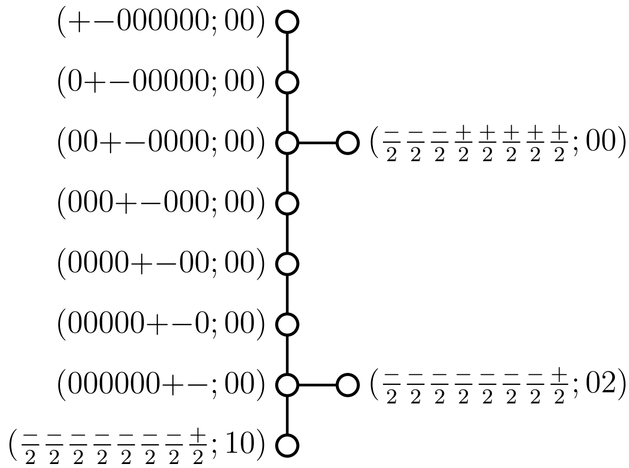

FIGURE 2.2. Simple roots for $M = \mathrm{O}^{\uparrow}(\Lambda)_{\bar{\rho}}$; see the proof of theorem 2.2.

The advantage of working with M rather than S is that it contains the Coxeter group M_0 whose simple roots are shown in figure 2.2. We will see later that in fact M_0 is all of M; for now we just prove $M_0 \subseteq M$. First, M contains W_{236} by definition. To see that M contains the reflection in the last root (the lower right one), note that $r = (\frac{-}{2}\frac{-}{2}\frac{-}{2}\frac{-}{2}\frac{-}{2}\frac{-}{2}\frac{-}{2}\frac{+}{2};0,0)$ is a root of Λ_0, so its reflection lies in W_{235}. Choose an element λ of Λ_0 having inner product -1 with it. Then $T_{2\lambda} \in S$ sends r to $(\frac{-}{2}\frac{-}{2}\frac{-}{2}\frac{-}{2}\frac{-}{2}\frac{-}{2}\frac{-}{2}\frac{+}{2};2,0)$. Now consider the conjugate of $T_{2\lambda}$ by the isometry of Λ which exchanges the last two coordinates. This lies in S by normality, and sends r to $(\frac{-}{2}\frac{-}{2}\frac{-}{2}\frac{-}{2}\frac{-}{2}\frac{-}{2}\frac{-}{2}\frac{+}{2};0,2)$. Therefore M contains the reflection in this root. This finishes the proof that M contains M_0.

Next we claim that M_0 is all of $\mathrm{O}^{\uparrow}(\Lambda)_{\bar{\rho}}$. The fact that $\mathrm{O}^{\uparrow}(\Lambda)_{\bar{\rho}}$ contains M_0 (even M) is obvious. Now observe that M_0's chamber has finite volume, with 3 cusps, corresponding to the $\widetilde{D}_8$ subdiagram and two $\widetilde{E}_8$ subdiagrams. Therefore M_0 has 3 orbits on null vectors. On the other hand, $\mathrm{O}^{\uparrow}(\Lambda)_{\bar{\rho}}$ has at least 3 orbits on null vectors, since it has three orbits on isotropic vectors in V. (Namely: $\bar{\rho}$ itself, the other isotropic vectors, and the isotropic vectors not orthogonal to $\bar{\rho}$.) So $\mathrm{O}^{\uparrow}(\Lambda)_{\bar{\rho}}$ and its subgroup M_0 have the same orbits on null vectors. The stabilizer of ρ in either of them is W_{236}, proving $M_0 = \mathrm{O}^{\uparrow}(\Lambda)_{\bar{\rho}}$. Since $M_0 \subseteq M \subseteq \mathrm{O}^{\uparrow}(\Lambda)_{\bar{\rho}}$, this also shows the equality of M with these groups.

