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Spotting Infinite Groups

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Abstract

We generalize a theorem of R. Thomas, which sometimes allows one to tell by inspection that a finitely presented group G is infinite. Groups to which his theorem applies have presentations with not too many more relators than generators, with at least some of the relators being proper powers. Our generalization provides lower bounds for the ranks of the abelianizations of certain normal subgroups of G in terms of their indices. We derive Thomas's theorem as a special case.

1. *Introduction*

There is a very simple theorem which states that any group with a presentation with at least one more generator than relations is infinite. To see this, one simply abelianizes the group and uses linear algebra to conclude that the abelianization is infinite. It is also classical that the (p, q, r) triangle group, given by the presentation

$$\langle a, b \,|\, 1 = a^p = b^q = (ab)^r \rangle,$$

is infinite when $1/p+1/q+1/r \leq 1$. One sees this by realizing the triangle group as a group of transformations of the Euclidean plane or the hyperbolic plane, and simply observing that the group is infinite. (See [**3**].)

It is remarkable that these two facts stem from the same source: In [**4**], R. Thomas showed that when a certain technical condition applies, a relator which is a pth power only counts as "$1/p$ of a relator" for the purpose of comparing the number of generators with the number of relations. His theorem sometimes allows one to conclude "at a glance" that a group is infinite. We will state his result precisely, as a corollary of our main theorem.

We generalize Thomas's theorem by obtaining information about the ranks of the abelianizations of certain subgroups of a group G, if G has a presentation with at least one more generator than relators (when relators which are powers are counted as "fractions of relators"). The proof is simple and geometric; our principal tool is elementary homology theory.

In [**2**] G. Bergman has obtained a further generalization of Thomas's theorem, which allows one to obtain results similar to ours for groups with somewhat more complicated presentations than the ones we treat. His techniques are algebraic and considerably more sophisticated than ours. He also surveys related results.

This paper is derived from part of the author's Ph.D. dissertation [**1**].

2. *Theorem and Proof*

Theorem. *Let G be a group with presentation*

$$\langle a_1 \ldots a_n \,|\, 1 = w_1^{r_1} = \cdots = w_m^{r_m} \rangle$$

where each w_j is a word in the a_i and their inverses. Suppose that H is a normal subgroup of G of index $N < \infty$ and that for each j, $w_j^k \notin H$ for $k = 1, \ldots, r_j - 1$. Then the rank of the abelianization of H is at least

$$1 + N \left(n - 1 - \sum_i \frac{1}{r_i} \right).$$

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Proof. We begin by building the 2-dimensional CW complex K associated to the presentation. K has one 0-cell, n 1-cells, and m 2-cells; the 1-cells are identified with the generators a_i of G, and the 2-cells are attached according to the relators $w_j^{r_j}$. The fundamental group of K is isomorphic to G. The Euler characteristic $\chi(K)$ is equal to $1-n+m$, and so the covering space $\tilde{K}$ associated to the subgroup H of G has Euler characteristic $\chi(\tilde{K}) = N(1-n+m)$. Since $\tilde{K}$ is connected and only has cells in dimensions ≤ 2, $H_0(\tilde{K}) \cong \mathbf{Z}$ and $H_i(\tilde{K}) = 0$ for $k > 2$. The abelianization of H is isomorphic to $H_1(\tilde{K})$; we will estimate the rank of $H_2(\tilde{K})$ and this will give us the needed information about the rank of $H_1(\tilde{K})$.

Take a close look at $\tilde{K}$. Its 1-skeleton is a copy of the Cayley graph Γ of G/H with respect to the generating set $\{a_1, \ldots, a_n\}$. Attached to it are Nm 2-cells, since there are N lifts of each 2-cell of K. For each vertex v of Γ and for each j there is a 2-cell of $\tilde{K}$ whose boundary is the closed edge-path beginning at v and tracing out the path $w_j^{r_j}$. Fix j and observe that the 2-cells associated to the vertices $v \cdot w_j^k$ (as k varies from 1 through r_j) all have the same boundary. Furthermore, since $w_j^k \notin H$ for each $k = 1, \ldots, r_j$, these are distinct vertices and so we obtain r_j disks all sharing a common boundary. There is one such set of r_j disks for every set $\{v \cdot w_j^k \mid k = 1, \ldots, r_j\}$, so there are a total of N/r_j such sets.

