New complex and quaternion-hyperbolic reflection groups

Daniel Allcock*
9 September 1997
allcock@math.utah.edu
web page: *http://www.math.utah.edu/∼allcock*
Department of Mathematics, University of Utah, Salt Lake City, UT 84112, USA

1991 mathematics subject classification: 22E40 (11F06, 11E39, 53C35)

Dedicated to my father John Allcock, 1940–1991

Abstract.
We consider the automorphism groups of various Lorentzian lattices over the Eisenstein, Gaussian, and Hurwitz integers, and in some of them we find reflection groups of finite index. These provide explicit constructions of new finite-covolume reflection groups acting on complex and quaternionic hyperbolic spaces of high dimensions. Specifically, we provide groups acting on $\mathbb{C}H^n$ for all $n < 6$ and $n = 7$, and on $\mathbb{H}H^n$ for $n = 1, 2, 3$, and 5. We compare our groups with those discovered by Deligne and Mostow and by Thurston, and show that our most interesting examples are new. For many of these Lorentzian lattices we show that the entire symmetry group is generated by reflections, and obtain a description of the group in terms of the combinatorics of a lower-dimensional positive-definite lattice. The techniques needed for our lower-dimensional examples are elementary; to construct our best examples we also use certain facts about the Leech lattice. We conjecture that Lorentzian lattices provide examples of hyperbolic reflection groups in dimensions even higher than those considered here, and mention connections to moduli of cubic surfaces. By studying orbits of norm 0 vectors in certain selfdual Lorentzian lattices we provide a new and geometric proof of the classifications of selfdual Eisenstein lattices of dimension ≤ 6 and of selfdual Hurwitz lattices of dimension ≤ 4.

1. Introduction

In this paper we carry out complex and quaternionic analogues of some of Vinberg's extensive study of reflection groups on real hyperbolic space. In [24] and [25] he investigated the symmetry groups of the integral quadratic forms diag $[-1, +1, \ldots, +1]$, or equivalently the Lorentzian lattices $I_{n,1}$. He was able to describe these groups quite concretely for $n \leq 17$, and extensions of his work by Vinberg and Kaplinskaja [26] and Borcherds [6] provide concrete descriptions for all $n \leq 23$. In particular, the subgroup of $\operatorname{Aut} I_{n,1}$ generated by reflections has finite index just when $n \leq 19$.

In our work, we study the symmetry groups of Lorentzian lattices over the rings $\mathcal{G}$ and $\mathcal{E}$ of Gaussian and Eisenstein integers and the ring $\mathcal{H}$ of Hurwitz integers (a discrete subring of the skew field $\mathbb{H}$ of quaternions). Most of the paper is devoted to the most natural of such lattices, the selfdual ones. The symmetry groups of these lattices provide a large number of discrete groups generated by reflections and acting with finite-volume quotient on on the hyperbolic spaces $\mathbb{C}H^n$ and $\mathbb{H}H^n$. We construct a total of 18 such groups, including groups acting on $\mathbb{C}H^7$ and $\mathbb{H}H^5$. At least one of our groups has been discovered before, in the work of Mostow and Deligne [17], Mostow [21] and Thurston [23], but we show that our "largest" examples are new. To the author's knowledge, quaternion-hyperbolic reflection groups not been studied before.

Our results and techniques have already found important application in work of the author, J. Carlson and D. Toledo [2] on moduli spaces of smooth cubic surfaces over $\mathbb{C}$. Namely, the (coarse) moduli space of such surfaces is isomorphic to the quotient of $\mathbb{C}H^4$ by a certain reflection group (studied here), minus the images of the mirrors of certain reflections. Furthermore, the usual (fine) moduli space of marked cubic surfaces may be realized as a quotient of $\mathbb{C}H^4$ by a congruence subgroup of this reflection group.

* Supported by National Science Foundation Graduate and Postdoctoral Fellowships.

The techniques used by Vinberg and others for the real hyperbolic case rely heavily on the fact that if a discrete group G is generated by reflections of $\mathbb{R}H^n$, then the mirrors of the reflections of G chop $\mathbb{R}H^n$ into pieces; each piece may be taken as a fundamental domain for G. Work with complex or quaternionic reflection groups is much more complicated, since hyperplanes have real codimension 2 or 4, and so the mirrors fail to chop hyperbolic space into pieces. Our solution to this problem is to avoid fundamental domains altogether. Each of our groups is defined as the subgroup $\operatorname{Reflec} L$ of $\operatorname{Aut} L$ generated by reflections, where L is a Lorentzian lattice over $\mathcal{G}$, $\mathcal{E}$ or $\mathcal{H}$. (A Lorentzian lattice is a free module equipped with a Hermitian form of signature $-+\cdots+$.) Since $\operatorname{Aut} L$ is an arithmetically defined group, to show that the quotient of $\mathbb{C}H^n$ or $\mathbb{H}H^n$ by $\operatorname{Reflec} L$ has finite volume, it suffices to show that $\operatorname{Reflec} L$ has finite index in $\operatorname{Aut} L$. In this case we say that L is reflective. Our basic strategy for proving a suitable lattice L to be reflective is to prove first that $\operatorname{Reflec} L$ acts with only finitely many orbits on the vectors of L of norm 0, and second that the stabilizer in $\operatorname{Reflec} L$ of one such vector has finite index in the stabilizer in $\operatorname{Aut} L$. That is, we work mostly arithmetically, avoiding use of such tools as the bisectors introduced by Mostow for his study [20] of reflection groups on $\mathbb{C}H^2$ (including some nonarithmetic ones).

However, there are certain steps in our constructions where geometric issues play a key role. We express each of our Lorentzian lattices L in the form $\Lambda \oplus II_{1,1}$, where Λ is positive-definite and $II_{1,1}$ is a certain 2-dimensional lattice, the "hyperbolic cell". It turns out that this description of L allows one to easily write down a large collection of reflections of L, parameterized by (a central extension of) the lattice Λ. It turns out that if Λ satisfies several properties, such as providing a good covering of Euclidean space by balls, then one can automatically deduce that L is reflective. This implication is the content of Theorem 6.2. The rest of Section 6 is devoted to the application of this theorem (and related ideas) in the study of various examples. In particular, we prove that each of the selfdual lattices

$$\begin{array}{ll} I^{\mathcal{E}}_{n,1} & n = 1, 2, 3, 4, 7, \\ II^{\mathcal{G}}_{n,1} & n = 1, 5, \\ I^{\mathcal{H}}_{n,1} & n = 1, 2, 3, 5 \end{array}$$

is reflective. (These lattices are defined in Section 3 and characterized in Theorem 7.1.) For some of these, we obtain more detailed information. In particular, we prove that $\operatorname{Reflec} I^{\mathcal{E}}_{n,1} = \operatorname{Aut} I^{\mathcal{E}}_{n,1}$ for $n = 2$, 3, 4 or 7 and that $\operatorname{Reflec} I^{\mathcal{H}}_{n,1}$ has index at most 4 in $\operatorname{Aut} I^{\mathcal{H}}_{n,1}$ for $n = 2$, 3 or 5. The group $\operatorname{Reflec} I^{\mathcal{E}}_{4,1}$ is particularly interesting because the quotient of $\mathbb{C}H^4$ by it is a partial compactification of the coarse moduli space of smooth cubic surfaces (see [2]). We also give explicit descriptions of the reflection groups of $I^{\mathcal{E}}_{1,1}$, $II^{\mathcal{G}}_{1,1}$ and $I^{\mathcal{H}}_{1,1}$ as subgroups of certain Coxeter groups, acting on $\mathbb{C}H^1 \cong \mathbb{R}H^2$ and $\mathbb{H}H^1 \cong \mathbb{R}H^4$.

We also note that the geometric idea we use, namely that good coverings of Euclidean space can lead to hyperbolic reflection groups, applies even when $\mathbb{C}$ and $\mathbb{H}$ are replaced by the nonassociative field $\mathbb{O}$ of octaves. In [1] we construct two octave reflection groups acting on $\mathbb{O}H^2$ and one acting on $\mathbb{O}H^1 \cong \mathbb{R}H^8$, and interpret these groups as the stabilizers of "lattices" over a certain discrete subring of $\mathbb{O}$.