Finally, we have

$$\begin{aligned}[\mathrm{O}^{\uparrow}(\Lambda):S] &= [\mathrm{O}^{\uparrow}(\Lambda):M][M:S]\\ &= \bigl[\mathrm{O}^{\uparrow}(\Lambda):\mathrm{O}^{\uparrow}(\Lambda)_{\bar\rho}\bigr]\,\bigl|\mathrm{O}(V)_{\bar\rho}\bigr|\\ &= \bigl[\mathrm{O}(V):\mathrm{O}(V)_{\bar\rho}\bigr]\,\bigl|\mathrm{O}(V)_{\bar\rho}\bigr|\\ &= \bigl|\mathrm{O}(V)\bigr|\end{aligned}$$

Since S lies in the kernel of the surjection $\mathrm{O}^{\uparrow}(\Lambda)\to\mathrm{O}(V)$, it must be the whole kernel, finishing the proof. □

3. Preparation for the 1-nodal case

This section can be summarized as "the same as section 2 with $E_7\oplus U$ in place of $E_8\oplus U$". To tighten the analogy it is necessary to use the "even" coordinate system for the E_8 lattice in place of the "odd" one we used in the previous section. So now we take the E_8 lattice to consist of the vectors $(x_1,\dots,x_8)$ with even coordinate sum and either all entries in $\mathbb{Z}$ or all in $\mathbb{Z}+\frac{1}{2}$. See [5, §8.1 of Ch. 4]; these coordinates differ from those of section 2 by negating any coordinate. We take Λ to consist of the vectors $(x_1,\dots,x_8;y,z)$ with $(x_1,\dots,x_8)$ in the E_8 lattice and $y,z\in\mathbb{Z}$. The norm is still $-x_1^2-\cdots-x_8^2+2yz$. Mimicking our notation from section 2, we write Λ_0 for the sublattice $\{(x_1,\dots,x_8;0,0)\}$ of Λ.

We write ν for the root $(\frac{+}{2}\frac{+}{2}\frac{+}{2}\frac{+}{2}\frac{+}{2}\frac{+}{2}\frac{+}{2}\frac{+}{2};00)$ of Λ_0. It stands for "nodal root", although for this section it is just a root. Its orthogonal complement in Λ_0 is a copy of the E_7 root lattice, and its full orthogonal complement $\nu^{\perp}$ in Λ is $E_7\oplus U$. It is easy to see that $\nu^{\perp}$ is spanned by the roots in figure 3.1, and that their inner products are indicated in the usual way by the edges of the diagram. In particular they form a set of simple roots for the Coxeter group W_{245} generated by their reflections. We also write W_{244} for the subgroup generated by the reflections in the top 8 roots, and regard both these groups as acting on all of Λ, not just $\nu^{\perp}=E_7\oplus U$. The next two results are proven the same way lemma 2.1 and theorem 2.2 were.

Lemma 3.1. *W_{244} is the full stabilizer of the null vector $\rho=(0;1,0)$ in $\mathrm{O}(\Lambda)_\nu$.* □

Theorem 3.2 (Vinberg)**.** *W_{245} is all of $\mathrm{O}^{\uparrow}(\Lambda)_\nu$.* □

Bertini involutions: from the presence of an E_8 diagram in figure 3.1 we see that $\nu^{\perp}$ has sublattices isomorphic to E_8. Every E_8 sublattice is unimodular, hence a direct summand of Λ, so the involution that negates the E_8 summand is an isometry. Since this summand was

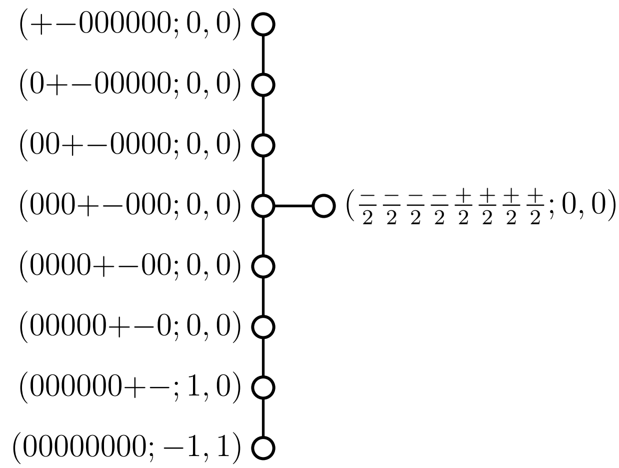

FIGURE 3.1. Simple roots for $\mathrm{O}^{\uparrow}(\Lambda)_{\nu}$, where ν is the root $(\frac{\pm}{2}\frac{\pm}{2}\frac{\pm}{2}\frac{\pm}{2}\frac{\pm}{2}\frac{\pm}{2}\frac{\pm}{2}\frac{\pm}{2};00)$ of Λ; see theorem 3.2.