We may form a new complex L from $\tilde{K}$ by removing $r_j - 1$ two-cells from each such set of r_j, for each $j = 1, \ldots, m$. We can rebuild $\tilde{K}$ from L by simply replacing the removed 2-cells. Since the boundary of each adjoined disk is already a boundary in L, a simple Mayer-Vietoris argument shows that adjoining any of these 2-cells to L increases the rank of the second homology of the complex. Therefore $\mathrm{rank}(H_2(\tilde{K}))$ is at least equal to the number of 2-cells added to L to obtain $\tilde{K}$. For each j, there are N/r_j sets of disks, and so after summing over j we obtain

$$\mathrm{rank}\,(H_2(\tilde{K})) \geq \sum_{j=1}^{m} \frac{N}{r_j}(r_j - 1).$$

Since

$$\chi(\tilde{K}) = N(1-n+m) = \mathrm{rank}\ H_0(\tilde{K}) - \mathrm{rank}\ H_1(\tilde{K}) + \mathrm{rank}\ H_2(\tilde{K}),$$

we see that

$$N(1-n+m) \geq 1 - \mathrm{rank}\ H_1(\tilde{K}) + \sum_{j=1}^{m} N(1 - 1/r_j).$$

Rearranging terms, we obtain the claimed result. ∎

Corollary [**4**]. *Let G be a group with presentation*

$$\langle a_1 \ldots a_n \mid 1 = w_1^{r_1} = \cdots = w_m^{r_m} \rangle$$

where each of the w_j is a word in the a_i and their inverses, and where the order in G of each w_j is actually r_j (i.e., no collapsing takes place). Then G is infinite if

$$n \geq 1 + \frac{1}{r_1} + \cdots + \frac{1}{r_m}.$$

Proof. If G were finite then the trivial subgroup would have finite index. The theorem would imply that this subgroup had abelianization of positive rank, which is absurd. ∎

3. *Examples*

We can use the theorem to study the groups (p, q, r) for $1/p + 1/q + 1/r \leq 1$. A presentation for this group G is

$$\langle a, b \mid 1 = a^p = b^q = (ab)^r \rangle.$$

The usual proof that $|G| = \infty$ involves realizing (p, q, r) as a group of isometries of either the Euclidean plane or hyperbolic plane. We proceed instead by exhibiting a representation of (p, q, r) to a finite group such that the images of a, b and ab have orders p, q and r, respectively. The corollary then yields the order of G, and we can use the theorem to estimate the ranks of the abelianizations of subgroups of G. Let F be a finite field containing primitive $2p$th, $2q$th and $2r$th roots of unity. (One can construct such a field by taking a prime ℓ not dividing $2p$, $2q$ or $2r$, and then taking F to be the field of order ℓ^n where ℓ^n is congruent to 1 modulo each of $2p$, $2q$ and $2r$.) Let α, β and γ be such roots. Let A and B be the matrices

$$A = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha^{-1} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \beta & 0 \\ x & \beta^{-1} \end{pmatrix},$$

where we will choose x later. Since the roots of the characteristic equations for A and B lie in F and are distinct, the matrices are diagonalizable and have orders $2p$ and $2q$. One checks that the characteristic polynomial of AB is (as a polynomial in the symbol λ)

$$\lambda^2 + \lambda(-\alpha\beta - \alpha^{-1}\beta^{-1}) + 1 - \lambda x.$$

We may obviously choose x so that γ is a root of this polynomial, so AB has an eigenvalue γ and thus is conjugate to an upper-triangular matrix M. Since AB has determinant 1, the diagonal entries of M are γ and γ^{-1}. Since these are distinct and lie in F, AB is diagonalizable and has order $2r$. Working modulo $\{\pm I\}$, we find that the images of A, B, and AB in $PSL_2(F)$ have orders p, q, and r. Therefore the map $a \mapsto A$ and $b \mapsto B$ defines a homomorphism from G to $PSL_2(F)$. We also see that a, b and ab have orders p, q and r in G, and by the corollary we conclude that (p, q, r) is infinite.

Now we use our theorem to estimate the ranks of abelianizations of subgroups of the kernel K of this representation. If $H \subseteq K$ is a normal subgroup of G then in the "Euclidean" case $(1/p + 1/q + 1/r = 1)$, we find that the abelianization of H has rank at least 1, and that in the "hyperbolic" case $(1/p+1/q+1/r < 1)$, the estimate of the rank grows linearly with the index of the subgroup. One can show that in the Euclidean case, $K \cong \mathbf{Z}^2$ and so actually has rank 2; in the hyperbolic case, K is isomorphic to the fundamental group of a surface of genus > 1 and so its proper subgroups are fundamental groups of surfaces of still larger genus, and so have larger abelianizations. Our theorem provides a purely algebraic approach to this topological fact.

For $n \geq 3$, the presentation $\langle a, b \,|\, 1 = a^n = a^{n+1} = b^n = b^{n+1} \rangle$ of the trivial group shows that the no-collapsing hypothesis of the corollary is essential. In [**4**], Thomas gives a much more subtle presentation of the trivial group that also shows the need for the the no-collapsing hypothesis.

REFERENCES

[**1**] D. J. Allcock. The Leech Lattice and Hyperbolic Geometry. Ph.D. thesis. U.C. Berkeley (1996).

[**2**] G. Bergman. Infinite groups arising from partial presentations of finite groups. Preprint (1996).

[**3**] H. S. M. Coxeter and W. O. J. Moser. *Generators and Relations for Discrete Groups.* Springer-Verlag, 1984.

[**4**] R. Thomas. Cayley graphs and group presentations. *Proc. Camb. Phil. Soc.* **103**(1988), 385–7.