ORTHOGONAL COMPLEX HYPERBOLIC ARRANGEMENTS

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To Herb Clemens on his 60th birthday

1. Introduction

The purpose of this note is to study the geometry of certain remarkable infinite arrangements of hyperplanes in complex hyperbolic space which we call *orthogonal arrangements*: whenever two hyperplanes meet, they meet at right angles. A natural example of such an arrangement appears in [3]; see also [2]. The concrete theorem that we prove here is that the fundamental group of the complement of an orthogonal arrangement has a presentation of a certain sort. As an application of this theorem we prove that the fundamental group of the quotient of the complement of an orthogonal arrangement by a lattice in $PU(n,1)$ is not a lattice in any Lie group with finitely many connected components. One special case of this result is that the fundamental group of the moduli space of smooth cubic surfaces is not a lattice in any Lie group with finitely many components. This last result was the motivation for the present note, but we think that the geometry of orthogonal arrangements is of independent interest.

To state our results, let B^n denote complex hyperbolic n-space, which can be described concretely as either the unit ball in $\mathbb{C}^n$ with its Bergmann metric, or as the set of lines in $\mathbb{C}^{n+1}$ on which the hermitian form $h(z) = -|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2$ is negative definite, with its unique (up to constant scaling factor) $PU(n,1)$-invariant metric. Let $\mathcal{A} = \{H_1, H_2, H_3, \dots\}$ be a non-empty locally finite collection of totally geodesic complex hyperplanes in B^n. We call $\mathcal{A}$ a complex hyperbolic arrangement and write $\mathcal{H}$ for $H_1 \cup H_2 \cup H_3 \cup \cdots$. We are interested in $\pi_1(B^n - \mathcal{H})$, the fundamental group of the complement. It is clear that if $\mathcal{A}$ is infinite (the case of interest here), this group is not finitely generated. (For instance, its abelianization $H_1(B^n - \mathcal{H}, \mathbb{Z})$ is the free abelian group on the set $\mathcal{A}$.) If $n = 1$, $\mathcal{H}$ is a discrete subset of B^1 and $\pi_1(B_1 - \mathcal{H})$ is a free group, and we have nothing further to say in this case. We thus assume throughout the paper that $n \geq 2$. We say that $\mathcal{A}$ is an *orthogonal arrangement* if any two distinct H_i's are either orthogonal or disjoint. The main purpose of this note is to prove the following theorem:

Theorem 1.1. *Let $\mathcal{A}$ be an orthogonal complex hyperbolic arrangement. Then the group $\pi_1(B^n - \mathcal{H})$ has a presentation $\langle \gamma_1, \gamma_2, \dots \mid r_1, r_2, \dots \rangle$ where each relator r_k has the form $r_k = [\gamma_i, l_{ij}\gamma_j l_{ij}^{-1}]$, where l_{ij} is a word in $\gamma_1, \dots, \gamma_{\max\{i,j\}-1}$.*

At this point the conclusion of the theorem may seem completely technical. We hope that a look at the proof will convince the reader that the conclusion reflects

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some beautiful complex hyperbolic geometry. We will use Theorem 1.1 to prove the following theorem.

Theorem 1.2. *Suppose an orthogonal hyperplane arrangement in B^n is preserved by a lattice $\Gamma \subseteq PU(n,1)$, $n > 1$. Then the orbifold fundamental group $\pi_1^{\mathrm{orb}}(\Gamma \backslash (B^n - \mathcal{H}))$ is not a lattice in any Lie group with finitely many connected components.*

This allows us to solve the problem which motivated this work, concerning the moduli space of cubic surfaces in $\mathbb{C}P^3$. Following [3], let $\mathcal{C}_0$ denote the space of smooth cubic forms in 4 variables, let $P\mathcal{C}_0$ denote its image in the projective space of all cubic forms in 4 variables, and let $M_0 = PGL(4,\mathbb{C}) \backslash P\mathcal{C}_0$ denote the moduli space of smooth cubic surfaces.