chosen in $\nu^{\perp}$, we obtain an element of $\mathrm{O}^{\uparrow}(\Lambda)_{\nu}$. These are called Bertini involutions, and act trivially on V.

Kantor involutions: by construction, $\nu^{\perp}$ has direct sum decompositions $E_7 \oplus U$. For any such decomposition, the central involution in $W(E_7)$ acts by negation on the E_7 summand and trivially on the U summand. These are called Kantor involutions. Every one acts on V by the transvection in $\bar{\nu}$. (Proof: the complement in Λ of the U summand is a copy of E_8 containing $\langle\nu\rangle \oplus E_7$. The Kantor involution is the product of the negation map of this E_8, which acts trivially on V, with the reflection in ν.)

Theorem 3.3. *The Kantor and Bertini involutions generate the subgroup* $\mathrm{O}^{\uparrow}(\Lambda)_{\nu,\bar{\nu}^{\perp}}$ *of* $\mathrm{O}^{\uparrow}(\Lambda)$ *that fixes* ν *and acts trivially on* $\bar{\nu}^{\perp} \subseteq V$.

Proof. We reuse our strategy from theorem 1.1. That is, we write S for the subgroup of $\mathrm{O}^{\uparrow}(\Lambda)_{\nu}$ generated by the Kantor and Bertini involutions, and think of it as "small". We think of $\mathrm{O}^{\uparrow}(\Lambda)_{\nu}$ as "large". Obviously S is normal in $\mathrm{O}(\Lambda)_{\nu}$. To relate these groups we define the "medium" group M to be generated by S and W_{244}.

Recall that W_{244} has structure $E_7 : W(E_7) = E_7 : (2 \times \mathrm{O}_7(2))$ where the initial E_7 indicates the root lattice regarded as a group. The central involution in $2 \times \mathrm{O}_7(2)$ is a Kantor involution. Mimicking the proof of theorem 1.1 shows that $M/(S \cap M)$ is a quotient of $2^7 : \mathrm{O}_7(2)$. Continuing the mimicry, the image of M in $\mathrm{O}(V)$ has structure $2^7 : (2 \times \mathrm{O}_7(2))$, which is the simultaneous stabilizer $\mathrm{O}(V)_{\bar{\nu},\bar{\rho}}$. (Note: 2^7 and $2 \times \mathrm{O}_7(2)$ are subgroups of 2^8 and $\mathrm{O}_8^+(2) : 2$ from the proof of

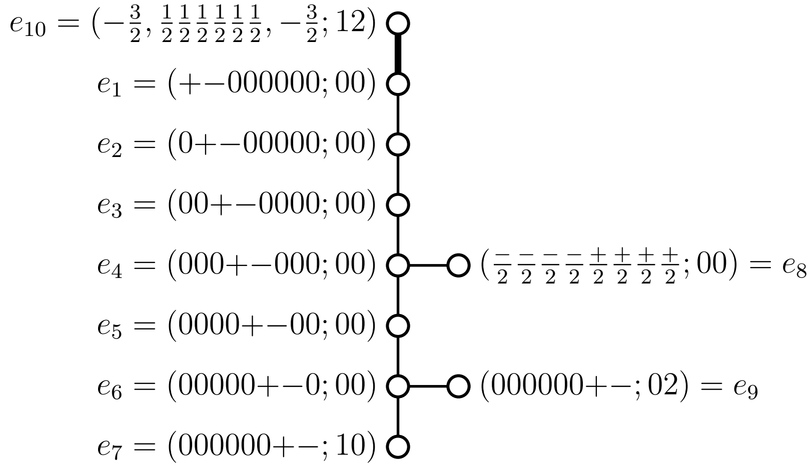

FIGURE 3.2. Simple roots for $M = \mathrm{O}^{\uparrow}(\Lambda)_{\nu,\bar{\rho}}$ where $\rho = (0; 1, 0)$ and $\nu = (\frac{+}{2}\frac{+}{2}\frac{+}{2}\frac{+}{2}\frac{+}{2}\frac{+}{2}\frac{+}{2}\frac{+}{2}; 00)$; see the proof of theorem 3.3.

