NORMALIZERS OF PARABOLIC SUBGROUPS OF COXETER GROUPS

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Abstract. We improve a bound of Borcherds on the virtual cohomological dimension of the non-reflection part of the normalizer of a parabolic subgroup of a Coxeter group. Our bound is in terms of the types of the components of the corresponding Coxeter subdiagram rather than the number of nodes. A consequence is an extension of Brink’s result that the non-reflection part of a reflection centralizer is free. Namely, the non-reflection part of the normalizer of parabolic subgroup of type $D_5$ or $A_{m\text{ odd}}$ is either free or has a free subgroup of index 2.

Suppose $\Pi$ is a Coxeter diagram, $J$ is a subdiagram and $W_J \subseteq W_\Pi$ is the corresponding inclusion of Coxeter groups. The normalizer $N_{W_\Pi}(W_J)$ has been described in detail by Borcherds [3] and Brink-Howlett [5]. Such normalizers have significant applications to working out the automorphism groups of Lorentzian lattices and K3 surfaces; see [3] and its references. $N_{W_\Pi}(W_J)$ falls into 3 pieces: $W_J$ itself, another Coxeter group $W_\Omega$, and a group $\Gamma_\Omega$ of diagram automorphisms of $W_\Omega$. The last two groups are called the “reflection” and “non-reflection” parts of the normalizer. Borcherds bounded the virtual cohomological dimension of $\Gamma_\Omega$ by $|J|$. Our theorems 1, 3 and 4 give stronger bounds, in terms of the types of the components of $J$ rather than the number of nodes. There are choices involved in the definition of $W_\Omega$ and $\Gamma_\Omega$, and our bound in theorem 3 applies regardless of how these choices are made (theorem 1 is a special case). Theorem 4 improves this bound when $W_\Omega$ is “maximal”. In this case, when $J = D_5$ or $A_{m\text{ odd}}$, $\Gamma_\Omega$ turns out to either be free or have an index 2 subgroup that is free. This extends Brink’s result [4] that $\Gamma_\Omega$ is free when $J = A_1$.

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We follow the notation of [3], and refer to [6] for general information about Coxeter groups. Suppose $(W_\Pi, \Pi)$ is a Coxeter system, which is

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to say that $W_\Pi$ is a Coxeter group and $\Pi$ is a standard set of generators. The Coxeter diagram is the graph whose nodes are $\Pi$, with an edge between $s_i, s_j \in \Pi$ labeled by the order $m_{ij}$ of $s_is_j$, when $m_{ij} > 2$. $W_\Pi$ acts isometrically on a real inner product space $V_\Pi$ with basis (the simple roots) $\Pi$ and inner products defined in terms of the $m_{ij}$. The (open) Tits cone $K$ is an open convex subset of $V_\Pi^*$ on which $W_\Pi$ acts properly discontinuously with fundamental chamber $C_\Pi$. (Our $C_\Pi$ and $K$ are “missing” the faces corresponding to infinite parabolic subgroups of $W_\Pi$.) The standard generators act on $V_\Pi^*$ by reflections across the hyperplanes containing the facets of $C_\Pi$, and they also act on $V_\Pi$ by reflections. For a root $\alpha$ (i.e., a $W_\Pi$-image of a simple root) we write $\alpha^\perp$ for $\alpha$’s mirror, meaning the fixed-point set in $K$ of the reflection associated to $\alpha$.

Now let $J \subseteq \Pi$ be a spherical subdiagram, i.e., one corresponding to a finite subgroup of $W_\Pi$, and let $W_{\text{min}}$ be the group generated by the reflections in $W_\Pi$ that act trivially on $V_J \subseteq V_\Pi$. This is the “reflection” part of $N_{W_\Pi}(W_J)$, or rather the strictest possible interpretation of this idea. It corresponds to Borcherds’ $W_\Omega$ in the case that the groups he calls $\Gamma_\Pi$ and $\Gamma_J$ are trivial; see the discussion after lemma 2. Let $J^\perp := \cap_{\alpha \in J} \alpha^\perp$, pick a component $C_\text{min}$ of the complement of $W_{\text{min}}$’s mirrors in $J^\perp$, and define $C_{\text{min}}$ as its closure (in $J^\perp$). By definition, $W_{\text{min}}$ is a Coxeter group, and the general theory of these groups shows that $C_{\text{min}}$ is a chamber for it. The “non-reflection” part of $N_{W_\Pi}(W_J)$ means the subgroup $\Gamma_\text{min}$ of $W_\Pi$ preserving $J$ (regarded as a set of roots) and sending $C_{\text{min}}$ to itself. The reason for the first condition is to discard the trivial part of $N_{W_\Pi}(W_J)$, namely $W_J$ itself. That is, $W_{\text{min}}:\Gamma_\text{min}$ is a complement to $W_J$ in $N_{W_\Pi}(W_J)$. We write $\Gamma_\text{min}^\vee$ for the subgroup of $\Gamma_{\text{min}}$ acting trivially on $J$ (equivalently, on $V_J$). The reason for passing to this (finite-index) subgroup is that $\Gamma_{\text{min}}$ often contains torsion and therefore has infinite cohomological dimension for boring reasons.