Corollary 1.3. *Neither $\pi_1(P\mathcal{C}_0)$ nor $\pi_1^{\mathrm{orb}}(M_0)$ is a lattice in any Lie group with finitely many connected components.*

The corollary follows from Theorem 1.2 because the main result of [3] is that there is an orbifold isomorphism $M_0 \cong \Gamma \backslash (B^4 - \mathcal{H})$ where Γ is a certain lattice in $U(4,1)$ and $\mathcal{H}$ is a certain Γ-invariant orthogonal arrangement in B^4. Namely, let $\mathcal{E}$ denote the ring of Eisenstein integers (the integers $\mathbb{Z}[\sqrt[3]{1}]$ in $\mathbb{Q}(\sqrt{-3})$), let h be the above Hermitian form in $n+1$ variables, let $\mathcal{A}$ be the arrangement $\{v^\perp : v \in \mathcal{E}^{n+1}, h(v) = 1\}$, and let $\mathcal{H}$ be the union of the hyperplanes. It is easy to see (see Lemma 7.29 of [3]) that $\mathcal{A}$ is an orthogonal arrangement. Let Γ denote the lattice $PU(h, \mathcal{E})$ in $PU(n,1)$, which obviously preserves $\mathcal{H}$. Then $\Gamma \backslash B^n$ is a quasi-projective variety and a complex analytic orbifold, and in the case $n = 4$ we have the orbifold isomorphism $M_0 \cong \Gamma \backslash (B^4 - \mathcal{H})$.

2. Proof of Theorem 1.1

Let $\mathcal{A}$ be an orthogonal arrangement, let $\mathcal{H}$ denote the union of the hyperplanes in $\mathcal{A}$, and choose the basepoint $p_0 \in B^n - \mathcal{H}$. We will now show that p_0 can be chosen to satisfy two genericity conditions: (G1) for any two distinct nonempty sub-balls X and Y of B^n, each of which is an intersection of members of $\mathcal{A}$, $d(p_0, X) \neq d(p_0, Y)$, and (G2) for any $i \neq j$, the minimal geodesic from p_0 to H_i does not meet H_j. (We write d for the complex hyperbolic distance.) It is clear that condition (G1) holds on the complement of a countable collection of equidistant hypersurfaces, which are closed real analytic subvarieties of real codimension one. The same holds for (G2), namely, let $V_{i,j}$ denote the union of all geodesic rays from H_i which are perpendicular to H_i and which meet H_j. Then $V_{i,j}$ is a locally closed real analytic subvariety of real codimension one, whose closure is the nowhere dense semi-analytic set of geodesics perpendicular to H_i and which meet the closure of H_j in the closed unit ball $\bar{B}^n$. Thus the complement of $\bigcup_{i,j} V_{i,j}$ is not empty. We denote $d(p_0, H_i)$ by d_i, and label the hyperplanes $H_1, H_2, \dots \in \mathcal{A}$ according to increasing distance from p_0, so that $d_1 < d_2 < \cdots$. We will keep this convention throughout the paper.

Now choose generators $\{\gamma_i\}$ for $\pi_1(B^n - \mathcal{H})$ as follows: for each i, let σ_i be the minimal geodesic segment from p_0 to H_i, let D_i be the complex hyperbolic line which contains σ_i, and note that D_i intersects H_i orthogonally at a single point p_i. Choose a geodesic subsegment λ_i of σ_i that starts at p_0 and ends at a point q_i very close to p_i but before reaching p_i. Let c_i be the loop in D_i based at q_i and running positively around the circle in D_i centered at p_i that passes through q_i. Finally let $\gamma_i = \lambda_i c_i \lambda_i^{-1}$. From (G2) it is clear that if q_i is chosen close enough to p_i then $\gamma_i \subset B^n - \mathcal{H}$.

Let $f : B^n \to \mathbb{R}$ be the function $f(x) = d(p_0, x)^2$. Then f has a unique critical point on B^n (a minimum, namely p_0), its restriction to each H_i has a unique critical point on H_i (also a minimum, the point p_i, defined above), for each non-empty intersection $H_i \cap H_j, i \neq j$ has a unique critical point (also a minimum, which we denote by p_{ij}), and so on for its restriction to any non-empty intersection $H_{i_1} \cap \cdots \cap H_{i_p}$. Condition (G1) assures us that distinct critical points correspond to different values of f. The stratified Morse theory of Goresky and MacPherson, see Theorem 10.8 of [5], implies that $B^n - \mathcal{H}$ is homotopy equivalent to a CW complex with one zero-cell, with one-cells in one to one correspondence with the H_i, two-cells in one to one correspondence with the non-empty intersections $H_i \cap H_j$, three-cells in one to one correspondence with non-empty intersections $H_i \cap H_j \cap H_k$, etc.