theorem 1.1. The 2^7 is the subgroup of $\mathrm{O}(V)_{\bar{\nu}}$ that fixes $\bar{\rho}$ and acts trivially on $\bar{\rho}^{\perp}/\langle\bar{\rho}\rangle$, and $2 \times \mathrm{O}_7(2)$ acts faithfully on $\bar{\rho}^{\perp}/\langle\bar{\rho}\rangle$.) Every Kantor involution acts trivially on $\bar{\nu}^{\perp}$, and the image of M in $\mathrm{O}(\bar{\nu}^{\perp}) \cong \mathrm{O}_9(2)$ has structure $2^7 : \mathrm{O}_7(2)$. It follows that S is the kernel of the action of M on $\bar{\nu}^{\perp} \subseteq V$. So we may identify M/S with the stabilizer of $\bar{\rho}$ in $\mathrm{O}(\bar{\nu}^{\perp})$.

Next we claim that M contains the Coxeter group M_0 with simple roots pictured in figure 3.2. First, $e_1, \ldots, e_8$ are the simple roots of W_{244}, whose reflections lie in M by definition. The proof that M contains the reflection in e_9 is exactly the same as in the proof of theorem 1.1. (Only the Kantor involutions are needed.) For e_{10}, observe that it and $e_2, \ldots, e_8$ span a copy of the lattice $A_1 \oplus E_7$. Furthermore, $(e_{10}+e_5+e_7+e_8)/2$ lies in Λ, so the saturation of this $A_1 \oplus E_7$ is a copy of E_8. The reflection in e_{10} is equal to the Bertini involution of this E_8, times the central involution of the copy of $W(E_7) \subseteq W_{244}$ generated by the reflections in $e_2, \ldots, e_8$. Therefore M contains this reflection.

The same argument as in the proof of theorem 1.1 shows that $M_0 = M = \mathrm{O}^{\uparrow}(\Lambda)_{\nu,\bar{\rho}}$. (This time the affine diagrams are $\widetilde{D}_6\widetilde{A}_1$ and two $\widetilde{E}_7$'s.) The final step of the proof is also conceptually the same as before. First,

$$\begin{aligned}
[\mathrm{O}^{\uparrow}(\Lambda)_{\nu} : S] &= [\mathrm{O}^{\uparrow}(\Lambda)_{\nu} : M][M : S] \\
&= [\mathrm{O}^{\uparrow}(\Lambda)_{\nu} : \mathrm{O}^{\uparrow}(\Lambda)_{\nu,\bar{\rho}}]\big|\mathrm{O}(\bar{\nu}^{\perp})_{\bar{\rho}}\big| \\
&= [\mathrm{O}(V)_{\bar{\nu}} : \mathrm{O}(V)_{\bar{\nu},\bar{\rho}}]\big|\mathrm{O}(\bar{\nu}^{\perp})_{\bar{\rho}}\big|
\end{aligned}$$