**Theorem 1.** $\Gamma_\text{min}^\vee$ acts freely on a contractible cell complex of dimension at most

$$\#A_1 + \#D_{m>4} + \#E_6 + \#I_2(5) + 2(\#A_{m>1} + \#D_4)$$

where $\#X_m$ means the number of components of $J$ isomorphic to a given Coxeter diagram $X_m$. In particular, $\Gamma^\vee_{\text{min}}$’s cohomological dimension is at most (1).

Borcherds’ result [3, thm. 4.1] has $|J|$ in place of (1), but treats a more general group $\Gamma_\Omega$, of which $\Gamma_{\text{min}}$ is a special case. The more general case follows from this one, in theorem 3 below.
Proof. First we prove for \( x \in C_{\min}^\circ \) that its stabilizer \( \Gamma_{\min,x}^\vee \) is trivial. The \( W_\Pi \)-stabilizer of \( x \) is some \( W_\Pi \)-conjugate \( W_x \) of a spherical parabolic subgroup of \( W_\Pi \). So \( W_x \) acts on \( V_\Pi \) as a finite Coxeter group. It is well-known that any vector stabilizer in such an action is generated by reflections, so the subgroup \( W_{x,J} \) fixing \( J \) pointwise is generated by reflections. Observe that any reflection in \( W_{x,J} \) lies in \( W_{\min} \). Since \( x \) lies in the interior \( C_{\min}^\circ \) of \( C_{\min} \), it is fixed by no reflection in \( W_{\min} \), so there can be no reflection in \( W_{x,J} \), so \( W_{x,J} = 1 \). It is easy to see that \( W_{x,J} \) contains \( \Gamma_{\min,x}^\vee \), so we have proven that \( \Gamma_{\min}^\vee \) acts freely on \( C_{\min}^\circ \).

\( C_{\min}^\circ \) is contractible because it is convex, and it obviously admits an equivariant deformation-retraction to its dual complex. So it suffices to show that the dual complex has dimension at most \( (1) \). Suppose \( \phi \subseteq J^\perp \) is a face of a chamber of \( W_\Pi \), with codimension in \( J^\perp \) larger than \( (1) \); we must show \( \phi \cap C_{\min}^\circ = \emptyset \). For some \( w \in W_\Pi \), \( w\phi \) is a face of \( C_\Pi \) whose corresponding set of simple roots \( I' \subseteq \Pi \) contains \( J' := w(J) \cong J \). By the codimension hypothesis on \( \phi \), \( |I'| - |J'| \) is more than \( (1) \). Applying the lemma below to \( J' \) and \( I' \), we see that \( W_{I'} \) contains a reflection \( r \) fixing \( J' \) pointwise. Since \( r \in W_{I'} \), its mirror contains \( w\phi \). So \( w^{-1}rw \) is a reflection fixing \( J \) pointwise (so it lies in \( W_{\min} \)), whose mirror contains \( \phi \). Since \( C_{\min}^\circ \) is a component of the complement of the mirrors of \( W_{\min} \), it is disjoint from \( \phi \), as desired. □

Lemma 2. If \( J \) lies in a spherical Coxeter diagram \( I \subseteq \Pi \), whose cardinality exceeds that of \( J \) by more than \( (1) \), then \( W_I \) contains a reflection fixing \( J \) pointwise.

Remark. Equality in \( (1) \) holds when \( I \) extends the \( A_m \), \( D_m \), \( E_6 \) and \( I_2(5) \) components of \( J \) by \( A_1 \rightarrow A_2 \), \( A_{m>1} \rightarrow D_{m+2} \), \( D_4 \rightarrow E_6 \), \( D_{m>4} \rightarrow D_{m+1} \), \( E_6 \rightarrow E_7 \) and \( I_2(5) \rightarrow H_3 \). One can check in this case that the conclusion of the lemma fails.

