Complex hyperbolic geometry and the monster simple group (conjectural)

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I explained the ideas and coincidences which led me to conjecture in [2] that a group closely related to the monster simple group is got from the orbifold fundamental group of a certain 13-dimensional complex-analytic variety by adjoining a certain relation.

Conway conjectured [6] that a group he called the bimonster is generated by 16 involutions satisfying certain braid and commutation relations (the ones specified by the Y_{555} diagram), together with one extra relation $w^{10} = 1$. The bimonster is $(M \times M)$:2, where M is the monster simple group. (Conway was working with the bimonster rather than the monster because it made working with a subgroup $\frac{1}{2}(S_5 \times S_{12})$ of M more convenient.) Ivanov [8] and Norton [9] proved this. I found the Y_{555} diagram appearing in my work in complex hyperbolic reflection groups [1], so naturally I wondered if there was a connection. For me, it appeared because one of my reflection groups contains 16 triflections (order 3 complex reflections) satisfying exactly the same commutation and braid relations.

How can one compare two groups, similar except with generators of different orders? One way is to find a larger group, with generators of infinite order, of which both groups are quotients. My reflection group approach suggested such a group. Call my group Γ; it acts on the complex 13-ball B^{13}. Write $\mathcal{H}$ for the union of the mirrors (fixed-point sets) of the triflections, and define $X = B^{13}/\Gamma$ and $X_0 = (B^{13} - \mathcal{H})/\Gamma$. Essentially by construction, B^{13} is the covering space of X which is universal among all those having ramification of degree 3 along $\Delta = \mathcal{H}/\Gamma$ and no other ramification. A way to express this is that Γ is got from $\pi_1(X_0)$ by demanding that a loop around Δ have order 3. (Remark: π_1 here means orbifold π_1.) If we instead demand that such a loop has order 2, then we get a group which satisfies all the relations of the bimonster, except perhaps the w^{10} relation. So I conjectured that this quotient actually is the bimonster.

Implicit in the last few sentences is the fact that $\pi_1(X_0)$ has 16 generators that satisfy the braid and commutator relations of the Y_{555} diagram. One may find the generators by picking a suitable point p of the ball and taking certain paths based at p. Each of these travels toward one of 16 nearby mirrors, travels 1/3 of the way around it, and then travels backwards along the translate of the first part of the path. Basak has recently established [5] that these loops do indeed satisfy the braid and commutation relations. It remains open whether they generate $\pi_1(X_0)$, and we don't know what other relations might be present in $\pi_1(X_0)$. It is known that the 16 triflections in Γ do generate Γ; see [4] and [3].

In fact, Basak found that there are 26 mirrors closest to p, so it's natural to adjoin the 10 extra generators to our 16, and it turns out that these 26 satisfy the braid and commutation relations of the incidence graph of the points and lines of $P^2(\mathbb{F}_3)$. Exactly the same thing happens in the bimonster! Conway's 16 involutions extend to 26, satisfying these same commutation and braid relations.

A model for the whole conjecture is the largest of the Deligne-Mostow ball quotients [7], which uniformizes the moduli space of unordered 12-tuples in $\mathbb{C}P^1$. One replaces B^{13} by B^9, our Γ by a discrete subgroup Γ^{DM} of $U(9,1)$ generated by triflections, and defines $X^{DM}, X_0^{DM}, \mathcal{H}^{DM}, \Delta^{DM}$ as above. Then $\pi_1(X_0^{DM})$ is the spherical braid group on 12 strands, and a loop around Δ^{DM} is one of the standard generators. Killing its cube reduces $\pi_1(X_0^{DM})$ to Γ^{DM}, while killing its square reduces it to S_{12}. In fact, this example is embedded in our situation, and the S_{12} corresponds to the factor of the $S_{12} \times S_5$ mentioned at the beginning.

This also suggests that X may be a moduli space of some sort of algebra-geometric objects. Whatever that type of objects is, it would have some sort of notion of marking, for which the monodromy group on markings would be the bimonster. The analogy in the Deligne-Mostow case is that an unordered 12-tuple admits a notion of marking for which the monodromy group is S_{12}—which it certainly does, namely an ordering of the points. Another suggestive moduli connection is that the 10-dimensional subvariety of X corresponding to the Y_{551} diagram is the moduli space of cubic threefolds.

There are some more consistency checks on the conjecture, notably that $w^{20} = 1$ in Γ; please refer to [2], [4] and [5] for more details.

I close with one thing that is not-well-enough known: in the setting of complex triflection groups, the A_4 Dynkin diagram should always make one pay attention. The reason is that it plays the same role as E_8 does in the usual setting of Coxeter groups. If you take 4 triflections satisfying the braid and commutation relations of the A_4 diagram, write G for the group generated and α for a "root" defining one of the triflections, then the G-translates of α span a copy of the E_8 lattice (equipped with a module structure over $\mathbb{Z}[\sqrt[3]{1}]$). You can see three A_4's in the Y_{555} diagram, and two in the $Y_{550} = A_{11}$ diagram, the latter being the one relevant to Γ^{DM}.

References

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