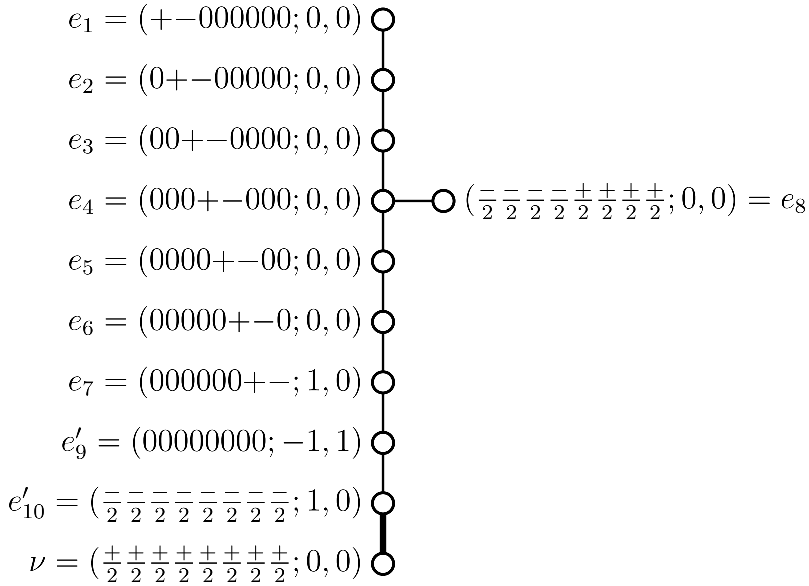

FIGURE 4.1. Simple roots for $\mathrm{O}^{\uparrow}(\Lambda)_{\bar{\nu}}$; see lemma 4.1.

$$\begin{aligned} &= [\mathrm{O}(\bar{\nu}^{\perp}) : \mathrm{O}(\bar{\nu}^{\perp})_{\bar{\rho}}]\big|\mathrm{O}(\bar{\nu}^{\perp})_{\bar{\rho}}\big| \\ &= |\mathrm{O}(\bar{\nu}^{\perp})|. \end{aligned}$$

From this and the fact that S acts trivially on $\bar{\nu}^{\perp} \subseteq V$, it follows that S is the full kernel of $\mathrm{O}^{\uparrow}(\Lambda)_{\nu}$'s action on $\bar{\nu}^{\perp}$. $\square$

4. The 1-nodal case

In this section we continue to use the previous section's model for Λ. We suppose X is an Enriques surface with a single nodal class $\bar{\nu} \in V = \Lambda/2\Lambda$, and we fix some nodal root $\nu \in \Lambda$ lying over it.

All roots of Λ are equivalent under isometries (since figure 2.1 is simply laced). So we may choose the identification between Λ and X's numerical lattice such that ν is any chosen root. We choose $\nu = (\tfrac{+}{2}\tfrac{+}{2}\tfrac{+}{2}\tfrac{+}{2}\tfrac{+}{2}\tfrac{+}{2}\tfrac{+}{2}\tfrac{+}{2}; 0, 0)$, which is compatible with the previous section's notation. To the simple roots from figure 3.1 we may adjoin two more roots, to obtain simple roots for the larger Coxeter group whose diagram appears in in figure 4.1. The extra simple roots are ν and e_{10}'. We write C for the fundamental chamber for these 11 simple roots. We define W_{246} as the group generated by the reflections in the top 10 roots, and continue writing W_{244} and W_{245} as before.

Lemma 4.1. $\mathrm{O}^{\uparrow}(\Lambda)_{\bar{\nu}}$ *is the Coxeter group with simple roots in figure 4.1.*

Proof. First, the affine subgroup $\widetilde{E}_7\widetilde{A}_1$ generated by the reflections in all the roots except e_9' is the full $\mathrm{O}^\uparrow(\Lambda)_{\bar\nu}$-stabilizer of the null vector $\rho = (0; 1, 0)$. The proof is just like lemma 2.1, and one can even use the same formula (2.1) for the translations. The main difference is that only half of the translations preserve $\bar\nu$. The ones that do correspond to the sublattice $E_7 \oplus A_1$ of E_8.