We provide background information on lattices in Section 2 and examples of them in Section 3; the latter should be referred to only as needed. Section 4 establishes our conventions regarding hyperbolic geometry. In Section 5 we relate certain geometric properties of a positive-definite lattice Λ to the reflection group of $\Lambda \oplus II_{1,1}$ and lay other foundations for Section 6, where we construct all of our examples. In Sections 5 and 6, statements of many results are complicated by the fact that while lattices over the three rings $\mathcal{G}$, $\mathcal{E}$ and $\mathcal{H}$ may be treated in parallel, the exact results one can obtain depend slightly on the ring under consideration. The reader may focus on just one of these rings (we suggest $\mathcal{E}$) and still fully understand all of the ideas presented. In Section 7 we explain the correspondence between primitive isotropic sublattices of $I_{n+1,1}$ and positive-definite selfdual lattices of dimension n. We use this correspondence to provide a quick geometric proof of the classification of selfdual lattices over $\mathcal{E}$ and $\mathcal{H}$ in dimensions ≤ 6 and ≤ 4, respectively. The only examples besides the trivial lattices are the Coxeter-Todd lattice $\Lambda_6^{\mathcal{E}}$ and a quaternionic form $\Lambda_4^{\mathcal{H}}$ of the Barnes-Wall lattice. In Section 8 we identify properties of three of our groups, namely $\operatorname{Reflec} L$ for $L = I_{7,1}^{\mathcal{E}}$, $I_{4,1}^{\mathcal{E}}$ and $II_{5,1}^{\mathcal{H}}$, that distinguish them from the 94 groups constructed in [17], [21] and [23]. We also sketch a proof that $\operatorname{Reflec} I_{3,1}^{\mathcal{E}}$ appears on the lists given there. Finally, in Section 9 we make a few closing comments and conjecture that the Lorentzian lattices $I_{n,1}$ over $\mathcal{E}$, $\mathcal{G}$ and $\mathcal{H}$ are reflective for n considerably larger the cases considered here.

The easiest route to a new reflection group is our treatment of $II_{5,1}^{\mathcal{G}} = E_8^{\mathcal{G}} \oplus II_{1,1}^{\mathcal{G}}$, which acts on $\mathbb{C}H^5$. The proof that $II_{5,1}^{\mathcal{G}}$ is reflective requires only Theorem 5.1, the Gaussian case of Theorem 5.2, and the relevant parts of Theorem 6.2 and Corollary 6.3. There are two "tracks" through Section 6, the first leading to a large number of low-dimensional examples (including $II_{5,1}^{\mathcal{G}}$) and the second leading to a detailed study of the lattices $I_{7,1}^{\mathcal{E}}$ and $I_{5,1}^{\mathcal{H}}$. See the comments there for more information. Here we mention only that these two lattices succumb to our techniques because of several special properties of the Coxeter-Todd and Barnes-Wall lattices. In particular, the automorphism group of each is generated by reflections, and (suitable scaled) each has a very nice embedding in the Leech lattice Λ_{24}. Our basic approach in this paper was inspired by Conway's remarkable description [9] of the isometry group of the $\mathbb{Z}$-lattice $II_{25,1} = \Lambda_{24} \oplus II_{1,1}$ in terms of the combinatorics of Λ_{24}, so it is pleasing to see Λ_{24} playing a role here as well.

This paper is derived in part from the author's Ph.D. thesis at Berkeley; he would like to thank his dissertation advisor, R. Borcherds, for his interest and suggestions—in particular for suggesting that the quaternionic Barnes-Wall lattice would provide a reflection group on $\mathbb{H}H^5$.

2. Lattices

We denote by $\mathcal{R}$ any one of the rings $\mathcal{E}$, $\mathcal{G}$, and $\mathcal{H}$—the Eisenstein, Gaussian, and Hurwitz integers. That is, $\mathcal{G} = \mathbb{Z}[\mathrm{i}]$ and $\mathcal{E} = \mathbb{Z}[\omega]$, where $\omega = (-1+\sqrt{-3})/2$ is a primitive cube root of unity. The ring $\mathcal{H}$ is the integral span of its 24 units ± 1, $\pm\mathrm{i}$, $\pm\mathrm{j}$, $\pm\mathrm{k}$ and $(\pm 1 \pm \mathrm{i} \pm \mathrm{j} \pm \mathrm{k})/2$ in the skew field $\mathbb{H}$ of quaternions. We write $\mathbb{K}$ for the field ($\mathbb{C}$ or $\mathbb{H}$) naturally containing $\mathcal{R}$. Conjugation $x \mapsto \bar{x}$ denotes complex or quaternionic conjugation, as appropriate. For any element x of $\mathbb{K}$, we write $\operatorname{Re} x = (x+\bar{x})/2$ and $\operatorname{Im} x = (x-\bar{x})/2$ for the real and imaginary parts of x, and say that x is imaginary if $\operatorname{Re} x = 0$. If $X \subseteq \mathbb{K}$ then we write $\operatorname{Im} X$ for the set of imaginary elements of X. For any $x \in \mathbb{K}$, $x\bar{x}$ is a positive real number, and the absolute value $|x|$ of x is defined to be $(x\bar{x})^{1/2}$. It is convenient to define the element $\theta = \omega - \bar{\omega} = \sqrt{-3}$ of $\mathcal{E}$. We will sometimes also consider ω and θ as elements of $\mathcal{H}$, via the embedding $\mathcal{E} \to \mathcal{H}$ defined by $\omega \mapsto (-1+\mathrm{i}+\mathrm{j}+\mathrm{k})/2$ or equally well by $\theta \mapsto \mathrm{i}+\mathrm{j}+\mathrm{k}$.

A lattice Λ over $\mathcal{R}$ is a free (right) module over $\mathcal{R}$ equipped with a Hermitian form, which is to say a $\mathbb{Z}$-bilinear pairing (the inner product) $\langle\cdot|\cdot\rangle : \Lambda \times \Lambda \to \mathbb{K}$ such that

$$\langle x|y\rangle = \overline{\langle y|x\rangle} \qquad \text{and} \qquad \langle x|y\alpha\rangle = \langle x|y\rangle\alpha$$

for all $\alpha \in \mathcal{R}$. A Hermitian form on a (right) vector space over $\mathbb{K}$ is defined similarly. Section 3 defines a number of interesting lattices and lists some of their properties. Sometimes we indicate that a lattice Λ is an $\mathcal{R}$-lattice by writing $\Lambda^{\mathcal{R}}$ or somesuch.

If $S \subseteq \Lambda$ then we denote by $S^\perp$ its orthogonal complement: those elements of Λ whose inner products with all elements of S vanish. We say that Λ is integral if for all $x, y \in \Lambda$, the inner product $\langle x|y\rangle$ lies in $\mathcal{R}$, and that Λ is nonsingular if $\Lambda^\perp = \{0\}$. All lattices we consider will be integral and nonsingular unless otherwise specified. The dual Λ^* of Λ is the set of all $\mathcal{R}$-linear maps from Λ to $\mathcal{R}$. An integral lattice Λ is called selfdual if the natural map from Λ to Λ^* is onto.

A selfdual lattice is sometimes called "unimodular", because the matrix of inner products of any basis for Λ has determinant ± 1; we use "selfdual" to avoid discussing determinants of quaternionic matrices.

The norm of a vector $v \in V$ is defined to be $v^2 = \langle v|v\rangle$; some authors call this the squared norm of v. We say that v is isotropic, or null, if $v^2 = 0$. A lattice is isotropic (or null) if each of its elements is. A lattice is called even if each of its elements has even norm and odd if it is not even. A sublattice Λ' of Λ is called primitive if $\Lambda' = \Lambda \cap (\Lambda' \otimes \mathbb{R})$. A vector v of Λ is called primitive if $v = w\alpha$ for $w \in \Lambda$ and $\alpha \in \mathcal{R}$ implies that α is a unit. Because the rings $\mathcal{G}$, $\mathcal{E}$ and $\mathcal{H}$ are principal ideal domains, v is primitive if and only if its $\mathcal{R}$-span is primitive as a sublattice. We will sometimes write $\langle v\rangle$ for the $\mathcal{R}$-span of v. In the context of Lorentzian lattices (see below), any isotropic sublattice has dimension ≤ 1. When we refer to null sublattices of Lorentzian lattices we implicitly restrict attention to those of dimension 1—that is, we exclude from consideration the zero-dimensional lattice. In Lorentzian lattices, the concepts of primitive null vectors and primitive null lattices almost coincide.

We sometimes define an $\mathcal{R}$-lattice by describing a Hermitian form on $\mathcal{R}^n$. We do this by giving an $n \times n$ matrix (ϕ_{ij}) with entries in $\mathcal{R}$ such that $\overline{\phi_{ij}} = \phi_{ji}$. Then the Hermitian form is given by

$$\langle (x_1, \ldots, x_n)|(y_1, \ldots, y_n)\rangle = \sum_{i,j=1}^{n} \bar{x}_i \phi_{ij} y_j \ .$$

We may also view a lattice as a subset of a vector space V over the field $\mathbb{K}$—simply take V to be the (right) vector space $\Lambda \otimes \mathbb{R}$ over $\mathbb{K} = \mathbb{R} \otimes \mathcal{R}$. The Hermitian form on Λ gives rise to one on V. If Λ is nonsingular then Λ^* may be identified with the set of vectors in V having $\mathcal{R}$-integral inner product with each element of Λ.