Proof. We may suppose \( I = \Pi \), by discarding the rest of \( \Pi \). Working one component at a time, it suffices to prove the lemma under the additional hypothesis that \( \Pi \) is connected. We now consider the various possibilities for \( \Pi \), and suppose \( W_\Pi \) contains no reflections fixing \( V_J \) pointwise. That is, we assume \( W_{\min} = 1 \). In each case we will show that \( |\Pi| - |J| \) is at most \( (1) \).

The \( \Pi = A_n \) case is a model for the rest. Suppose the component of \( J \) nearest one end of \( \Pi \) has type \( A_m \) and does not contain that end. Then it must be adjacent to that end (since \( W_{\min} = 1 \)), so together with the end it forms an \( A_{m+1} \). We conjugate by the long word in \( W(A_{m+1}) \), which exchanges the two \( A_m \) diagrams in \( A_{m+1} \) and fixes the roots in the other components of \( J \). The result is that we may suppose without loss that \( J \) contains that end of \( \Pi \). Repeating the argument to move
the other components of $J$ toward that end, we may suppose that there is exactly one node of $\Pi$ between any two consecutive components of $J$. And the other end of $\Pi$ is either in $J$ or adjacent to it. It is now clear that $|\Pi| - |J|$ is the number of components of $J$, or one less than this. Since every component of $J$ has type $A$, $|\Pi| - |J|$ is at most (1). This finishes the proof in the $\Pi = A_n$ case.

If $\Pi = B_n = C_n$ then we begin by shifting any type $A$ components of $J$ as far as possible from the double bond. If $J$ has no $B_m$ then $J$ contains one end of the double bond, and we get $|\Pi| - |J|$ equal to the number of components of $J$, all of which have type $A$. If $J$ has a $B_m$ then the node after it (if there is one) must be adjacent to some type $A$ component of $J$. This is because $W(B_{m+1})$ contains a reflection acting trivially on $V_{B_m}$. This is easy to see in the model of $W(B_{m+1})$ as the isometry group of $\mathbb{Z}^{m+1}$. It follows that $|\Pi| - |J|$ is the number of components of $J$ of type $A$.

In the $\Pi = D_{n>3}$ case, one can use the shifting trick to reduce to one of the cases

\[
\begin{array}{cccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

where the filled nodes are those in $J$ and the dashes indicate a chain of nodes with no two consecutive unfilled nodes. (Except for the dashes on the left in the last 3 diagrams, which indicate chains of filled nodes.)

In every case we get

$$|\Pi| - |J| \leq \#A_1 + \#D_{m\geq 4} + 2 \#A_{m>1}.$$ 

The most interesting case is $A_{n-2} \rightarrow D_n$, at the top left.

We will treat the case $\Pi = E_8$ and leave the similar $E_6$ and $E_7$ cases to the reader. If $J$ has a $D_4$, $D_5$ or $E_6$ component, then it must also have a type $A$ component, and then $|\Pi| - |J| \leq 2 \#D_4 + \#D_5 + \#A_{m \geq 1}$, as desired. $J$ cannot be $D_6$ or $E_7$, because then $W_{\min}$ would contain the reflection in the lowest root of $E_8$, which extends $E_8$ to the affine diagram $\tilde{E}_8$. So we may suppose $J$’s components have type $A$. In order for $|\Pi| - |J|$ to exceed (1), we must have $J = A_{m \leq 5}$, $A_3A_1$, $A_2A_1$ or $A_1^m \leq 3$. But none of these cases can occur, because in each of them we may shift $J$’s components around so that some node of $\Pi$ is not joined to $J$.

The remaining cases are $\Pi = F_4$, $H_3$, $H_4$ and $I_2$, the last case including $G_2 = I_2(6)$. The facts required to treat these cases are that if $J = B_2$ or $B_3$ in $\Pi = F_4$ then $W_{\min}$ contains a reflection, and similarly in the $J = H_3 \subseteq H_4 = \Pi$ case. The first fact is visible inside a $B_3$ or
B_4 root system inside F_4. To see the second, observe that the root stabilizer in H_4 contains Coxeter groups of types A_2 and I_2(5), visible in the centralizers of the two end reflections of H_4 (which are conjugate). So the root stabilizer can only be W(H_3), which is to say that the H_3 root system is orthogonal to a root. □