The Coxeter group lies in $\mathrm{O}^\uparrow(\Lambda)_{\bar\nu}$ because every simple root has even inner product with ν. Cusp-counting shows that the Coxeter group has two orbits on null vectors, corresponding to the diagrams $\widetilde{E}_8$ and $\widetilde{E}_7\widetilde{A}_1$. And $\mathrm{O}^\uparrow(\Lambda)_{\bar\nu}$ has at least two orbits on null vectors, since it has two orbits on isotropic vectors in V (those orthogonal to $\bar\nu$ and those not). It follows that $\mathrm{O}^\uparrow(\Lambda)_{\bar\nu}$ and this reflection subgroup have the same orbits on null vectors. The equality of these groups follows because their subgroups stabilizing ρ are equal. □

Lemma 4.2. *The cone* $\mathrm{nef}(X)$ *contains the entire future cone of* $\nu^\perp$*.*

Recall that $\mathrm{nef}(X)$ is the Weyl chamber for the group $W_{\mathrm{nodal}} \subseteq \mathrm{O}(\Lambda)$ generated by the reflections in the nodal roots. (These reflections do not arise from symmetries of X, but they are isometries of Λ.)

Proof. Because X has a single nodal class $\bar\nu$, every nodal root represents it. In particular, if ν' is any nodal root then Λ contains $(\nu - \nu')/2$, and $\nu \cdot \nu'$ is even (since $\bar\nu$ is isotropic). We cannot have $\nu \cdot \nu' = 0$, because then $(\nu - \nu')/2$ would have norm -1, which is impossible in the even lattice Λ. For $\nu' \neq \nu$ this implies $\nu \cdot \nu' \geq 2$, so their orthogonal complements do not intersect in hyperbolic space H^9. Since the orthogonal complements of the nodal roots bound $\mathrm{nef}(X)$, their future cones lie in it. In particular, $\mathrm{nef}(X)$ contains the future cone of $\nu^\perp$. □

Lemma 4.3. *All the* W_{246}*-translates of* C *lie in* $\mathrm{nef}(X)$*.*

Proof. Every nodal root has the same nodal class $\bar\nu$, so all their reflections preserve it. So the mirrors of W_{nodal} are among the mirrors of $\mathrm{O}^\uparrow(\Lambda)_{\bar\nu}$. It follows that every chamber for W_{nodal} is a union of chambers for $\mathrm{O}^\uparrow(\Lambda)_{\bar\nu}$. In particular, $\mathrm{nef}(X)$ is such a union.

From this and lemma 4.2 it follows that $\mathrm{nef}(X)$ contains every chamber of $\mathrm{O}^\uparrow(\Lambda)_{\bar\nu}$ that lies on the positive side of $\nu^\perp$ and has one of its facets lying in $\nu^\perp$. In particular, $\mathrm{nef}(X)$ contains C.

Now consider the following Coxeter group W' lying between W_{nodal} and $\mathrm{O}^\uparrow(\Lambda)_{\bar\nu}$: the one generated by the reflections in all the roots of Λ lying over $\bar\nu$. Writing C' for its chamber containing C, we obviously have $C' \subseteq \mathrm{nef}(X)$. Therefore it suffices to show that C' contains the W_{246}-translates of C. The advantage of W' over W_{nodal} is that W_{246}

visibly normalizes it, hence permutes its chambers. Suppose r is any simple root from figure 4.1 other than ν. Then a generic point of $r^\perp$ is orthogonal to no roots except r, hence to no roots of W'. So r's reflection preserves an interior point of C', hence C' itself. It follows that W_{246} preserves C', so C' contains the W_{246}-translates of C, as desired. (Parts (1) and (4) of theorem 1.2 imply $W' = W_{\text{nodal}}$.) □

Next we will describe some symmetries of X, called Geiser, Bertini and Kantor involutions. They are defined in terms of nef classes and nodal curves with certain properties; see the proof of theorem 5 in [9] for the details of their construction. We will describe their actions on Λ. Their actions on V follow easily: Geiser and Bertini involutions act trivially and Kantor involutions act by the transvection in $\bar{\nu}$.

Geiser involutions: suppose E_1, E_2 are nef divisors with $E_1^2 = E_2^2 = 0$ and $E_1 \cdot E_2 = 1$, such that $E_1 + E_2$ is ample. Then the linear system $|2E_1 + 2E_2|$ realizes X as a 2-fold branched cover of the unique 4-nodal quartic del Pezzo surface. (It can be defined in $\mathbb{P}^4$ by $0 = x_0x_1 + x_2^2 = x_3x_4 + x_2^2$.) The deck transformation of this covering is an automorphism G of X, called a Geiser involution. Its action on Λ can be described as follows. The classes of E_1, E_2 in Λ span a summand isometric to $U \cong \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and G acts by the negation map of this summand's orthogonal complement.