Every nonsingular Hermitian form on a vector space V over $\mathbb{K}$ is equivalent under $\mathrm{GL}(V)$ to one given by a diagonal matrix, with each diagonal entry being ± 1. The signature of a form Φ is the number of $+1$'s minus the number of -1's. The signature of Φ characterizes Φ up to equivalence under $\mathrm{GL}(V)$. We write $\mathbb{K}^{n,m}$ for the vector space $\mathbb{K}^{n+m}$ equipped with a Hermitian form of signature $n - m$. The isometry group of $\mathbb{K}^{n,m}$ is the unitary group $U(n, m; \mathbb{K})$. The term "Lorentzian" is applied to various concepts in the study of real Minkowski space $\mathbb{R}^{n,1}$. By analogy with this we call an n-dimensional lattice Lorentzian if its signature is $n - 2$.

If Λ is positive-definite then $\Lambda \otimes \mathbb{R}$ is a copy of Euclidean space, under the metric $d(x, y) = \sqrt{(x - y)^2}$. Points of $\Lambda \otimes \mathbb{R}$ at maximal distance from Λ are called deep holes of Λ. The maximal distance is called the covering radius of Λ, because closed balls of that radius placed at lattice points exactly cover $\Lambda \otimes \mathbb{R}$. The lattice points nearest a deep hole are called the vertices of the hole. The covering radii of the $\mathbb{Z}$-lattices $\mathrm{Im}\,\mathcal{G}$, $\mathrm{Im}\,\mathcal{E}$ and $\mathrm{Im}\,\mathcal{H}$ are $1/2$, $3^{1/2}/2$ and $3^{1/2}/2$, respectively. The first two are obvious and the last follows because $\mathrm{Im}\,\mathcal{H}$ is the 3-dimensional cubic lattice spanned by i, j and k. Any two deep holes of $\mathrm{Im}\,\mathcal{R}$ are equivalent under translation by some element of $\mathrm{Im}\,\mathcal{R}$.

Suppose that V is a $\mathbb{K}$-vector space, $\xi \in \mathbb{K}$ is a root of unity and $v \in V$ has nonzero norm. We define ξ-reflection in v to be the map

$$v \mapsto v - r(1 - \xi)\frac{\langle r|v\rangle}{r^2} \ . \tag{2.1}$$

This is an automorphism of V as a right vector space equipped with Hermitian form $\langle\cdot|\cdot\rangle$; it fixes $r^\perp$ pointwise and carries r to $r\xi$. (*Warning:* if $\mathbb{K} = \mathbb{H}$ then although the reflection acts by right scalar multiplication on r, it does not act this way on all of the $\mathbb{H}$-span of r. This is due to the noncommutativity of multiplication in $\mathbb{H}$.) Unless otherwise specified, we will use the term "reflection" to mean "reflection in a vector of positive norm". Under the conventions of Section 4, (-1)-reflections in negative norm vectors act on hyperbolic space as inversions in points, rather than by reflections in hyperplanes. This is why we focus on positive-norm vectors. We call $r^\perp$ the mirror of the reflection. Reflections of order 2, 3, ... are sometimes called biflections, triflections, etc. A ξ-reflection is a biflection just if $\xi = -1$; in this case we recover the classical notion of a reflection.

Suppose L is an integral lattice. If $v \in L$ has norm 1 (resp. 2) then we say that v is a short (resp. long) root of L. Inspection of Eq. (2.1) reveals that if ξ is a unit of $\mathcal{R}$ then ξ-reflection in any short root of L preserves L. Furthermore, biflections in long roots of L also preserve L. We define the reflection group $\operatorname{Reflec} L$ to be the subgroup of $\operatorname{Aut} L$ generated by reflections (in positive-norm vectors), and we say that L is reflective if $\operatorname{Reflec} L$ has finite index in $\operatorname{Aut} L$. In general, a group generated by reflections is called a reflection group. Since $\operatorname{Aut} L$ is an arithmetically defined subgroup of the semisimple real Lie group $U(L \otimes \mathbb{R}; \mathbb{K})$, a theorem of Borel and Harish-Chandra [8] implies that it has finite covolume therein. It follows that L is reflective if and only if $\operatorname{Reflec} L$ also has finite covolume. We define $\operatorname{Reflec}_0 L$ to be the subgroup of $\operatorname{Reflec} L$ generated by reflections in the short roots of L. It may happen that $\operatorname{Reflec} L$ contains reflections other than those in its roots, but we will not use such reflections except briefly in the proof of Theorem 5.1.

3. Reference: examples of lattices

This section contains background information on the various specific complex and quaternionic lattices we use; it should be referred to only as necessary. We briefly define each lattice as a module over $\mathcal{R}$, list a few important properties, and give references to the literature. The main source is [14, Chap. 4]. All lattices described here are integral. When lattices are described as subsets of $\mathbb{K}^n$, it should be understood that the Hermitian form is $\langle (x_1, \ldots, x_n) | (y_1, \ldots, y_n) \rangle = \sum \bar{x}_i y_i$.

The simplest lattice is $\mathcal{R}^n$, which is obviously selfdual. Its symmetry group contains the left-multiplication by each diagonal matrix all of whose diagonal entries are units of $\mathcal{R}$. It is easy to see that this is the entire group preserving each of the scalar classes of short roots, and thus must contain the reflections in these roots. The group generated by these reflections has the same order as the group of diagonal matrices, so the groups coincide. (The reason the argument is this complicated is that the diagonal matrices act on the left, whereas reflections are defined in terms of right-multiplication. This is only important if $\mathcal{R} = \mathcal{H}$.) Adjoining to this group the permutations of coordinates, which are generated by biflections in long roots such as $(1, -1, 0, \ldots, 0)$, we see that $\operatorname{Aut} \mathcal{R}^n$ is a reflection group.

If Λ is a lattice, then its real form is the $\mathbb{Z}$-module Λ equipped with the inner product $(x, y) = \operatorname{Re}\langle x|y \rangle$. Here are three forms of the E_8 root lattice:

$$E_8 = \frac{1}{2} \left\{ (x_1, \ldots, x_8) \in \mathbb{Z}^8 \;\middle|\; x_i \equiv x_j \pmod 2,\ \sum x_i \in 4\mathbb{Z} \right\} ,$$
$$E_8^{\mathcal{G}} = \frac{1}{1+\mathrm{i}} \left\{ (x_1, \ldots, x_4) \in \mathcal{G}^4 \;\middle|\; x_i \equiv x_j \pmod{1+\mathrm{i}},\ \sum x_i \in 2\mathcal{G} \right\} ,$$
$$E_8^{\mathcal{H}} = \left\{ (x_1, x_2) \in \mathcal{H}^2 \;\middle|\; x_1 + x_2 \in (1+\mathrm{i})\mathcal{H} \right\} .$$

It is straightforward to identify the real forms of these $\mathcal{R}$-lattices with each other; each has covering radius 1 and minimal norm 2. Often the dimension of a lattice is indicated by a subscript. Unfortunately, as for E_8, this sometimes refers to its dimension as a $\mathbb{Z}$-lattice and sometimes to its dimension as an $\mathcal{R}$-lattice. There seems to be no universal solution to this notational problem.

Another set of useful even Gaussian lattices are

$$D_{2n}^{\mathcal{G}} = \left\{ (x_1, \ldots, x_n) \in \mathcal{G}^n \;\middle|\; \sum x_i \equiv 0 \pmod{1+\mathrm{i}} \right\} ,$$

whose real forms are the D_{2n} root lattices. The D_4 lattice is also the real form of $\mathcal{H}$, scaled up by a factor of $2^{1/2}$. The covering radius of D_{2n} is $(n/2)^{1/2}$.

The Eisenstein lattice

$$D_3(\sqrt{-3}) = \left\{ (x,y,z) \in \mathcal{E}^3 \;\middle|\; x+y+z \equiv 0 \pmod{\theta} \right\}$$

was introduced by Feit [18]. It has 54 long roots and 72 vectors of norm 3; biflections in the former and triflections in the latter preserve the lattice. Its covering radius is 1; this can be seen as follows. According to [14, p. 126], the real form of the lattice $\{ (x,y,z) \in \mathcal{E}^3 \mid x \equiv y \equiv z \pmod{\theta} \}$ is the E_6 root lattice scaled up by $(3/2)^{1/2}$. This identification can be used to show that the real form of $D_3(\sqrt{-3})$ is the real form of E_6^* scaled up by $(3/2)^{1/2}$, where E_6^* is the dual (over $\mathbb{Z}$) of E_6. By [14, p. 127], the covering radius of E_6^* is $(2/3)^{1/2}$, so the covering radius of $D_3(\sqrt{-3})$ is 1.

The Coxeter-Todd lattice $\Lambda_6^{\mathcal{E}}$ is a selfdual $\mathcal{E}$-lattice with minimal norm 2. It is discussed at length in [13]; we quote just one of the definitions given there.

$$\Lambda_6^{\mathcal{E}} = \frac{1}{\theta} \left\{ (x_1, \ldots, x_6) \in \mathcal{E} \;\middle|\; x_i \equiv x_j \pmod{\theta\mathcal{E}}, \ \sum x_i \in 3\mathcal{E} \right\} .$$

It automorphism group is the finite complex reflection group $6 \cdot U_4(3){:}2$, and $\Lambda_6^{\mathcal{E}}$ shares many interesting properties with E_8 and the Leech lattice Λ_{24}. We refer to [13] for details.