The greater generality obtained by Borcherds is the following. Let Γ_{Π} be a group of diagram automorphisms of Π, acting on V_{Π} and K in the obvious way. The goal is to understand \( N_{W_ΠΓ_{Π}}(W_J) \). Again we discard the boring part of this normalizer by passing to the subgroup \( W_J' \) preserving the set of roots \( J \subseteq Π \). Let \( W_{Ω} \) be any subgroup of \( W_J' \) which contains \( W_{min} \) and is generated by elements which act on \( J^⊥ \) by reflections. We define \( C^Ω_{Ω}, C_{Ω} \) and \( Γ_{Ω} \) as for \( C^min_{min}, C_{min} \) and \( Γ_{min} \), and define \( Γ^Ω_{min} \) as the subgroup of \( Γ_{Ω} \cap W_Π \) acting trivially on \( J \). (Borcherds left \( Γ^Ω_{min} \) unnamed and defined \( W_{Ω} \) in terms of auxiliary groups \( R \lhd Γ_J \subseteq Aut J \); his \( W_{Ω} \) has the properties assumed here.) The inclusion \( W_{min} \subseteq W_{Ω} \) is the source of the subscript “min”, but note that \( C_{min} \) and \( Γ_{min} \) are larger than \( C_{Ω} \) and \( Γ_{Ω} \). We can now recover Borcherds’ result [3, thm. 4.1] with our (1) in place of \(|J|\).

Theorem 3. Theorem 1 holds with \( Γ^Ω_{min} \) replaced by \( Γ^Ω_{min} \).

Proof. The freeness of the action follows from the same argument. (This is why \( Γ^Ω_{Ω} \) is defined as a subgroup of \( Γ_{Ω} \cap W_Π \) rather than just \( Γ_{Ω}. \)) The essential point for the rest of the proof is that \( W_{Ω} \) contains \( W_{min} \), so the decomposition of \( J^⊥ \) into chambers of \( W_{Ω} \) refines that of \( W_{min} \). This shows \( C^Ω_{Ω} \subseteq C_{min}^Ω \). So the dual complex of \( C^Ω_{Ω} \) has dimension at most that of \( C_{min}^Ω \), and we can apply theorem 1. □

The point of considering \( W_{Ω} \) rather than \( W_{min} \) is that it is larger and so \( Γ_{Ω} \) will be smaller than \( Γ_{min} \). This is good since the nonreflection part is more mysterious than the reflection part. So it is natural to define \( W_{max} \) by setting \( Γ_{Π} = 1 \) and taking \( W_{Ω} \) as large as possible, i.e., \( W_{max} \) is the subgroup of \( W_J' \) generated by the transformations which act on \( J^⊥ \) by reflections.

This is the largest possible “universal” \( W_{Ω} \), although a larger \( W_{Ω} \) is possible if Π admits suitable diagram automorphisms. For example, \( Γ_{Π} \) might contain elements acting on \( C_{Π} \) by reflections. I don’t know other examples, although probably there are some.

We define \( C_{max}^Ω, C_{max}, Γ_{max} \) and \( Γ^Ω_{max} \) as above. The next theorem follows from lemma 5 in exactly the same way that theorem 1 follows from lemma 2.
Theorem 4. The dimension of the dual complex of $C_{\text{max}}^\circ$, hence the cohomological dimension of $\Gamma_{\text{max}}^\vee$, is bounded above by

(3) \[ \#D_5 + \#A_{m \text{ odd}} + 2 \#A_{m \text{ even}}. \]

Remarks. (i) If $J$ has no $A_m$ or $D_5$ component then $\Gamma_{\text{max}}^\vee = 1$ and $\Gamma_{\text{max}}$ is finite. This is Borcherds’ [3, example 5.6]. (ii) If $J = D_5$ or $A_{m \text{ odd}}$ then $\Gamma_{\text{max}}^\vee \subseteq N_{W_I}(W_J)$ is free. Also, since $|\text{Aut } J| \leq 2$, $\Gamma_{\text{max}}^\vee$ has index 1 or 2 in $\Gamma_{\text{max}}$. Therefore the non-reflection part $\Gamma_{\text{max}}$ of $N_{W_I}(W_J)$ has a free subgroup of index 1 or 2. (iii) If $J = A_1$ then $\Gamma_{\text{min}}^\vee = \Gamma_{\text{min}} = \Gamma_{\text{max}} = \Gamma_{\text{max}}^\vee$ has cohomological dimension $\leq 1$. This recovers Brink’s result [4] that $\Gamma_{\text{min}}^\vee$ is free. (iv) If $J = A_{m \text{ even}}$ then the conclusion $\dim(\text{dual of } C_{\text{min}}^\circ) \leq 2$ suggests that $\Gamma_{\text{max}}$ is often comprehensible, like the $J = A_6$ example of [3, example 5.4].