Bertini involutions: now suppose E is a nef divisor with $E^2 = 0$, and that R is a nodal curve having intersection number 1 with it. Then the linear system $|4E + 2R|$ realizes X as a 2-fold branched cover of a degenerate form of the previous paragraph's del Pezzo surface. (Its equations in $\mathbb{P}^4$ are $0 = x_0x_1 + x_2^2 = x_3x_4 + x_0^2$.) The deck transformation is an automorphism B of X, called a Bertini involution. Its action on Λ can be described as follows. The classes of E and R span a summand U of Λ, and B acts by negating its orthogonal complement. (This resembles the Geiser involution case if one thinks of E_1 as E and E_2 as the image of E under reflection in R. The differences are that E_2 is not nef and $R \cdot (E_1 + E_2) = 0$. In particular, $|2E_1 + 2E_2|$ collapses R to a point.)

Kantor involutions: now suppose E_1 and E_2 are nef divisors with $E_1^2 = E_2^2 = 0$ and $E_1 \cdot E_2 = 1$, and that R is a nodal curve disjoint from them. Then the linear system $|2E_1 + 2E_2 - R|$ realizes X as a 2-fold branched cover of the Cayley cubic (the unique cubic surface with four A_1 singularities). The deck transformation of this covering is an automorphism K of X, called a Kantor involution. Its action on Λ can be described as follows. The classes of E_1 and E_2 in Λ generate a summand isometric to U, and K acts as the composition of the negation map on its orthogonal complement and the reflection in the nodal root corresponding to R.

Remarks. In section 3 we introduced some isometries of Λ that we called Bertini and Kantor involutions. As the language suggests, they are special cases of the Bertini and Kantor involutions given here:

A Bertini involution in section 3 meant an involution of Λ whose negated lattice is isometric to E_8 and orthogonal to ν. First we give an example of such an involution arising from the construction above. Consider the sublattice L of Λ spanned by the roots of the E_8 subdiagram of figure 4.1. Write B for its Bertini involution in the sense of section 3: it negates L and fixes $L^\perp$ pointwise. Computation shows that $L^\perp$ is spanned by ν and the null vector $E = (-1, 0, 0, 0, 0, 0, 0, -1; 1, 1)$, which have inner product 1. It is easy to check that E has inner product ≥ 0 with the simple roots in figure 4, so it lies in C and hence is nef (lemma 4.3). The Bertini involution constructed above, using E and $R = \nu$, is exactly B.

Now consider any Bertini involution B' in the sense of section 3, and write $L' \cong E_8$ for its negated lattice. Any two copies of the E_8 lattice in $\nu^\perp$ are equivalent under isometries of $\nu^\perp$, because each is a direct summand (being unimodular), with its 1-dimesional complement in $\nu^\perp$ determined by $\det(\nu^\perp)$. Therefore some $g \in \mathrm{O}^\uparrow(\nu^\perp)$ sends L to L'. Since $\mathrm{O}^\uparrow(\nu^\perp) = W_{245}$ (theorem 3.2) and W_{245} preseves $\mathrm{nef}(X)$ (lemma 4.3), $g(E)$ is also nef. The Bertini involution constructed above, using $g(E)$ and $R = \nu$, is B'.

Finally, suppose K is a Kantor involution in the sense of section 3, so its negated lattice is the first summand of some decomposition $\nu^\perp = E_7 \oplus U$. We take E_1, E_2 to be null vectors spanning the U summand. By lemma 4.2 they are nef. Then the Kantor involution constructed above from E_1 and E_2 is K.

Proof of theorem 1.2. The main step is to prove (3). For this we reuse the strategy of theorem 1.1. We think of W_{246} as the "large" group, and for the "small" subgroup S we take the subgroup generated by the Geiser, Bertini and Kantor involutions of X that lie in W_{246}. (In fact these are all the Geiser, Bertini and Kantor involutions, but we have not proven this.) To relate these groups we define the "medium" group M as the subgroup generated by S and W_{245}. Obviously M normalizes S.