The quaternionic Barnes-Wall lattice is

$$\Lambda_4^{\mathcal{H}} = \frac{1}{1+\mathrm{i}} \left\{ (x_1, \ldots, x_4) \in \mathcal{H} \;\middle|\; x_i \equiv x_j \pmod{(1+\mathrm{i})\mathcal{H}}, \ \sum x_i \in 2\mathcal{H} \right\} .$$

We may recognize the real form of $2^{1/2}\Lambda_4^{\mathcal{H}}$ by identifying the vector

$$(a_1 + b_1\mathrm{i} + c_1\mathrm{j} + d_1\mathrm{k}, \ldots, a_4 + b_4\mathrm{i} + c_4\mathrm{j} + d_4\mathrm{k})$$

with the vector in $\mathbb{R}^{16}$ whose coordinates we arrange in the square array

$$\frac{4}{\sqrt{8}} \begin{array}{|cc|cc|} \hline a_1 & a_2 & a_3 & a_4 \\ d_1 & c_2 & d_3 & c_4 \\ \hline b_1 & d_2 & b_3 & d_4 \\ c_1 & b_2 & c_3 & b_4 \\ \hline \end{array}$$

where the inner product is the usual one on $\mathbb{R}^{16}$. This array may be taken to be (say) the left 4 columns of the 4×6 array of the MOG description [14, Chaps. 4, 11] of the Leech lattice Λ_{24}, and then the real form of $2^{1/2}\Lambda_4^{\mathcal{H}}$ is visibly the real Barnes-Wall lattice BW_{16} [14, Chap. 4].

Theorem 3.1. *The lattice $\Lambda_4^{\mathcal{H}}$ is selfdual and spanned by its minimal vectors, which have norm 2. Its automorphism group is generated by the biflections in its minimal vectors. Each class of $\Lambda_4^{\mathcal{H}}$ modulo $\Lambda_4^{\mathcal{H}}(1+\mathrm{i})$ is represented by a vector of norm at most 3. The deep holes of $\Lambda_4^{\mathcal{H}}$ coincide with the set $\{ \lambda(1+\mathrm{i})^{-1} \mid \lambda \in \Lambda_4^{\mathcal{H}}, \lambda^2 \equiv 1(2) \}$.*

Proof: Proofs of all claims except the last appear in [3, Sect. 4.6]. Most of the rest of the work has been done for us by Conway and Sloane [11, Sect. 5]. They showed that the deep holes of BW_{16} nearest 0 are the halves of certain vectors $v \in BW_{16}$ of norm 12, and further that such v are not congruent modulo 2 to minimal vectors of BW_{16}. (They write Λ_{16} for BW_{16}.) After rescaling, we find that the deep holes of $\Lambda_4^{\mathcal{H}}$ nearest 0 are the halves of certain elements v of norm 6 in $\Lambda_4^{\mathcal{H}}$. Since each such v has even norm and is not congruent modulo $2 = (1-\mathrm{i})(1+\mathrm{i})$ to any root, it must map to 0 in $\Lambda_4^{\mathcal{H}}/\Lambda_4^{\mathcal{H}}(1+\mathrm{i})$. Therefore $v = \lambda(1+\mathrm{i})$ for some λ of norm 3 in $\Lambda_4^{\mathcal{H}}$ and so the deep holes nearest 0 have the form $v/2 = \lambda(1+\mathrm{i})/2 = \lambda\mathrm{i}(1+\mathrm{i})^{-1}$.

The deep holes of $\Lambda_4^{\mathcal{H}}$ are the translates by lattice vectors of the deep holes nearest zero. That is, the set of deep holes coincides with the set

$$\left\{ \lambda(1+\mathrm{i})^{-1} \;\middle|\; \lambda \in \Lambda_4^{\mathcal{H}} \text{ is congruent modulo } 1+\mathrm{i} \text{ to a norm 3 lattice vector} \right\} .$$

The norms of any two lattice vectors that are congruent modulo $1+\mathrm{i}$ have the same parity. Since each lattice vector is congruent to some vector of norm 0, 2 or 3, the set above coincides with the one in the statement of the theorem. □

Now we describe some indefinite selfdual lattices. The lattice $I_{n,m}^{\mathcal{R}}$ is the $\mathcal{R}$-module $\mathcal{R}^{n+m}$ equipped with the inner product given by the diagonal matrix

$$\operatorname{diag}[+1,\ldots,+1,-1,\ldots,-1]$$

with n (resp. m) +1's (resp. (−1)'s). The lattice $II_{1,1}^{\mathcal{R}}$ is the module $\mathcal{R}^2$ with inner product matrix $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. If $\mathcal{R} = \mathcal{E}$ or $\mathcal{H}$ then $II_{1,1}^{\mathcal{R}} \cong I_{1,1}^{\mathcal{R}}$. If $\mathcal{R} = \mathcal{G}$ then this lattice is even, whereas $I_{1,1}^{\mathcal{G}}$ is odd. We define the Gaussian lattices $II_{4m+n,n}^{\mathcal{G}}$ to be the lattices

$$II_{4m+n,n}^{\mathcal{G}} = E_8^{\mathcal{G}} \oplus \cdots \oplus E_8^{\mathcal{G}} \oplus II_{1,1}^{\mathcal{G}} \oplus \cdots \oplus II_{1,1}^{\mathcal{G}} ,$$

where there are m summands $E_8^{\mathcal{G}}$ and n summands $II_{1,1}^{\mathcal{G}}$. These lattices are even and selfdual. By Theorem 7.1, every indefinite selfdual lattice over $\mathcal{R}$ appears among the examples just given. In particular, $\Lambda_6^{\mathcal{E}} \oplus II_{1,1}^{\mathcal{E}} \cong I_{7,1}^{\mathcal{E}}$ and $\Lambda_4^{\mathcal{H}} \oplus II_{1,1}^{\mathcal{H}} \cong I_{5,1}^{\mathcal{H}}$.

4. Hyperbolic space

The hyperbolic space $\mathbb{K}H^{n+1}$ ($n \geq 0$) is defined as the image in projective space $\mathbb{K}P^{n+1}$ of the set of vectors of negative norm in $\mathbb{K}^{n+1,1}$; its boundary $\partial\mathbb{K}H^{n+1}$ is the image of the (nonzero) null vectors. We write elements of $\mathbb{K}^{n+1,1}$ in the form $(\lambda;\mu,\nu)$ with $\lambda \in \mathbb{K}^{n,0}$ and $\mu,\nu \in \mathbb{K}$, with inner product

$$\langle(\lambda_1;\mu_1,\nu_1)|(\lambda_2;\mu_2,\nu_2)\rangle = \langle\lambda_1|\lambda_2\rangle + \bar{\mu}_1\nu_2 + \bar{\nu}_1\mu_2 .$$

This corresponds to a decomposition $\mathbb{K}^{n+1,1} \cong \mathbb{K}^{n,0} \oplus \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. We will sometimes refer to points in projective space by naming vectors in the underlying vector space.

It is convenient to distinguish the isotropic vector $(0;0,1)$ and give it the name ρ. Every point of $\mathbb{K}H^{n+1} \cup \partial\mathbb{K}H^{n+1}$ except ρ has a unique preimage in $\mathbb{K}^{n+1,1}$ with inner product 1 with ρ, and so we may make the identifications

$$\begin{aligned} \mathbb{K}H^{n+1} &= \{(\lambda;1,z) : \lambda \in \mathbb{K}^n, \lambda^2 + 2\operatorname{Re}(z) > 0\} . \\ \partial\mathbb{K}H^{n+1} \smallsetminus \{\rho\} &= \{(\lambda;1,z) : \lambda \in \mathbb{K}^n, \lambda^2 + 2\operatorname{Re}(z) = 0\} . \end{aligned} \tag{4.1}$$

We define the height of a vector $v \in \mathbb{K}^{n+1,1}$ to be $\operatorname{ht} v = \langle\rho|v\rangle$. For $v = (\lambda;\mu,\nu)$ we simply have $\operatorname{ht} v = \mu$. For vectors of any fixed norm, the height function measures how far away from ρ the corresponding points in projective space are; the smaller the height, the closer to ρ. We will sometimes say that for vectors v and v' the height of v' is less than that of v. By this we will mean that $|\operatorname{ht} v'| < |\operatorname{ht} v|$.