Since the fundamental group of $B^n - \mathcal{H}$ is isomorphic to the fundamental group of the two-skeleton of this complex, we must look in more detail at the one- and two-cells of this complex. First, when passing the critical value $f(p_i)$ correspoding to the unique critical point p_i of $f|H_i$, the change in homotopy type of the two sublevel sets is described by attaching a one-cell. Namely, the larger sublevel set is obtained from the smaller by attaching an interval that completes the loop c_i defined above. From this it is clear that the loops γ_i generate $\pi_1(B^n - \mathcal{H}^n)$.

Now we must look at the two-cells. Suppose that $H_i \cap H_j \neq \emptyset$ and that $i \neq j$. When crossing the critical level $f(p_{ij})$ corresponding to the unique critical point p_{ij} of $f|_{H_i \cap H_j}$, the bigger sublevel set is obtained from the smaller by attaching a two-cell that can be visualized as follows. Let q_{ij} be a point in the smaller sublevel set very close to p_{ij} and let e_i and e_j be loops in the smaller sublevel set based at q_{ij} and encircling H_i and H_j respectively. Then the larger sublevel set is homotopically equivalent to the smaller union a two cell, which can be visualized as a square whose boundary is attached to the union of e_i and e_j by the commutator map, so that e_i and e_j commute in the larger sublevel set.

Since the loops e_i and e_j are freely homotopic to the loops γ_i and γ_j, it is clear that the relators are commutativity between γ_i and some conjugate of γ_j, as in the first half of the assertion of Theorem 1.1. It remains to prove the more subtle assertion on the expression of l_{ij} as a word in the γ's.

To this end, for the remainder of this section, fix a pair i, j with $i \neq j$ so that $H_i \cap H_j \neq \emptyset$. Observe that the points p_0, p_i, p_{ij}, p_j form the vertices of a totally real quadrilateral Q, two of whose sides are the geodesic segments σ_i and σ_j; the remaining two sides are the geodesics joining p_i and p_j to p_{ij} (which lie in H_i, respectively H_j). Moreover Q is a Lambert quadrilateral: the angles at p_i, p_j, p_{ij} are all right angles, as in Figure 1. See Lemma 3.2.14 of [4] for details (after an easy reduction to the case $n = 2$). It is easy to see that that Q and H_i (repectively H_j) meet everywhere at right angles, and that Q is foliated by the geodesic segments in Q perpendicular to the side $H_i \cap Q$, which we call *vertical segments* (there are respectively horizontal segments, which we will not need); see Figure 1.

Lemma 2.1. *Suppose that $k \neq i, j$ and that the hyperplane H_k meets either H_i or H_j. Then $H_k \cap Q = \emptyset$.*

Proof. Suppose, say, that $H_k \cap H_i \neq \emptyset$. We will first show that $H_k \cap Q$ is a vertical segment. Note that H_k and H_i intersect at right angles, and that H_k is foliated by totally geodesic discs (complex lines) perpendicular to H_i at the points of $H_i \cap H_k$. Call these discs *vertical discs*. Suppose that $H_k \cap Q \neq \emptyset$, and let $q \in H_k \cap Q$. Then q is in a unique vertical segment and a unique vertical disc, and it is easy to see

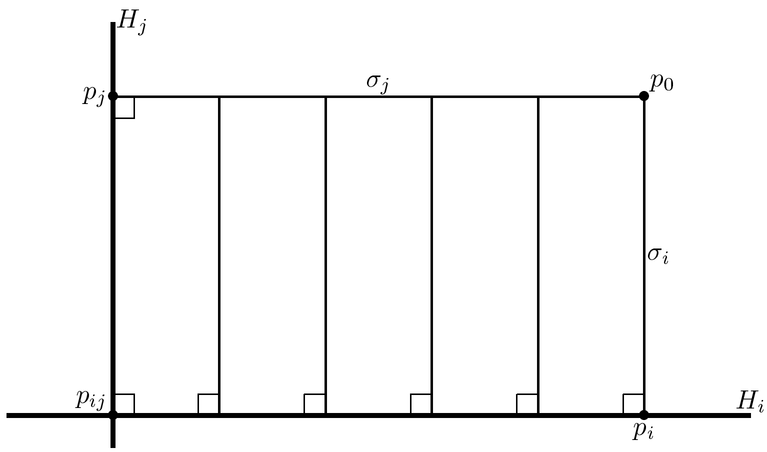

FIGURE 1. A picture of Q.