Theorem 3.3 says that $S \cap W_{245}$ contains the subgroup of W_{245} that acts trivially on $\bar{\nu}^\perp$. Theorem 3.2 says that $W_{245} = \mathrm{O}^\uparrow(\Lambda)_\nu$. And the image of W_{245} in $\mathrm{O}(V)$ acts on $\bar{\nu}^\perp$ by its full isometry group. Therefore the action of M on V identifies M/S with $\mathrm{O}(\bar{\nu}^\perp) = \mathrm{O}_9(2)$.

Now we claim that M is all of W_{246}. It contains W_{245} by definition, so it suffices to show that M contains the reflection in e'_{10}. Observe that $e_2, \ldots, e_8, e'_{10}$ span a root lattice E_7A_1. Its saturation is strictly larger, hence a copy of E_8, because Λ contains $(e_9 - e'_{10} + e_7 + e_5)/2$. We will show in the next paragraph that the negation map G of this E_8 summand of Λ is a Geiser involution. It lies in W_{246} since it is the product of the reflection in e'_{10} and the central involution of $W(E_7)$. Therefore $G \in S \subseteq M$. Since M contains the central involution of $W(E_7)$, the same decomposition of G shows that M contains the reflection in e'_{10}. Modulo the fact that G is a Geiser involution, this completes the proof that $M = W_{246}$. From our understanding of M/S it follows that S is exactly the subgroup of W_{246} that acts trivially on $\bar{\nu}^{\perp} \subseteq V$. This is definition of $\overline{W}_{246}(2)$, proving (3).

We must still show that G is a Geiser involution. To do this we seek $E_1, E_2 \in \mathrm{nef}(X)$ with $E_1^2 = E_2^2 = 0$ and $E_1 \cdot E_2 = 1$ and $E_1 + E_2$ ample, which span a copy of U orthogonal to $e_2, \ldots, e_8, e'_{10}$. Now, the orthogonal complement of $e_2, \ldots, e_8, e'_{10}$ has signature $(1, 1)$, so it has only two isotropic lines. This determines E_1 and E_2 up to scaling. In fact they are obvious from the Dynkin diagram: they must be the null vectors representing the $\widetilde{E}_7\widetilde{A}_1$ and $\widetilde{E}_8$ cusps. We already know that the first is $(0, \ldots, 0; 1, 0)$, and one checks that the second is $(-1, 0, \ldots, 0, -1; 1, 1)$. They have inner product 1, hence span a copy of U. They are nef because they lie in C. Their sum is ample because it lies in the interior of the edge joining these cusps, hence lies in C and is not orthogonal to any nodal root. So G is a Geiser involution. This completes the proof that $M = W_{246}$ and hence the proof of (3).

Next we prove the rest. The main point is that $W_{246,\nu}/S_{\nu}$ maps isomorphically to W_{246}/S, since $W_{246,\nu} = W_{245}$ acts on $\bar{\nu}^{\perp}$ as $\mathrm{O}_9(2)$. It follows that the S-orbit of ν coincides with the W_{246}-orbit of ν. This shows simultaneously that every facet of $\cup_{w \in W_{246}} w(C)$ is the orthogonal complement of a nodal root, and that S (hence Γ) acts transitively on them. These are claims (1) and (4) of the theorem. Claim (5) follows from (4) and the bijection between nodal roots and nodal curves.

For claim (2), recall that every nodal root lies over $\bar{\nu}$. So the full $\mathrm{O}^{\uparrow}(\Lambda)$-stabilizer of $\mathrm{nef}(X)$ must preserve $\bar{\nu}$. Since C is a fundamental domain for $\mathrm{O}^{\uparrow}(\Lambda)_{\bar{\nu}}$ by lemma 4.1, the stabilizer of $\mathrm{nef}(X)$ is exactly the subgroup of $\mathrm{O}^{\uparrow}(\Lambda)_{\bar{\nu}}$ which preserves $\mathrm{nef}(X)$. This is obviously W_{246}. □

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