We say that a vector $(\lambda;\mu,\nu)$ with $\mu\neq 0$ lies over $\lambda\mu^{-1}\in\mathbb{K}^n$. It is obvious that all the scalar multiples of any given vector of nonzero height lie over the same point of $\mathbb{K}^n$, so we may think of points in projective space (except for those in $\rho^\perp$) as lying over elements of $\mathbb{K}^n$. The geometric content of this definition is that the lines in $\mathbb{K}P^{n+1}$ passing through ρ and meeting $\mathbb{K}H^{n+1}$ are in one-to-one correspondence with $\mathbb{K}^n$. The points in the line associated to $\lambda\in\mathbb{K}^n$ are just the scalar multiples of those of the form $(\lambda;1,z)$ with $z\in\mathbb{K}$. Special cases are given in Eq. (4.1). In particular, the family of isotropic vectors of height one lying over λ are parameterized by the elements of $\operatorname{Im}\mathbb{K}$. This description of $\partial\mathbb{K}H^{n+1}\smallsetminus\{\rho\}$ as a bundle over $\mathbb{K}^n$ with fiber $\operatorname{Im}\mathbb{K}$ will help us relate the properties of lattices in $\mathbb{K}^n$ to properties of groups acting on $\mathbb{K}H^{n+1}$.

The subgroup of $U(n+1,1;\mathbb{K})$ fixing ρ contains transformations $T_{x,z}$ (with $x\in\mathbb{K}^n$, $z\in\operatorname{Im}\mathbb{K}$) defined by

$$T_{x,z}\colon\qquad \begin{aligned} \rho &\mapsto \rho \\ (0;1,0) &\mapsto (x;1,z-x^2/2) \\ (\lambda;0,0) &\mapsto (\lambda;0,-\langle x|\lambda\rangle) \qquad \text{for each } \lambda\in\mathbb{K}^n. \end{aligned} \tag{4.2}$$

(The map is defined in terms of some unspecified but fixed inner product on $\mathbb{K}^n$.) We call these maps translations. If we regard elements of $\mathbb{K}^{n+1,1}$ as column vectors then $T_{x,z}$ acts by multiplication on the left by the matrix

$$\begin{pmatrix} I_n & x & 0 \\ 0 & 1 & 0 \\ -x^* & z-x^2/2 & 1 \end{pmatrix}.$$

(We have written x^* for the linear function $y\mapsto\langle x|y\rangle$ on $\mathbb{K}^{n,0}$ defined by x.) We have the relations

$$T_{x,z}\circ T_{x',z'} = T_{x+x',z+z'+\operatorname{Im}\langle x'|x\rangle} \tag{4.3}$$

$$T_{x,z}^{-1} = T_{-x,-z} \tag{4.4}$$

$$T_{x,z}^{-1}\circ T_{x',z'}^{-1}\circ T_{x,z}\circ T_{x',z'} = T_{0,2\operatorname{Im}\langle x'|x\rangle}\ , \tag{4.5}$$

which are most easily verified in the order listed. These relations make it clear that the translations form a group and that its center and commutator subgroup coincide and consist of the $T_{0,z}$. We call elements of this subgroup central translations. The translations form a (complex or quaternionic) Heisenberg group which acts freely and transitively on $\partial\mathbb{K}H^{n+1}\smallsetminus\{\rho\}$. If $v\in\mathbb{K}^{n+1,1}$ lies over $\lambda\in\mathbb{K}^n$ then $T_{x,z}(v)$ lies over $\lambda+x$. That is, the translations act in the natural way (by translations) on the points of $\mathbb{K}^n$ over which vectors in $\mathbb{K}^{n+1,1}$ lie.

We note that these constructions all make sense with $\mathbb{K}=\mathbb{R}$, and in fact simplify. Since $\operatorname{Im}\mathbb{R}=0$, the translations form an abelian group, which is just the obvious set of translations in the usual upper half-space model for $\mathbb{R}H^{n+1}$. The obvious projection map from the upper half-space to $\mathbb{R}^n$ carries points of $\mathbb{R}H^{n+1}$ to the points of $\mathbb{R}^n$ over which they lie, in the sense defined above. This is the source of the terminology.

The simultaneous stabilizer of $(0;1,0)$ and $(0;0,1)$ is the unitary group $U(n,0;\mathbb{K})$, which fixes pointwise the second summand in the decomposition $\mathbb{K}^{n+1,1}=\mathbb{K}^{n,0}\oplus\mathbb{K}^{1,1}$. If S is an element of this unitary group then matrix computations reveal

$$S\circ T_{x,z}\circ S^{-1} = T_{Sx,z}\ . \tag{4.6}$$

This is useful in its own right and also shows that the group of translations is normal in the full stabilizer of ρ.

5. Reflections in Lorentzian lattices

The Lorentzian lattices we will consider all have the form $\Lambda \oplus II_{1,1}$, where Λ is a positive-definite $\mathcal{R}$-lattice and $II_{1,1}$ is the 2-dimensional selfdual lattice defined by the matrix $II_{1,1} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. In general we will write L for a Lorentzian lattice $\Lambda \oplus II_{1,1}$, where Λ and even $\mathcal{R}$ may be left unspecified, except that Λ will always be positive-definite. We write elements of $L = \Lambda \oplus II_{1,1}$ in the form $(\lambda; \mu, \nu)$ with $\lambda \in \Lambda$ and $\mu, \nu \in \mathcal{R}$. This embeds L in the description of $\mathbb{K}^{n+1,1}$ given in Section 4 and allows us to transfer to L several important concepts defined there. In particular, $\rho = (0; 0, 1)$ is an element of L and we define the height of elements of L as before. For $v \in L$ of nonzero height we can speak of the point of $\Lambda \otimes \mathbb{R}$ over which v lies (which need not be an element of Λ).

There are two basic ideas in this section. First, that this description of L provides a way to write down a large collection of reflections of L, essentially parameterized by the elements of a discrete Heisenberg group of translations. The second idea is that if (i) r is a root of L, (ii) v is a null vector in $\mathbb{K}^{n+1,1}$, (iii) neither r nor v has height zero, and (iv) the points of $\mathbb{K}^n$ over which r and v lie are sufficiently close, then by applying a reflection of L one can reduce the height of v. (This reflection might be in some root other than r.)

Both of these ideas can be found, in the simpler setting of real hyperbolic space, in Conway's study [9] of the automorphism group of the Lorentzian $\mathbb{Z}$-lattice $II_{25,1} = \Lambda_{24} \oplus II_{1,1}$. Here Λ_{24} is the famous Leech lattice, and Conway found a set of reflections permuted freely by a group of translations naturally isomorphic to the additive group of Λ_{24}. By using facts about the covering radius of Λ_{24} and using the second idea described above, he was able to prove that these reflections generate the entire reflection group of $II_{25,1}$.

The major complication in transferring this approach to our setting is that the discrete group of translations is no longer a copy of Λ but a central extension of Λ by $\operatorname{Im}\mathcal{R}$. This issue dramatically complicates the precise formulation (Theorems 5.2 and 5.3) of the second main idea. For example, it is complicated to state exactly what happens when one can't *quite* reduce the height of $v \in \mathbb{K}^{n+1,1}$ by using a reflection.

We begin by finding the translations in $\operatorname{Aut} L$ and showing that under simple conditions, $\operatorname{Reflec} L$ contains a large number of them. The translation $T_{x,z}$ preserves L just if $x \in \Lambda$ and $z - x^2/2 \in \mathcal{R}$. If $\mathcal{R} = \mathcal{E}$ or $\mathcal{H}$ then for any given $x \in \Lambda$ we may choose $z \in \operatorname{Im}\mathbb{K}$ such that $T_{x,z} \in \operatorname{Aut} L$, by taking $z = 0$ or $\theta/2$ according as x^2 is even or odd. If $\mathcal{R} = \mathcal{G}$ then such a z exists if and only if x^2 is even; z may then be taken to be zero. The different rings behave differently because $\mathcal{E}$ and $\mathcal{H}$ contain elements with half-integral real parts, while $\mathcal{G}$ does not. All the central translations $T_{0,z}$ with $z \in \mathcal{R}$ lie in $\operatorname{Aut} L$—they fix Λ pointwise and act by isometries on the $II_{1,1}$ summand. The assertions of the next theorem are precise formulations of the idea that if $\operatorname{Aut}\Lambda$ contains many reflections then $\operatorname{Reflec} L$ contains many translations.

Theorem 5.1. *Let $L = \Lambda \oplus II_{1,1}$ for some positive-definite $\mathcal{R}$-lattice Λ. Define*

$$\Lambda_0 = \{\, x \in \Lambda \mid T_{x,z} \in \operatorname{Reflec} L \text{ for some } z \in \operatorname{Im}\mathbb{K} \,\} \text{ and}$$
$$\mathcal{S} = \{ z \in \operatorname{Im}\mathcal{R} \mid T_{0,z} \in \operatorname{Reflec} L \} .$$

(i) If no element of Λ is fixed by every reflection of Λ then Λ_0 has finite index in Λ.

(ii) If r is a root of Λ (a long root, if $\mathcal{R} = \mathcal{G}$) having inner product 1 with some element of Λ, then $r \in \Lambda_0$.

