LINEAR ALGEBRAIC GROUPS WITHOUT THE NORMALIZER THEOREM

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Abstract. One can develop the basic structure theory of linear algebraic groups (the root system, Bruhat decomposition, etc.) in a way that bypasses several major steps of the standard development, including the self-normalizing property of Borel subgroups.

An awkwardness of the theory of linear algebraic groups is that one must develop a lot of material about general linear algebraic groups before one can really get started. Our goal here is to show how to develop the root system, etc., using only the completeness of the flag variety and some facts about solvable groups. In particular, one can skip over the usual analysis of Cartan subgroups, the fact that G is the union of its Borel subgroups, the connectedness of torus centralizers, and the normalizer theorem (i.e., a Borel subgroup is self-normalizing). The main idea is a new approach to the structure of rank 1 groups; the key step is lemma 5.

All algebraic geometry is over a fixed algebraically closed field. G always denotes a connected linear algebraic group with Lie algebra $\mathfrak{g}$, T a maximal torus, and B a Borel subgroup containing it. We assume the structure theory for connected solvable groups, and the completeness of the flag variety G/B and some of its consequences. Namely: that all Borel subgroups (resp. maximal tori) are conjugate; that G is nilpotent if one of its Borel subgroups is nilpotent; that $C_G(T)_0$ lies in every Borel subgroup containing T; and that $N_G(B)$ contains B of finite index and (therefore) is self-normalizing. We also assume known that the centralizer of a torus has the expected dimension, namely, that of the subspace of $\mathfrak{g}$ where the torus acts trivially. For these results we refer to Borel [1], Humphreys [2] and Springer [3].

In section 1 we develop a few properties of solvable groups, and in section 2 we treat the structure of rank 1 groups. The root system, etc., can then be developed in essentially the standard way, so after

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the rank 1 analysis we restrict ourselves to brief comments. We are grateful to J. Humphreys, G. McNinch and T. Springer for their helpful comments.

1. Lemmas about solvable groups

First we recall from [3, §13.4] the groups that we call the positive and negative subgroups of G. Fix a 1-parameter group $\phi : \mathbb{G}_m \to G$. For $g \in G$ and $\lambda \in \mathbb{G}_m$, define $\mathrm{Cl}_g(\lambda) = \phi(\lambda) g \phi(\lambda)^{-1}$ and

$$G^+ = \left\{ g \in G \;\middle|\; \lim_{\lambda \to 0} \mathrm{Cl}_g(\lambda) = 1 \right\}.$$

Of course, the condition $\lim_{\lambda\to 0} \mathrm{Cl}_g(\lambda) = 1$ means that $\mathrm{Cl}_g : \mathbb{G}_m \to G$ extends to a regular map $\mathbb{G}_m \cup \{0\} \to G$ sending 0 to 1. We call G^+ the positive part of G (with respect to ϕ). It is a group because

$$\lim_{\lambda \to 0} \phi(\lambda) g g' \phi(\lambda)^{-1} = \lim_{\lambda \to 0} \phi(\lambda) g \phi(\lambda)^{-1} \cdot \lim_{\lambda \to 0} \phi(\lambda) g' \phi(\lambda)^{-1}$$

and

$$\lim_{\lambda \to 0} \phi(\lambda) g^{-1} \phi(\lambda)^{-1} = \Bigl(\lim_{\lambda \to 0} \phi(\lambda) g \phi(\lambda)^{-1} \Bigr)^{-1}$$

when the limits on the right hand sides exist. It is closed and connected because it is generated by irreducible curves containing 1. Of course, there is a corresponding subgroup G^- got by considering limits as λ approaches ∞. All of our discussion applies equally well to G^-. The key properties of $G^\pm$ are that they are unipotent and "large":

Proposition 1. *G^+ is unipotent, and every weight of $\mathbb{G}_m$ on the Lie algebra of G^+ is positive. If G is solvable, then the Lie algebra of G^+ contains all the positive weight spaces for $\mathbb{G}_m$ on $\mathfrak{g}$.*

We remark that the solvability hypothesis in the last part is unnecessary. The general case requires the structure theory, which depends on theorem 3, which in our development depends on the solvable case.

Proof. See [3, theorem 13.4.2] for the first claim; the idea is to embed G in GL_n and diagonalize the $\mathbb{G}_m$ subgroup. We will prove the second claim, since the proof in [3] includes the nonsolvable case and therefore relies on properties of the root system. We will use induction on $\dim G_u$; the 0-dimensional case is trivial.