Lemma 5. If $J$ lies in a spherical Coxeter diagram $I \subseteq \Pi$, whose cardinality exceeds that of $J$ by more than (3), then $W_I$ contains an element preserving the set $J$ of roots and acting on $J^\perp$ by a reflection.

Proof. This is essentially the same as for lemma 2, using the following additional ingredients. For example, when $I = D_n$ one can use them to show that the 5th, 7th, 8th and 10th cases of (2) are impossible, while the first can only occur when $n$ is even.

First, if $J = E_6 \subseteq E_7 = I$ then $W_I$ contains the negation of $V_I$, which we follow by the long word in $W_I$ to send $-J$ back to $J$. The composition is the claimed element of $W_I$. The same argument applies if $J = I_2(5) \subseteq H_3 = I$.

Second, if $J = A_{m \text{ odd}} \subseteq D_{m+2} = I$ as in the first diagram of (2), then consider the long word in $W_I$. It negates $J$ and exchanges and negates the two simple roots in $I - J$. Following this by the long word in $W_J$ yields the claimed element of $W_I$. (When $m$ is even, the long word in $W_I$ negates the simple roots in $I - J$ without exchanging them, so it doesn’t act on $J^\perp$ by a reflection.)

Third, if $J = D_{m \geq 3} \subseteq D_{m+1} = I$ then consider the model of $W_I$ as the group generated by permutations and evenly many negations of $m + 1$ coordinates, with $W_J$ the corresponding subgroup for the first $m$ coordinates. Letting $\sigma$ be the negation of the last two coordinates, and following it by the element of $W_J$ sending $\sigma(J)$ back to $J$, gives the claimed element of $W_I$. □

There is a nice geometrical interpretation of the freeness of $\Gamma_{\text{min}}^\vee$ in the case $J = A_1$, developed further in [1]. Namely, the natural map $C_{\text{min}}^\circ \to C_{\text{min}}^\circ/\Gamma_{\text{min}} \subseteq K/W_{\Pi} = C_{\Pi}$ is the universal cover of its image.
The image is got by discarding all the codimension 2 faces of \( C_{\Pi} \) corresponding to even bonds in \( \Pi \), discarding all codimension 3 faces, and taking the component corresponding to \( J \). This identifies \( \Gamma_{\text{min}} \) with the fundamental group of \( J \)’s component of the “odd” subgraph of \( \Pi \) in a natural manner.

One can extend this picture to the case \( J \neq A_1 \), but complications arise. First, one must take \( W_{\Omega} \) to be normal in \( W_{\Pi}:\Gamma_{\Pi} \). Second, while \( C^\circ_{\Omega} \to C^\circ_{\Omega}/\Gamma_{\Omega}^\vee \) is a covering space, the image \( C^\circ_{\Omega}/\Gamma_{\Omega} \) of \( C^\circ_{\Omega} \) in \( C_{\Pi} \) is the quotient of \( C^\circ_{\Omega}/\Gamma_{\Omega}^\vee \) by the finite group \( \Gamma_{\Omega}/\Gamma_{\Omega}^\vee \). Usually, \( C^\circ_{\Omega} \to C^\circ_{\Omega}/\Gamma_{\Omega} \) is only an orbifold cover since \( \Gamma_{\Omega} \) often has torsion. The top-dimensional strata of \( C^\circ_{\Omega}/\Gamma_{\Omega}^\vee \) correspond to the “associates” of the inclusion \( J \to \Pi \) in the sense of [3] and [5]. Suppose \( J' \subseteq \Pi \) is (the image of) an associate and \( I' \) is a spherical diagram containing it. Then the face of \( C_{\Pi} \) corresponding to \( I' \), minus lower-dimensional faces, lies in \( C^\circ_{\Omega}/\Gamma_{\Omega} \) just if \( W_{I'} \) contains no element preserving \( J' \), acting on it in a manner constrained by the choice of \( W_{\Omega} \), and acting on \( J'^\perp \) by a reflection. From this perspective, lemmas 2 and 5 amount to working out two cases of Borcherds’ notion of “R-reflectivity”. The orbifold structure on \( C^\circ_{\Omega}/\Gamma_{\Omega} \) is essentially the same information as Borcherds’ classifying category for \( \Gamma_{\Omega} \).

REFERENCES


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