(iii) $\mathcal{S}$ contains the integral span of the elements of the form $2\operatorname{Im}\langle x|y\rangle$ with $x, y \in \Lambda_0$.

(iv) Under the hypothesis of (i), the stabilizer of ρ in $\operatorname{Reflec} L$ has finite index in the stabilizer in $\operatorname{Aut} L$.

Remarks: By Eq. (4.3) and (4.4), Λ_0 is closed under addition and negation, so assertion (i) makes sense. We will not need an analogue of (ii) for short roots in Gaussian lattices. In Theorem 6.4, similar but stronger hypotheses are used to obtain similar but stronger conclusions.

Proof: Let R be a ξ-reflection of Λ with mirror M. We regard R as acting on L, fixing the summand $II_{1,1}$ pointwise. If $T_{x,z} \in \operatorname{Aut} L$ then $T_{x,z}^{-1} \circ R \circ T_{x,z} \in \operatorname{Reflec} L$. By Eqs. (4.4), (4.6) and (4.3),

$$T_{x,z}^{-1} \circ R \circ T_{x,z} \circ R^{-1} = T_{-x,-z} \circ T_{Rx,z} = T_{Rx-x,-\operatorname{Im}\langle Rx|x\rangle} ,$$

proving that

$$Rx - x \in \Lambda_0 \tag{5.1}$$

for all $x \in \Lambda$ and all reflections R of Λ. The geometric picture behind this computation is that both M and its translate by $T_{x,z}^{-1}$ pass through ρ and are parallel there. It is not surprising that one can construct translations out of reflections in pairs of parallel mirrors.

(i) We will show first that Λ_0 contains an $\mathcal{R}$-sublattice orthogonal to M. Suppose ξ is an nth root of unity. Then the self-map (n times the projection to M) of Λ given by $x \mapsto \sum_{a=1}^{n} R^a(x)$ is an $\mathcal{R}$-lattice endomorphism. It rank as a map on Λ is the same as its rank as a map on the $\mathbb{K}$-vector space $\Lambda \otimes \mathbb{R}$. Since R is a reflection, this rank is $\dim \Lambda - 1$, and we see that Λ contains a 1-dimensional $\mathcal{R}$-sublattice orthogonal to M. If x is a nonzero element of this lattice then $Rx - x$ is also, and Eq. (5.1) shows that $Rx - x$ lies in Λ_0. This shows that Λ_0 contains a 1-dimensional $\mathcal{R}$-sublattice orthogonal to M. By hypothesis, no vector of Λ is orthogonal to every mirror of Λ, so Λ_0 has finite index in Λ.

(ii) If $r^2 = 1$ then we only need to prove anything in the cases $\mathcal{R} = \mathcal{E}$ or $\mathcal{H}$. Take $x = r$ and R to be the $(-\omega)$-reflection in r. Then Eq. (5.1) shows that Λ_0 contains $Rx - x = r(-\omega) - r = r\bar{\omega}$. Applying the same argument to $r\omega$ we see that $r \in \Lambda_0$.

If $r^2 = 2$ then in Eq. (5.1) we take R to be the biflection in r and by hypothesis we may take $x \in \Lambda$ with $\langle x|r\rangle = 1$. We find that $Rx - x$ is a multiple of r because it lies in the (-1)-eigenspace of R. Since it has inner product -2 with r, we have $Rx - x = -r$, which implies $r \in \Lambda_0$.

(iii) Follows immediately from Eq. (4.5) by taking commutators of translations of $\operatorname{Reflec} L$.

(iv) The null vectors of height 1 in L are exactly those vectors $(\lambda; 1, z)$ with $\lambda \in \Lambda$, $z \in \mathcal{R}$ and $\operatorname{Re} z = -\lambda^2/2$; the translations in $\operatorname{Aut} L$ permute them transitively. Since the simultaneous stabilizer of ρ and one of these, say $(0; 1, 0)$, is the finite group $\operatorname{Aut} \Lambda$, it suffices to prove that the group of translations in $\operatorname{Reflec} L$ has finite index in the group of those in $\operatorname{Aut} L$. This follows from the facts that Λ_0 has finite index in Λ and $\mathcal{S}$ has finite index in $\operatorname{Im} \mathcal{R}$. The former fact is ($i$) and the latter follows from (i) and (iii). □

It is straightforward to enumerate the roots of L of any given height h; For $h = 1$ one finds that these are the vectors

Norm 2:	$(\lambda; 1, z)$,	$\operatorname{Re} z = (2 - \lambda^2)/2$
Norm 1:	$(\lambda; 1, z)$,	$\operatorname{Re} z = (1 - \lambda^2)/2$,

with $\lambda \in \Lambda$ and $z \in \mathcal{R}$. If $\mathcal{R} = \mathcal{E}$ or $\mathcal{H}$, then height 1 roots of both norms lie over each $\lambda \in \Lambda$, and the translations of L act simply transitively on each set. If $\mathcal{R} = \mathcal{G}$ then height one roots lie over each $\lambda \in \Lambda$: long roots over λ of even norm and short roots over λ of odd norm. Again, the translations act simply transitively on each set. This is another manifestation of the fact that $\mathcal{E}$ and $\mathcal{H}$ but not $\mathcal{G}$ have elements with half-integer real part. One may also enumerate roots of larger heights—for example, if Λ is an $\mathcal{E}$-lattice, then there are short (resp. long) roots of L of height θ over each $\lambda\theta^{-1} \in \Lambda\theta^{-1}$ with $\lambda^2 \equiv 1$ (resp. 2) modulo 3. For more information see Table 6.1 and the discussion concerning it.

Now we will develop the second main idea of this section, by investigating the effects of reflections in roots of small height h. We first deal with long roots and then with short ones. The results are complicated to state because the exact results one can obtain vary slightly with the choice of $\mathcal{R}$ and h. The essential ideas are all present in the $\mathcal{R} = \mathcal{E}$, $h = 1$ case.

Theorem 5.2. *Suppose Λ is a positive-definite $\mathcal{R}$-lattice and $L = \Lambda \oplus II_{1,1}$. Let h be*

$$\begin{array}{ll} 1 & \text{if } \mathcal{R} = \mathcal{G} \text{ or } \mathcal{H}, \text{ or} \\ 1 \text{ or } \theta & \text{if } \mathcal{R} = \mathcal{E}. \end{array}$$

Suppose r is a long root of L of height h, lying over λh^{-1} for $\lambda \in \Lambda$. Let $v \in L \otimes \mathbb{R}$ be isotropic, have height one and lie over $\ell \in \Lambda \otimes \mathbb{R}$. Set $D^2 = (\ell - \lambda h^{-1})^2$ and suppose $D^2 \leq 1/|h|^2$ (or $D^2 \leq 3^{1/2}$ if $\mathcal{R} = \mathcal{G}$). Then there exists a long root r' of L of height h also lying over λh^{-1} such that either
(i) biflection in r' carries v to a vector of height smaller than that of v,
or else one of the following holds:
(ii) $\mathcal{R} = \mathcal{G}$, $D^2 = 3^{1/2}$ and $\langle r'|v\rangle = 1 - 3^{1/2}/2 + \mathrm{i}/2$.
(iii) $\mathcal{R} = \mathcal{E}$ or $\mathcal{H}$, $D^2 = 1/|h|^2$ and $\langle r'|v\rangle = h^{-1}(\omega + 1)$.

Proof: Since v has height 1 and norm 0 and lies over ℓ, we know that for some $w \in \operatorname{Im} \mathbb{K}$ we have

$$v = (\ell; 1, w - \ell^2/2)\ .$$

Similarly, we deduce that

$$r = \left(\lambda; h, z_0 + \frac{2 - \lambda^2}{2|h|^2} h\right) \tag{5.2}$$

for some $z_0 \in \mathbb{K}$ such that $\operatorname{Re}(\bar{h} z_0) = 0$. Every other long root r' of L with height h that lies over λh^{-1} has the form $r' = r + (0; 0, z)$ for some $z \in \mathcal{R}$ satisfying $\operatorname{Re}(\bar{h} z) = 0$. We will obtain the theorem by choosing z carefully.

