Identifying Models of the Octave Projective Plane

Daniel Allcock
15 August 1995

allcock@math.berkeley.edu
Department of Mathematics,
University of California,
Berkeley, CA 94720.

1991 mathematics subject classification: 51A35 (17C40)

Published in *Geometriae Dedicata* **65**(1997) 215–217.

Abstract.
We provide a convenient identification between two models of the projective plane over the alternative field of octaves: Aslaksen's coordinate approach and the classic approach via Jordan algebras. We do this by modifying a 1949 lemma of P. Jordan.

The Octave Plane

The projective plane $\mathbb{O}P^2$ over the alternative field $\mathbb{O}$ of octaves (also called Cayley numbers) may be viewed from several perspectives. Two particularly attractive models are the elegant coordinatization due to H. Aslaksen using 'restricted homogeneous coordinates' [1], and the model developed extensively by H. Freudenthal, in which the points of $\mathbb{O}P^2$ are identified with a set of idempotents in $\mathcal{J}$, a certain Jordan algebra [2]. What is missing is a convenient means to pass between these two languages. This paper makes the observation that a lemma due to P. Jordan [3], when suitably modified, yields a beautiful identification. Jordan's paper seems to have received little attention, despite being the first paper linking $\mathcal{J}$ and $\mathbb{O}P^2$.

Briefly, here are the models. The points of Aslaksen's plane are the nonzero triples (x_1, x_2, x_3) of octaves with at least one real element, modulo the relation that two such triples are equivalent if they differ by left multiplication by an element of $\mathbb{O}$. Lines may be defined as follows. Declare two points to be *orthogonal* if we have $x_1\bar{y}_1 + x_2\bar{y}_2 + x_3\bar{y}_3 = 0$ when we choose representative triples (x_i), (y_i) for the points, with at least 2 of the sets $\{x_i, y_i\}$ $(i = 1, 2, 3)$ containing a real number. (This choice may always be made.) The lines of the geometry are the sets orthogonal to the points. Clever computations in [1] show that these conditions do actually yield a projective plane. (*Note:* Aslaksen required one coordinate to be unity, but this is inessential; he also defined the same set of lines without reference to the "innner product" above. His lines and ours coincide.)

The exceptional Jordan algebra $\mathcal{J}$ is the (real) algebra of 3×3 Hermitian matrices with elements in $\mathbb{O}$, under the multiplication defined by $A * B = (AB + BA)/2$. The points of $\mathbb{O}P^2$ are the trace 1 idempotents, and two such idempotents are called *orthogonal* if their Jordan product vanishes. Again, the lines of the geometry are the point-sets orthogonal to points. It is convenient to identify an idempotent of $\mathcal{J}$ with the vector subspace of $\mathcal{J}$ consisting of its real scalar multiples.

Generalizing a construction of P. Jordan, we define a map from Aslaksen's plane to $\mathcal{J}$ by $(x_1, x_2, x_3) \mapsto e$ where e is the matrix defined by $(e_{ij}) = \bar{x}_i x_j$. This is well-defined up to real scalar multiplication, and it is easy to check that e is a trace 1 idempotent exactly when $|x_1|^2 + |x_2|^2 + |x_3|^2 = 1$.

Theorem. *The map defined above is an isomorphism from Aslaksen's model of $\mathbb{O}P^2$ to Freudenthal's.*

Proof: It is observed above that the map is well-defined, and it is trivial to check that it is injective. To show that it is surjective, one need only find suitable (x_1, x_2, x_3), given a trace 1 idempotent in $\mathcal{J}$, which is easy. The heart of the theorem is proving that the notions of orthogonality between points of $\mathbb{O}P^2$ coincide. We accomplish this in the following lemma, which is a modification of Jordan's Hilfsatz 2. We indicate a proof (Jordan didn't) because the calculation is very tedious if approached incorrectly. □

Lemma. *Let (x_1, x_2, x_3), (y_1, y_2, y_3) be two triples of elements of $\mathbb{O}$, each with at least one real element, and such that at least two of the sets $\{x_i, y_i\}$ $(i = 1, 2, 3)$ contain a real number. Then, defining elements e, f of $\mathcal{J}$ by $e_{ij} = \bar{x}_i x_j$, $f_{ij} = \bar{y}_i y_j$, we have $e * f = 0$ if and only if $x_1\bar{y}_1 + x_2\bar{y}_2 + x_3\bar{y}_3 = 0$.*

Proof: We know that e and f are scalar multiples of trace 1 idempotents, and therefore (see [2]) $e * f = 0$ if and only if $\mathrm{Tr}(e * f) = 0$. We have

$$2\mathrm{Tr}(e * f) = \sum_{i,j}(e_{ij}f_{ji} + f_{ij}e_{ji}) = 2\sum_{i,j}\mathrm{Re}(e_{ij}f_{ji}) = 2\sum_{i,j}\mathrm{Re}((\bar{x}_i x_j)(\bar{y}_j y_i)).$$

Without loss of generality we may assume that x_1 and y_2 are real, and so every term (except the $i = j = 3$ term) contains a real number. By using the octave identities $\mathrm{Re}(a(bc)) = \mathrm{Re}((ab)c) = \mathrm{Re}((bc)a)$, we may replace each such term of the sum by $\mathrm{Re}((x_j\bar{y}_j)(y_i\bar{x}_i))$. We may also do this in the case $i = j = 3$, for the reason that any two elements of $\mathbb{O}$ lie in an associative subalgebra of $\mathbb{O}$. So we have

$$\begin{aligned}\mathrm{Tr}(e * f) &= \sum_{i,j}\mathrm{Re}((x_j\bar{y}_j)(y_i\bar{x}_i)) = \mathrm{Re}\Big(\Big(\sum_j x_j\bar{y}_j\Big)\Big(\sum_i y_i\bar{x}_i\Big)\Big)\\ &= |x_1\bar{y}_1 + x_2\bar{y}_2 + x_3\bar{y}_3|^2,\end{aligned}$$

which completes the proof. □

References

[1] H. Aslaksen. Restricted homogeneous coordinates for the Cayley projective plane. *Geometriae Dedicata*, 40:245–50, 1991.

[2] H. Freudenthal. Oktaven, Ausnahmegruppen, und Oktavengeometrie. *Geometriae Dedicata*, 19:1–73, 1985. (Informally published, Utrecht 1951).

[3] P. Jordan. Über eine nicht-desaurguessche ebene projective Geometrie. *Abhandlungen Mathematischen Seminar der Universität Hamburg*, 16:74–6, 1949.