that the segment is contained in the disc. Thus $H_k \cap Q$ is a vertical segment. Since Q is foliated by these vertical segments, the segment $H_k \cap Q$ must intersect the side σ_j of Q opposite to $H_i \cap Q$; see Figure 1. Thus $H_k \cap \lambda_j \neq \emptyset$, contradicting assumption (G2) of the choice of p_0. Thus we must have $H_k \cap Q = \emptyset$, and the lemma is proved. □

Lemma 2.2. *Suppose that $k \neq i, j$ and $H_k \cap Q \neq \emptyset$. Then $H_k \cap Q$ consists of a single point in the interior of Q, and hence the intersection is transverse.*

Proof. It is clear that H_k cannot intersect the boundary of Q: by the choice (G2) of p_0 it cannot intersect the edges σ_i or σ_j of Q, and by the previous lemma it cannot intersect the edges $H_i \cap Q$, $H_j \cap Q$. If $H_k \cap Q \neq \emptyset$, then since the intersection is totally geodesic in Q, it must be either a point or the intersection of Q with a full geodesic in the real 2-ball containing Q. Since H_k cannot intersect the boundary of Q, the intersection must be an interior point, and the proof is complete. □

Lemma 2.3. *Suppose that $k \neq i, j$ and $H_k \cap Q \neq \emptyset$. Then $d(p_0, H_k \cap Q) < \max\{d_i, d_j\}$.*

Proof. Observe that the quadrilateral Q lies in a unique totally geodesic complex hyperbolic plane $B^2 \subset B^n$, so by intersecting all these hyperplanes with this B^2 we may reduce to the case of lines in B^2. Consider this B^2 concretely as the unit ball $\{|z|^2 + |w|^2 < 1\} \subset \mathbb{C}^2$, with p_{ij} at the origin, H_i and H_j as the intersection of B^2 with the z and w-axes respectively, and with the quadrilateral Q lying in the quadrant

$$\{(x, y) : x, y \in \mathbb{R},\ x^2 + y^2 < 1,\ x \geq 0,\ y \geq 0\}$$

of the real subspace $B^2_{\mathbb{R}}$. The complex hyperbolic geometry of B^2 restricts to the Klein model of real hyperbolic geometry of $B^2_{\mathbb{R}}$, so geodesics in $B^2_{\mathbb{R}}$ are real line segments and convex sets are the Euclidean convex sets.

We suppose without loss of generality that $i < j$. The ball $B(p_0, d_j) \cap Q$ is a convex subset of $Q \subset B^2_{\mathbb{R}}$ which is bounded by the geodesic segment σ_j, a convex curve C from p_j to some point on the x-axis, a segment along the x-axis, and the segment σ_i. Let L be the line joining the endpoints of C. Then $x + y < 1$ on L

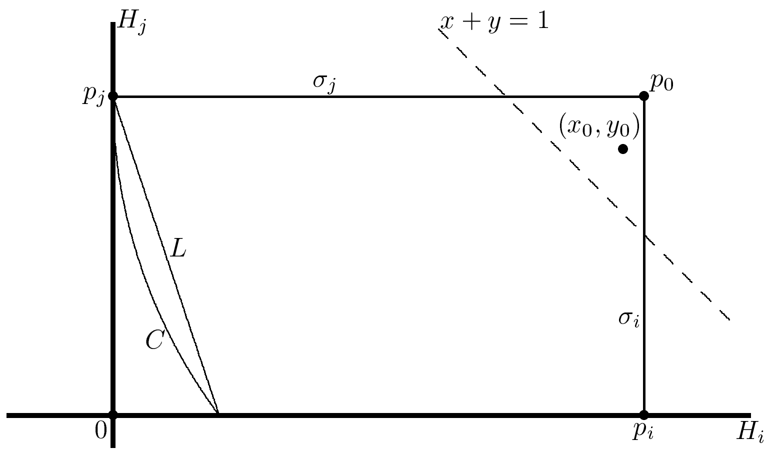

FIGURE 2

since this inequality holds at both endpoints. Thus the same inequality holds on C.