We have

$$\begin{aligned}
\langle r'|v\rangle &= \langle\lambda|\ell\rangle + \bar{h}(w - \ell^2/2) + \left(\frac{(2-\lambda^2)\bar{h}}{2|h|^2} + \bar{z}_0 + \bar{z}\right) \\
&= h^{-1}\left[h\langle\lambda|\ell\rangle + |h|^2 w - \frac{|h|^2\ell^2}{2} + \frac{2-\lambda^2}{2} + h\bar{z}_0 + h\bar{z}\right] \\
&= h^{-1}\left[1 - \frac{|h|^2}{2}\Big(\ell^2 - 2\langle\lambda h^{-1}|\ell\rangle + (\lambda h^{-1})^2\Big) + |h|^2 w + h\bar{z}_0 + h\bar{z}\right] \\
&= h^{-1}\bigg[1 - \frac{|h|^2}{2}\Big(\ell^2 - \langle\lambda h^{-1}|\ell\rangle - \langle\ell|\lambda h^{-1}\rangle + (\lambda h^{-1})^2\Big) \\
&\qquad - \frac{|h|^2}{2}\Big(-\langle\lambda h^{-1}|\ell\rangle + \langle\ell|\lambda h^{-1}\rangle\Big) + |h|^2 w + h\bar{z}_0 + h\bar{z}\bigg] \\
&= h^{-1}\left[\left(1 - \frac{|h|^2}{2}D^2\right) + |h|^2 \operatorname{Im}\langle\lambda h^{-1}|\ell\rangle + |h|^2 w + h\bar{z}_0 + h\bar{z}\right] \\
&= h^{-1}[a + B] \qquad\qquad (5.3)\ ,
\end{aligned}$$

where $a = 1 - |h|^2 D^2/2$ is the real part of the term in brackets and B is the imaginary part. (Note that $h\bar{z}_0$ and $h\bar{z}$ are imaginary because for all $x, y \in \mathbb{K}$, $\operatorname{Re}(x\bar{y}) = \operatorname{Re}(\bar{y}x) = \operatorname{Re}(\overline{\bar{y}x}) = \operatorname{Re}(\bar{x}y)$ and we know that $\bar{h}z_0$ and $\bar{h}z$ are imaginary.) The important thing to observe here is that a contains information about D^2, which is bounded by hypothesis, and B contains a term $h\bar{z}$, over which we have some influence.

Now let v' be the image of v under biflection in r'. Since $v' = v - r'\langle r'|v\rangle$, we have

$$\begin{aligned}\langle\rho|v'\rangle &= \langle\rho|v\rangle - \langle\rho|r'\rangle\langle r'|v\rangle \\ &= 1 - hh^{-1}(a+B) \\ &= |h|^2D^2/2 - B \\ &= a' - B\ , \end{aligned} \tag{5.4}$$

where $a' = 1 - a = |h|^2D^2/2$ is real and B is as above. We may take z to be any element of $\mathcal{R}$ satisfying $\mathrm{Re}(\bar{h}z) = 0$; this condition may be written $z \in \mathcal{R} \cap (h \cdot \mathrm{Im}\,\mathbb{K})$. Since the definition of B involves the term $h\bar{z}$ we see that by changing our choice of z we may change B by any element of

$$\begin{aligned}h \cdot \overline{\mathcal{R} \cap (h \cdot \mathrm{Im}\,\mathbb{K})} &= h \cdot (\mathcal{R} \cap ((\mathrm{Im}\,\mathbb{K}) \cdot \bar{h})) \\ &= (h\mathcal{R}) \cap (h(\mathrm{Im}\,\mathbb{K})\bar{h}) \\ &= (h\mathcal{R}) \cap \mathrm{Im}\,\mathbb{K} \\ &= \mathrm{Im}(h\mathcal{R})\ .\end{aligned}$$

We will now consider the possibilities for $\mathcal{R}$ separately.

If $\mathcal{R} = \mathcal{G}$ then we only need to prove anything for the case $h = 1$; then $\mathrm{Im}(h\mathcal{G}) = \mathrm{Im}\,\mathcal{G}$, so by making an appropriate choice of z we may suppose that $B = b\mathrm{i}$ with $b \in (-1/2, 1/2]$. By hypothesis, $a' \le 3^{1/2}/2$. Therefore $|\mathrm{ht}\,(v')|^2 = (a')^2 + |B|^2$ is less than $1 = |\mathrm{ht}\,v|^2$ (so that conclusion (i) applies) unless $a' = 3^{1/2}/2$ and $b = 1/2$. In this case, $D^2 = 3^{1/2}$ and by Eq. (5.3) we have $\langle r'|v\rangle = h^{-1}(a+B)$, so conclusion (ii) applies.

In each of the remaining cases ($\mathcal{R} = \mathcal{H}$ and $h = 1$; $\mathcal{R} = \mathcal{E}$ and $h = 1$ or θ), we observe that $\mathrm{Im}(h\mathcal{R}) = \mathrm{Im}\,\mathcal{R}$ and recall that $\mathrm{Im}\,\mathcal{R}$ has covering radius $3^{1/2}/2$. That is, by making an appropriate choice of z we may suppose $|B|^2 \le 3/4$. We have assumed that $D^2 \le 1/|h|^2$, so $a' \le 1/2$. Therefore $|\mathrm{ht}\,(v')|^2 = (a')^2 + |B|^2$ is less than $1 = |\mathrm{ht}\,(v)|^2$ (so that conclusion (i) applies) unless $a' = 1/2$ and $|B|^2 = 3/4$. In this case we see that $D^2 = 1/|h|^2$ and that B is a deep hole of $\mathrm{Im}\,\mathcal{R}$. Since all deep holes of $\mathrm{Im}\,\mathcal{R}$ are equivalent by translations of $\mathrm{Im}\,\mathcal{R}$, by choice of z we may take $B = \theta/2$. Then by Eq. (5.3) we have $\langle r'|v\rangle = h^{-1}(a+B) = h^{-1}(\omega+1)$, so conclusion (iii) applies. □

Theorem 5.3. *Let $L = \Lambda \oplus II_{1,1}$ for some positive-definite $\mathcal{R}$-lattice Λ. Let h be one of*

1, θ, 2, *or* 2θ	*if* $\mathcal{R} = \mathcal{E}$, *or*
1, $1+\mathrm{i}$, *or* 2	*if* $\mathcal{R} = \mathcal{H}$ *or* $\mathcal{G}$.

Let r be a short root of L of height h lying over λh^{-1}, with $\lambda \in \Lambda$. Let $v \in L \otimes \mathbb{R}$ be isotropic, have height one and lie over $\ell \in \Lambda \otimes \mathbb{R}$. Set $D^2 = (\ell - \lambda h^{-1})^2$ and suppose $D^2 \le 1/|h|^2$. Then there exists a short root r' of L also of height h and lying over λh^{-1} such that either

(i) some reflection in r' carries v to a vector of height smaller than that of v,

or else $D^2 = 1/|h|^2$ and (exactly) one of the following holds:

(ii) $\langle r'|v\rangle = 0$.

(iii) $\mathcal{R} = \mathcal{G}$, $h = 1+\mathrm{i}$ *or* 2, *and* $\langle r'|v\rangle = h^{-1}\mathrm{i}$.

(iv) $\mathcal{R} = \mathcal{E}$, $h = 2$ *or* 2θ, *and* $\langle r'|v\rangle = h^{-1}\theta$.

(v) $\mathcal{R} = \mathcal{H}$, $h = 1+\mathrm{i}$ *and* $\langle r'|v\rangle = h^{-1}\mathrm{i} = (\mathrm{i}+1)/2$.

(vi) $\mathcal{R} = \mathcal{H}$, $h = 2$ *and* $\langle r'|v\rangle = h^{-1}(b\mathrm{i} + c\mathrm{j} + d\mathrm{k})$ *for some* $b, c, d \in \{0, 1\}$, *not all zero.*

Remarks: For each h treated, the elements of $\mathcal{R}$ with absolute value $|h|$ are unit scalar multiples of each other and generate a 2-sided ideal in $\mathcal{R}$. It appears that the values of h treated here are the only ones for which conclusions similar to these can be obtained. The only place in this paper that cases (*iii*), (*iv*) and (*vi*) are used is in the proof of Theorem 6.2. Furthermore, none of these cases are required for the examples treated with that theorem. Thus, the reader may ignore these cases if desired. Omitting their treatment here would not significantly shorten or simplify the proof.

Proof: Following the first part of the proof of Theorem 5.2, we have

$$v = (\ell; 1, w - \ell^2/2)$$

for some $w \in \operatorname{Im}\mathbb{K}$, and

$$r = \left(\lambda; h, z_0 + \frac{1-\lambda^2}{2|h|^2}h\right) \tag{5.5}$$

for some $z_0 \in \mathbb{K}$ such that $\operatorname{Re}(\bar{h}z_0) = 0$. Continuing to follow that argument, the other short roots r' of L lying over λh^{-1} all have the form $r + (0; 0, z)$ for $z \in \mathcal{R}$ such that $\operatorname{Re}(\bar{h}z) = 0$. Modifying the derivation of Eq. (5.3) slightly we find

$$\begin{aligned}\langle r'|v\rangle =& \langle\lambda|\ell\rangle + \bar{h}\left(w - \frac{\ell^2}{2}\right) + \overline{\left(z_0 + z + \frac{(1-\lambda^2)h}{2|h|^2}\right)} \\ =& h^{-1}\left[\left(\frac{1}{2} - \frac{|h|^2}{2}D^2\right) + \left(|h|^2 \operatorname{Im}\langle\lambda h^{-1}|\ell\rangle + |h|^2 w + h\bar{z}_0 + h\bar{z}\right)\right] \\ =& h^{-1}[a + B] \end{aligned} \tag{5.6}$$

where $a = (1 - |h|^2D^2)/2$ is the real part of the term in brackets and B is the imaginary part. The slight difference between the terms a in Eqs. (5.3) and (5.6) is due to the replacement of $(2 - \lambda^2)$ in Eq. (5.2) by $(1 - \lambda^2)$ in Eq. (5.5), which is due to the fact that r is now a short root.