So suppose $\dim G_u > 0$ and choose a connected subgroup N of G_u, normal in G, with $G_u/N \cong \mathbb{G}_a$. We write π for the associated map $G_u \to \mathbb{A}^1$. By the action of $\mathbb{G}_m$ on G_u/N, we have $\pi \circ \mathrm{Cl}_g(\lambda) = \pi(g) \cdot \lambda^n$ for some $n \in \mathbb{Z}$. If $n \leq 0$ then $\pi \circ \mathrm{Cl}_g$ does not extend to a regular map $\mathbb{G}_m \cup \{0\} \to \mathbb{A}^1$ sending 0 to 0 unless $\pi(g) = 0$. Therefore a group element outside N cannot lie in G^+. So $G^+ = N^+$ and we use induction.

So suppose $n > 0$, i.e., the action is by a positive character. Choose a linear representation V of $\mathbb{G}_m$ and an embedding (as a variety) of G_u into PV which is equivariant with respect to the conjugation action of $\mathbb{G}_m$. Choose a $\mathbb{G}_m$-invariant linear subspace of PV containing $1 \in G_u$, whose tangent space there is complementary to that of N. By dimension considerations, its intersection with G_u contains an invariant curve C that passes through 1 and does not lie in N. By passing to a component we may assume that C is irreducible, so it is the closure of the orbit of some $g \in C$. The map $\mathrm{Cl}_g : \mathbb{G}_m \to G_u$ extends to a regular map from P^1 to PV. Because 1 lies in the closure of the orbit, Cl_g sends at least one of 0 or ∞ to 1. It cannot send ∞ to 1, because $\pi \circ \mathrm{Cl}_g(\lambda) = (\text{nonzero constant}) \cdot \lambda^n$ for some $n > 0$, which admits no regular extension $\mathbb{G}_m \cup \{\infty\} \to \mathbb{A}^1$. Therefore $\lim_{\lambda\to 0} \mathrm{Cl}_g(\lambda) = 1$, so $g \in G^+$. This shows that G^+ projects onto G_u/N. G^+ also contains N^+, to which the inductive hypothesis applies. The proposition follows. □

An immediate consequence is that a connected solvable group G is generated by its subgroups G^+, G^- and $C_G(\phi(\mathbb{G}_m))_0$, since together their Lie algebras span $\mathfrak{g}$. Next, we need a theorem on orbits of solvable groups. The structure theorem for solvable groups is not needed for this result, and can even be derived from it.

Theorem 2. *If G is solvable and acts on a variety, then no orbit contains any complete subvariety of dimension > 0.*

Proof. We assume the result known for solvable groups of smaller dimension than G. (The 0-dimensional case is trivial.) It suffices to treat the case that G acts transitively on the variety, say X. If $x \in X$ has stabilizer G_x, then the natural map $G/G_x \to X$ is generically finite (since G/G_x and X have the same dimension), hence finite (by homogeneity). If X contained a complete subvariety of positive dimension then the preimage in G/G_x would also be complete. So it suffices to treat the case $X = G/G_x$.

We consider two cases. First, if G_x surjects to $G/[G,G]$, then $[G,G]$ acts transitively on G/G_x, and by the inductive hypothesis applied to $[G,G]$, G/G_x cannot contain a complete subvariety of positive dimension. Second, suppose G_x does not surject to $G/[G,G]$, and set H equal to the group generated by G_x and $[G,G]$. We will use the fact that G/G_x maps to G/H with fibers that are copies of H/G_x. Since H is normal in G, G/H is an affine variety, so any complete subvariety is a finite set of points. Therefore any complete subvariety of G/G_x lies in the union of finitely many copies of H/G_x. But the inductive hypothesis applied to H shows that every complete subvariety of H/G_x is a finite set of points. Therefore the same conclusion applies to G/G_x. □

2. Rank One Groups

In this section, G is connected and non-solvable of rank 1. The goal is:

Theorem 3. *G modulo its unipotent radical admits an isogeny to* PGL_2.

There is a standard argument that reduces this to proving that T lies in exactly two Borel subgroups. We must modify this slightly because we are not assuming the normalizer theorem. Since $N := N_G(B)$ is self-normalizing, it fixes only one point of G/N, so the stabilizers of distinct points of G/N are the normalizers of distinct Borel subgroups. The fixed points of T in G/N correspond to Borel subgroups that T normalizes, hence lies in. Now we use the theorem that a torus acting on a d-dimensional projective variety has at least $d+1$ fixed points. Since G is not solvable, G/B has dimension > 0, so G/N does too. Therefore T lies in at least two Borel subgroups. And if we prove that it lies in exactly two, then we can also deduce $\dim G/N = 1$. Then it is easy to see that $G/N \cong P^1$ and derive theorem 3. So our aim is to prove that T lies in exactly two Borel subgroups.

Using the positive and negative subgroups, we will construct two Borel subgroups containing T, and then show that there are no more. Suppose $\phi : \mathbb{G}_m \to T$ is a parametrization of T (meaning ϕ is an isomorphism) and B a Borel subgroup containing T. Call B positive (with respect to ϕ) if it contains G^+ and negative if it contains G^-. Obviously, B is positive with respect to one parametrization of T if and only if it is negative with respect to the other. Here are the basic properties of positive and negative Borel subgroups.