Now let $H_k \cap Q = \{(x_0, y_0)\}$. We must show that $d(p_0, (x_0, y_0)) < d_j$. The line H_k has an equation $\alpha z + \beta w = 1$. By Lemma 2.1, H_k cannot meet either axis, thus $\max\{|\alpha|, |\beta|\} < 1$. Since $\alpha x_0 + \beta y_0 = 1$, by the duality between the l_1 and l_∞ norms we must have that $x_0 + y_0 > 1$. Therefore (x_0, y_0) lies in the connected component of $Q - C$ containing p_0, so (x_0, y_0) is interior to the d_j-ball about p_0, and the lemma is proven. □

We can now finish the proof of Theorem 1.1; we continue to assume $i < j$. Recall that the change in topology in crossing the critical level $f(p_{ij})$ is described by attaching a two cell to the union of the two loops e_i, e_j as described above, and this introduces the relation $[e_i, e_j] = 1$ in the fundamental group based at the point q_{ij}. Now the point q_{ij} can be chosen to lie in the quadrilateral Q. Let λ_{ij} and λ_{ji} be arcs in the interior of Q, close to the boundary of Q, joining q_i and q_j to q_{ij}. Observe that e_i is homotopic to $\lambda_{ij}^{-1} c_i \lambda_{ij}$ and e_j is homotopic to $\lambda_{ji}^{-1} c_j \lambda_{ji}$, as loops based at q_{ij} and homotopies relative to q_{ij}. Thus the relation $[e_i, e_j] = 1$ at q_{ij} can be rewritten as a relation at p_0 by changing the basepoint using the path $\lambda_i \lambda_{ij}$, and it reads:

$$[\gamma_i, l_{ij} \gamma_j l_{ij}^{-1}] = 1 \quad \text{where} \quad l_{ij} = \lambda_i \lambda_{ij} \lambda_{ji}^{-1} \lambda_j^{-1}.$$

Observe that l_{ij} is approximately the boundary of the quadrilateral Q, thus l_{ij} is homotopic to $l_1 \dots l_r$ where $l_1, \dots, l_r$ are loops in Q based p_0 encircling the points $Q \cap H_{k_1}, \dots, Q \cap H_{k_r}$, where $H_{k_1}, \dots, H_{k_r}$ are the hyperplanes that have non-empty intersection with the interior of Q. By Lemma 2.3, all these intersections lie at distance $< d_j$ from p_0, so each of $l_1, \dots, l_r$ is a word in $\gamma_1, \dots, \gamma_{j-1}$, and Theorem 1.1 is proven.

The only consequence of Theorem 1.1 we will use is the following corollary. For each $m = 1, 2, \dots$, we define $\mathcal{H}_m = H_1 \cup \dots \cup H_m$, and observe that the inclusion $B^n - \mathcal{H} \subset B^n - \mathcal{H}_m$ induces a surjection $\pi_1(B^n - \mathcal{H}) \to \pi_1(B^n - \mathcal{H}_m)$.

Corollary 2.4. *Let $\gamma \in \pi_1(B^n - \mathcal{H})$, $\gamma \neq 1$. Then there exists an m such that γ has non-trivial image in $\pi_1(B^n - \mathcal{H}_m)$.*

