Identifying Models of the Octave Projective Plane

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Abstract.
We provide a convenient identification between two models of the projective plane over the alternative field of octaves: Aslaksen's coordinate approach and the classic approach via Jordan algebras. We do this by modifying a 1949 lemma of P. Jordan.

The Octave Plane
The projective plane \( O P^2 \) over the alternative field \( O \) of octaves (also called Cayley numbers) may be viewed from several perspectives. Two particularly attractive models are the elegant coordinatization due to H. Aslaksen using ‘restricted homogeneous coordinates’ [1], and the model developed extensively by H. Freudenthal, in which the points of \( O P^2 \) are identified with a set of idempotents in \( \mathcal{J} \), a certain Jordan algebra [2]. What is missing is a convenient means to pass between these two languages. This paper makes the observation that a lemma due to P. Jordan [3], when suitably modified, yields a beautiful identification. Jordan’s paper seems to have received little attention, despite being the first paper linking \( \mathcal{J} \) and \( O P^2 \).

Briefly, here are the models. The points of Aslaksen’s plane are the nonzero triples \((x_1, x_2, x_3)\) of octaves with at least one real element, modulo the relation that two such triples are equivalent if they differ by left multiplication by an element of \( O \). Lines may be defined as follows. Declare two points to be orthogonal if we have \( x_1 \tilde{y}_1 + x_2 \tilde{y}_2 + x_3 \tilde{y}_3 = 0 \) when we choose representative triples \((x_i), (y_i)\) for the points, with at least 2 of the sets \( \{x_i, y_i\} \) \((i = 1, 2, 3)\) containing a real number. (This choice may always be made.) The lines of the geometry are the sets orthogonal to the points. Clever computations in [1] show that these conditions do actually yield a projective plane. (Note: Aslaksen required one coordinate to be unity, but this is inessential; he also defined the same set of lines without reference to the “inner product” above. His lines and ours coincide.)

The exceptional Jordan algebra \( \mathcal{J} \) is the (real) algebra of \( 3 \times 3 \) Hermitian matrices with elements in \( O \), under the multiplication defined by \( A \ast B = (AB + BA)/2 \). The points of \( O P^2 \) are the trace 1 idempotents, and two such idempotents are called orthogonal if their Jordan product vanishes. Again, the lines of the geometry are the point-sets orthogonal to points. It is convenient to identify an idempotent of \( \mathcal{J} \) with the vector subspace of \( \mathcal{J} \) consisting of its real scalar multiples.

Generalizing a construction of P. Jordan, we define a map from Aslaksen’s plane to \( \mathcal{J} \) by \((x_1, x_2, x_3) \rightarrow e\) where \( e \) is the matrix defined by \((e_{ij}) = \tilde{x}_i x_j\). This is well-defined up to real scalar multiplication, and it is easy to check that \( e \) is a trace 1 idempotent exactly when \( |x_1|^2 + |x_2|^2 + |x_3|^2 = 1 \).

**Theorem.** The map defined above is an isomorphism from Aslaksen’s model of \( O P^2 \) to Freudenthal’s.

**Proof:** It is observed above that the map is well-defined, and it is trivial to check that it is injective. To show that it is surjective, one need only find suitable \((x_1, x_2, x_3)\), given a trace 1
idempotent in $\mathcal{J}$, which is easy. The heart of the theorem is proving that the notions of orthogonality between points of $\mathcal{O}P^2$ coincide. We accomplish this in the following lemma, which is a modification of Jordan's Hilsatz 2. We indicate a proof (Jordan didn't) because the calculation is very tedious if approached incorrectly. 

**Lemma.** Let $(x_1, x_2, x_3), (y_1, y_2, y_3)$ be two triples of elements of $\mathcal{O}$, each with at least one real element, and such that at least two of the sets $\{x_i, y_i\}$ $(i = 1, 2, 3)$ contain a real number. Then, defining elements $e, f$ of $\mathcal{J}$ by $e_{ij} = \bar{x}_i x_j, f_{ij} = \bar{y}_i y_j$, we have $e \ast f = 0$ if and only if $x_1 \bar{y}_1 + x_2 \bar{y}_2 + x_3 \bar{y}_3 = 0$.

**Proof:** We know that $e$ and $f$ are scalar multiples of trace 1 idempotents, and therefore (see [2]) $e \ast f = 0$ if and only if $\text{Tr}(e \ast f) = 0$. We have

$$2\text{Tr}(e \ast f) = \sum_{i,j} (e_{ij} f_{ji} + f_{ij} e_{ji}) = 2 \sum_{i,j} \text{Re}(e_{ij} f_{ji}) = 2 \sum_{i,j} \text{Re}((\bar{x}_i x_j)(\bar{y}_j y_i)).$$

Without loss of generality we may assume that $x_1$ and $y_2$ are real, and so every term (except the $i = j = 3$ term) contains a real number. By using the octave identities $\text{Re}(ab^*) = \text{Re}((ab)c) = \text{Re}((bc)a)$, we may replace each such term of the sum by $\text{Re}((x_j \bar{y}_j)(y_i \bar{x}_i))$. We may also do this in the case $i = j = 3$, for the reason that any two elements of $\mathcal{O}$ lie in an associative subalgebra of $\mathcal{O}$. So we have

$$\text{Tr}(e \ast f) = \sum_{i,j} \text{Re}((x_j \bar{y}_j)(y_i \bar{x}_i)) = \text{Re} \left( \left( \sum_j x_j \bar{y}_j \right) \left( \sum_i y_i \bar{x}_i \right) \right)$$

$$= |x_1 \bar{y}_1 + x_2 \bar{y}_2 + x_3 \bar{y}_3|^2,$$

which completes the proof. 

**References**