Lemma 4. *Suppose $\phi : \mathbb{G}_m \to T$ a parametrization of the maximal torus T. Then*

(1) *T lies in a positive and in a negative Borel subgroup;*
(2) *if B (resp. B') is a positive (resp. negative) Borel subgroup containing T, then every Borel subgroup containing T lies in $\langle B, B' \rangle$;*
(3) *no Borel subgroup containing T is both positive and negative;*
(4) *$N_G(T)$ contains an element acting on T by inversion.*

Proof. (1) G^+ is connected, unipotent and normalized by T. Therefore TG^+ lies in some Borel subgroup, which is then positive. And similarly for G^-.

(2) Suppose B'' is a Borel subgroup containing T. Then $B''^+ \subseteq G^+$ lies in B since B is positive, $B''^- \subseteq G^-$ lies in B' since B' is negative, and $C_{B''}(T)_0$ lies in both B and B' because $C_G(T)_0$ lies in every Borel subgroup containing T. Since B'' is generated by B''^+, B''^- and $C_{B''}(T)_0$, it lies in $\langle B, B' \rangle$.

(3) If a Borel subgroup B containing T were both positive and negative, then (2) would imply that it is the only Borel subgroup containing T, contradicting the fact that T lies in at least 2 of them.

(4) By (1), positive and negative Borel subgroups exist, and by (3) they are distinct. The result follows from the fact that $N_G(T)$ acts transitively on the Borel subgroups containing T. □

Now we can give the key step in our approach to the structure theorem for rank 1 groups.

Lemma 5. *Every maximal torus of G lies in exactly two Borel subgroups, one positive and one negative.*

Proof. Choose a maximal torus T and a parametrization of it. We will use induction on the dimension of a Borel subgroup B. If this is 1 then B is abelian, so $G = B$. Therefore the base case is dimension 2. We already know that T lies in a positive and in a negative Borel subgroup. The key point is that any two positive Borel subgroups coincide. For otherwise their unipotent radicals would be distinct subgroups of G^+, hence generate a unipotent group of dimension > 1. This is impossible because $\dim B_u = 1$. Similarly, there is only one negative Borel subgroup and the base case is proven.

Now we prove the inductive step; suppose B has dimension at least 3. We may suppose without loss of generality that B contains T and is positive. Consider the action of B on G/N; there is a unique fixed point because the only Borel subgroup that B normalizes is itself. Next consider an orbit of minimal positive dimension. By theorem 2, it contains no complete subvarieties of dimension > 0. On the other hand, its closure is complete and is got by adjoining lower-dimensional orbits. By minimality, this means that its closure is got by adjoining a single point, so the orbit is a curve. Therefore there exists a Borel subgroup B' for which $B \cap N(B')$ has codimension 1 in B. That is, $I := (B \cap B')_0$ has codimension 1 in each of B and B'. There are two possibilities: $I = B_u = B'_u$, or I contains a torus. In the first case, $\langle B, B' \rangle$ normalizes I, and a Borel subgroup in $\langle B, B' \rangle / I$ has no unipotent part. This forces $\langle B, B' \rangle$ to be solvable, which is impossible.

Therefore I contains a torus, and I_u has codimension 1 in each of B_u and B'_u. By replacing B' and I by their conjugates by an element of B, we may suppose without loss of generality that B' contains T. Now, T normalizes B_u and B'_u, hence their intersection, hence I_u. Also, B_u normalizes I_u because it is only one dimension larger and is nilpotent. Similarly for B'_u. Therefore $\langle B, B'\rangle$ normalizes I_u, which has dimension > 0 since $\dim B > 2$. We apply induction to $\langle B, B'\rangle/I_u$ and then pull back to conclude the following. B and B' are the only Borel subgroups of $\langle B, B'\rangle$ containing T, and they are exchanged by an element of $N_{\langle B,B'\rangle}(T)$ that inverts T. The latter implies that B' is a negative Borel subgroup of G. Finally, lemma 4(2) implies that any Borel subgroup of G containing T lies in $\langle B, B'\rangle$, hence equals B or B'. □

This lemma implies theorem 3, and from then on one can follow the standard development. We make only the following remarks.

Bruhat decomposition: in the absence of the normalizer theorem, one should define the Weyl group W as the subgroup of $N_G(T)/C_G(T)$ generated by the reflections coming from roots. Then one can prove $G = BWB$ as in [1, §14], [2, §28] or [3, §8.3].

Normalizers: The theorem $N_G(B) = B$ follows immediately from the Bruhat decomposition, and implies that W, as defined here, is all of $N_G(T)/C_G(T)$, so that our definition agrees with the usual one.

Connectedness of torus centralizers: this can be deduced from the Bruhat decomposition and a standard fact about reflection groups: the pointwise stabilizer of a linear subspace is generated by the reflections that fix it pointwise.

References

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