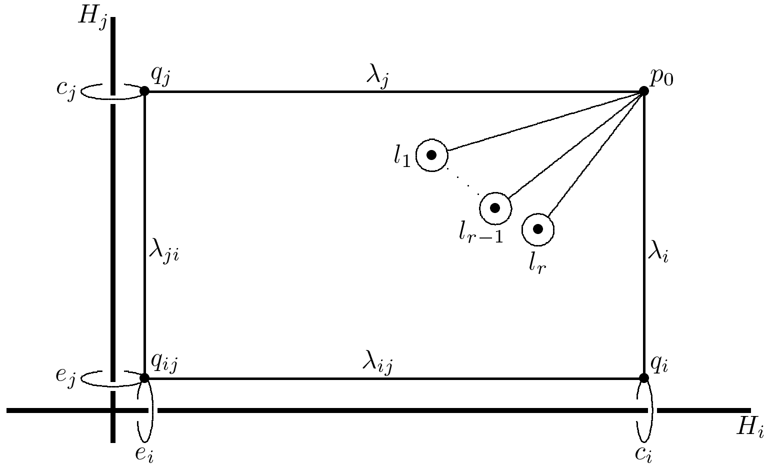

Figure 3

Proof. Theorem 1.1 clearly implies that for each m the map that sends each generator $\gamma_1, \ldots, \gamma_m$ of $\pi_1(B^n - \mathcal{H}_m)$ to the same element of $\pi_1(B^n - \mathcal{H})$ defines a group homorphism $\pi_1(B^n - \mathcal{H}_m) \to \pi_1(B^n - \mathcal{H})$ which splits the surjection $\pi_1(B^n - \mathcal{H}) \to \pi_1(B^n - \mathcal{H}_m)$ and has image $\langle \gamma_1, \ldots, \gamma_m \rangle$, the subgroup of $\pi_1(B^n - \mathcal{H})$ generated by $\gamma_1, \ldots, \gamma_m$. Thus the subgroup $\langle \gamma_1, \ldots, \gamma_m \rangle$ maps isomorphically onto $\pi_1(B^n - \mathcal{H}_m)$. Given $\gamma \neq 1$ as in the statement of the corollary, choose an m so that $\gamma \in \langle \gamma_1, \ldots, \gamma_m \rangle$, and the corollary follows. □

Remark. It is easy to see that the relations $[\gamma_i, l_{ij}\gamma_j l_{ij}^{-1}]$ of Theorem 1.1 in general cannot be simplified to $[\gamma_i, \gamma_j]$ by choosing the basepoint p_0 so that all the loops l_{ij} are homotopically trivial. We show this for the arrangement $\mathcal{A}$ associated to the Eisenstein integers that we defined in the introduction. To see this for $n = 2$, let us follow the conventions of the proof of Lemma 2.3, so that H_i and H_j are the z and w axes respectively. Choose a third hyperplane H_k to have equation $z + w = 1$, and note that this is also in the collection $\mathcal{H}$. Choose a point $(z_1, w_1) \in \partial B^2$ so that 1 is in the interior of parallelogram in $\mathbb{C}$ spanned by z_1 and w_1. Observe that this is equivalent to saying that the ideal Lambert quadrilateral $\{sz_1 + tw_1 : 0 \leq s, t \leq 1\}$ meets the hyperplane $H_k = \{z + w = 1\}$ in the interior of the quadrilateral. By the density of $\mathbb{Q}(\sqrt{-3})$-rational points in ∂B^2, we may assume that (z_1, w_1) has coordinates in $\mathbb{Q}(\sqrt{-3})$, for instance we may take $(z_1, w_1) = ((9 + 3\sqrt{-3})/16, (11 - 3\sqrt{-3})/16)$. Then there exists a neighborhood N of (z_1, w_1) so that for all $p_0 = (z_0, w_0) \in N \cap B^2$, 1 is in the interior of the parallelogram spanned by z_0 and w_0. Thus for all $p_0 \in N \cap B^2$, the Lambert quadrilateral $Q = \{sz_0 + tw_0 : 0 \leq s, t \leq 1\}$ with acute angle at p_0 meets the hyperplane H_k. Since every neighborhood $N \cap B^2$ of a cusp point in ∂B^2 contains a fundamental domain for Γ_2, it follows that for any choice of basepoint p_0 there are hyperplanes H_i, H_j so that the loop l_{ij} is not homotopically trivial. Suitable modifications of this argument show the necessity of the l_{ij} for any $n > 2$.

3. Proof of Theorem 1.2

In this section we write K for $\pi_1(B^n - \mathcal{H})$ and Φ for $\pi_1^{\text{orb}}(\Gamma\backslash(B^n - \mathcal{H})$. Observe that these groups fit into the exact sequence

$$1 \to K \to \Phi \to \Gamma \to 1,$$

which gives rise to a natural homomorphism $\Gamma \to \text{Out}(K)$, where Out denotes the group of outer automorphisms. Theorem 1.2 follows from well known theorems on lattices in semi-simple groups, together with the following lemmas.

Lemma 3.1. *For $n > 1$, the centralizer in Φ of each generator $\gamma_i \in K$ contains a non-abelian free group.*

Proof. Let $M_i \subset B^n - H_i$ be the boundary of a tubular neighborhood of the hyperplane H_i, and suppose that M_i contains the point q_i of §2. Let Γ_i denote the subgroup of Γ that preserves H_i. Then γ_i is freely homotopic to the loop c_i of §2 based at q_i, which is clearly central in the orbifold fundamental group $\pi_1^{\text{orb}}(\Gamma_i\backslash(M_i - \mathcal{H}))$. Since this latter group surjects (to $\pi_1^{\text{orb}}(\Gamma_i\backslash M_i)$ and hence) to Γ_i, which is a lattice in $PU(n-1,1)$, it contains a non-abelian free group if $n > 1$. It then follows that the centralizer of γ_i in Φ contains a non-abelian free group if $n > 1$. □

Lemma 3.2. *Let $H \neq \{1\}$ be a subgroup of K which is normal in Φ. Then*

1. *H contains a non-abelian free group.*
2. *The image of the natural map $\Gamma \to \text{Out}(H)$ contains elements of infinite order.*

Proof. Let $\mu \in H$ and $\mu \neq id$. We will produce $\nu \in H$ so that μ, ν generate a non-abelian free group. By Corollary 2.4, there exists an m so that μ has non-trivial image in $\pi_1(B^n - \mathcal{H}_m)$. Choose a hyperbolic element of Γ neither of whose limit points in ∂B^n lie in any of the boundaries $\partial H_i \subset \partial B^n$ for $i = 1, \ldots, m$. This can be done because Γ is a lattice. Sufficiently high powers of this transformation map $\mathcal{H}_m$ far away from itself; choose one such power and denote it by ϕ. Then B^n can be written as the union of two topological n-sub-balls intersecting along a topological $(n-1)$-sub-ball, where one of the balls contains $\mathcal{H}_m$ and the other contains $\phi(\mathcal{H}_m)$. It follows that $\pi_1(B^n - (\mathcal{H}_m \cup \phi(\mathcal{H}_m))$ is the free product of the groups $K_m = \pi_1(B^n - \mathcal{H}_m)$ and $\tilde{K}_m = \pi_1(B^n - \phi(\mathcal{H}_m))$. Since $\phi \in \Gamma$, $\phi(\mathcal{H}_m) \subset \mathcal{H}$. Thus there is a natural surjection $K \to K_m * \tilde{K}_m$. Since H is normal in $\pi_1^{\text{orb}}(\Gamma\backslash(B^n - \mathcal{H}))$, the group Γ acts by outer automorphisms on H, and there must be an element $\nu \in K$ which maps to the image of $\phi(\mu)$ in $K_m * \tilde{K}_m$. Thus H contains the free product of the cyclic groups generated by μ and ν, which is a free non-abelian group since μ and ν must be of infinite order, say by the result of [1] that $B^n - \mathcal{H}$ is aspherical. (One may avoid using [1] by first applying the above construction to show that H has an element of infinite order, and then applying the construction to this element.) This proves the first part of the lemma. Iterating this reasoning, it is clear that ϕ has infinite order in $\text{Out}(H)$, so the second part of the lemma is proved. □

It is now easy to prove Theorem 1.2. First we show that the first part of Lemma 3.2 precludes Φ from being a lattice in a connected Lie group G with non-trivial solvable radical R. Otherwise, in the exact sequence of Lie groups

$$1 \to R \to G \to G/R \to 1$$

there would be two possibilities: (1) The image of Φ in G/R is discrete. In this case, by Theorem 1.13 of [7], $R \cap \Phi$ would be a lattice in R, thus $R \cap \Phi$ would be a non-trivial normal solvable subgroup of Φ, which is precluded by the first part of Lemma 3.2. (2) The image of Φ in G/R is not discrete. In this case, by a theorem of Auslander, Theorem 8.24 of [7], the identity component of the closure of its image is a solvable group. The intersection of Φ with the pre-image in G of this group is then a non-trivial normal solvable subgroup of Φ, which we have already excluded. We can then exclude Φ from being a lattice in a Lie group G with finitely many components and with identity component having non-trivial radical by observing that Lemma 3.2 also holds for subgroups of finite index in Φ.

It remains only to exclude the possibility that Φ is a lattice in a semisimple group G, again easily reduced to a connected G. First, from the the fact that Γ is not virtually a product and from the second part of Lemma 3.2 (which also holds for subgroups of finite index of Φ), it is clear that Φ is not virtually a product. Thus if it were a lattice in G, it would be an irreducible one. If G had real rank at least 2, by a deep theorem of Margulis [6] any normal subgroup would have to be either of finite index or central (note that this holds even if G is not linear), which is precluded by the infinite-index non-central subgroup K. If G were a rank 1 group, linear or not, the centralizer of any non-central element would be virtually nilpotent. Since the γ_i are clearly not central, this contradicts Lemma 3.1 (Note this is the only point where the hypothesis $n > 1$ is used, and that Theorem 1.2 does not hold for $n = 1$). This completes the proof of Theorem 1.2.

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