We take v' to be the image of v under ξ-reflection in r' (we will choose ξ later). Since $v' = v - r'(1 - \xi)\langle r'|v\rangle$, we have

$$\begin{aligned}\langle\rho|v'\rangle &= \langle\rho|v\rangle - \langle\rho|r'\rangle(1-\xi)\langle r'|v\rangle \\ &= 1 + \frac{h(\xi-1)\bar{h}}{|h|^2}[a+B]\ .\end{aligned} \tag{5.7}$$

By hypothesis, $D^2 \leq 1/|h|^2$, so $a \in [0, 1/2]$. As before, by changing our choice of z we may change B by any element of $\operatorname{Im}(h\mathcal{R})$. Now we will treat $\mathcal{G}$, $\mathcal{E}$ and $\mathcal{H}$ separately; the analysis is just like that of the proof of Theorem 5.2 except that our freedom to choose ξ creates opportunities which require more complicated computations to exploit.

Suppose $\mathcal{R} = \mathcal{G}$. If $h = 1$ (resp. $1 + \mathrm{i}$ or 2) then $\operatorname{Im}(h\mathcal{G})$ is the set of integral multiples of i (resp. 2i). Writing $B = b\mathrm{i}$ with $b \in \mathbb{R}$, we may take $b \in [0, 1)$ (resp. $b \in (-1, 1]$). Taking $\xi = \pm\mathrm{i}$ we find by Eq. (5.7) that $\langle\rho|v'\rangle = 1 + (\pm\mathrm{i} - 1)[a + b\mathrm{i}]$. Computing the norm of this, or drawing pictures in $\mathbb{C}$, shows that if $b \in [0, 1)$ then upon taking $\xi = +\mathrm{i}$ we find $\operatorname{ht}(v') < 1 = \operatorname{ht}(v)$ (so conclusion (*i*) applies) unless $a = b = 0$, in which case Eq. (5.6) shows that conclusion (*ii*) applies. If $b \in (-1, 0]$ then taking $\xi = -\mathrm{i}$ we obtain the same result. Finally, if $b = 1$, which can only happen if $h = 1 + \mathrm{i}$ or 2, then taking $\xi = +\mathrm{i}$ yields $\operatorname{ht}(v') < \operatorname{ht}(v)$ unless $a = 0$. If $a = 0$ then $D^2 = 1/|h|^2$ and by Eq. (5.6) we find that

$$\langle r'|v\rangle = h^{-1}[a + b\mathrm{i}] = h^{-1}(0 + \mathrm{i})\ ,$$

so conclusion (*iii*) applies.

Now suppose $\mathcal{R} = \mathcal{E}$. If $h = 1$ (resp. θ, 2, 2θ) then $\operatorname{Im}(h\mathcal{E})$ is the set of integer multiples of θ (resp. θ, 2θ, 2θ). Writing $B = b\mathrm{i}$ with $b \in \mathbb{R}$ we may take $b \in [0, 3^{1/2})$ (resp. $b \in [0, 3^{1/2})$,

$b \in (-3^{1/2}, 3^{1/2}]$, $b \in (-3^{1/2}, 3^{1/2}]$). Using Eq. (5.7) and drawing pictures in $\mathbb{C}$, we see that if $b \in [0, 3^{1/2})$ then upon taking $\xi = -\bar{\omega}$ we find $|\mathrm{ht}\,(v')| < |\mathrm{ht}\,(v)|$ (so that conclusion (i) applies) unless $a = b = 0$, in which case (ii) applies. If $b \in (-3^{1/2}, 0]$ then taking $\xi = -\omega$ we obtain the same result. Finally, if $b = 3^{1/2}$, which can only happen if $h = 2$ or 2θ, then taking $\xi = -\bar{\omega}$ yields $|\mathrm{ht}\,(v')| < |\mathrm{ht}\,(v)|$ unless $a = 0$. If $a = 0$ then $D^2 = 1/|h|^2$ and by Eq. (5.6) we have

$$\langle r'|v\rangle = h^{-1}[0 + 3^{1/2}\mathrm{i}] = h^{-1}\theta \ ,$$

so conclusion (iv) applies.

Now suppose $\mathcal{R} = \mathcal{H}$. We write B as $b\mathrm{i} + c\mathrm{j} + d\mathrm{k}$ with $b, c, d \in \mathbb{R}$. We first carry out a computation that will allow us to use the 24 units of $\mathcal{H}$ effectively: we claim that there is a unit ξ' of $\mathcal{H}$ with $\mathrm{Re}\,\xi' = -1/2$ such that

$$|1 + \xi'(a + B)|^2 = (a - 1/2)^2 + (|b| - 1/2)^2 + (|c| - 1/2)^2 + (|d| - 1/2)^2 \ . \tag{5.8}$$

For any unit ξ', the left side is just the square of the distance between $a + B$ and $-\bar{\xi}'$ (proof: left-multiply by $1 = |-\bar{\xi}'|^2$). Setting $-\bar{\xi}' = (1 \pm \mathrm{i} \pm \mathrm{j} \pm \mathrm{k})/2$, with each of its i, j and k components having the same sign as the corresponding component of B (or a random sign if that component of B vanishes), the right hand side becomes another expression for this squared distance, proving the claim.

If $h = 1$ (resp. $h = 2$) then $\mathrm{Im}(h\mathcal{H})$ is the integral span of i, j and k (resp. of 2i, 2j and 2k), so we may take each of b, c and d to lie in $[0, 1)$ (resp. in $(-1, 1]$). Now suppose $h = 1 + \mathrm{i}$. It is easy to check that

$$\mathrm{Im}((1 + \mathrm{i})\mathcal{H}) = \{\, b\mathrm{i} + c\mathrm{j} + d\mathrm{k} \mid b, c, d \in \mathbb{Z},\ b + c + d \equiv 0 \pmod 2 \,\} \ .$$

That is, $\mathrm{Im}(h\mathcal{H})$ is spanned by $\mathrm{j} + \mathrm{k}$, $\mathrm{k} - \mathrm{i}$ and $\mathrm{i} + \mathrm{k}$, so by choice of z we may take $b \in (-1, 1]$ and $c, d \in [0, 1)$.

Let ξ' be a unit of $\mathcal{H}$ with $\mathrm{Re}\,\xi' = -1/2$ satisfying Eq. (5.8), and suppose for a moment that there is a unit ξ of $\mathcal{H}$ such that

$$\xi' = h(\xi - 1)\bar{h}/|h|^2 \ . \tag{5.9}$$

Then by Eqs. (5.7), (5.9) and (5.8),

$$\begin{aligned} |\mathrm{ht}\,v'|^2 &= |1 + \xi'(a + B)|^2 \\ &= (a - 1/2)^2 + (|b| - 1/2)^2 + (|c| - 1/2)^2 + (|d| - 1/2)^2 \ . \end{aligned} \tag{5.10}$$

By hypothesis $D^2 \le 1/|h|^2$, so $a \in [0, 1/2]$. By this and our constraints on b, c and d obtained above, we see that the right hand side of Eq. (5.10) is less than $1 = |\mathrm{ht}\,v|^2$ (so that conclusion (i) applies) unless $a = 0$ and $b, c, d \in \{0, 1\}$. In each of these cases, $a = 0$ implies that $D^2 = 1/|h|^2$, and $\langle r'|v\rangle$ can be read from Eq. (5.6). We obtain the following possibilities:

possibility	**value of h**	$\langle r'\|v\rangle$	**conclusion**
$b = c = d = 0$	1, $1 + \mathrm{i}$ or 2	0	(ii)
$b = 1$, $c = d = 0$	$1 + \mathrm{i}$ or 2	$h^{-1}\mathrm{i}$	(v) or (vi)
any other	2	$2^{-1}(b\mathrm{i} + c\mathrm{j} + d\mathrm{k})$	(vi).

(The last column refers to the various cases listed in the statement of the theorem.)

It remains only to show that given a unit ξ' of $\mathcal{H}$ with $\operatorname{Re}\xi' = -1/2$, there is another unit ξ of $\mathcal{H}$ satisfying Eq. (5.9). If $h = 1$ or 2 then this is trivial: take $\xi = \xi' + 1$. If $h = 1 + \mathrm{i}$ then we solve Eq. (5.9) for ξ:

$$\xi = |h|^2 \cdot h^{-1}\xi'\bar{h}^{-1} + 1 = \frac{(1-\mathrm{i})}{\sqrt{2}}\frac{\xi'(1+\mathrm{i})}{\sqrt{2}} + 1\,. \tag{5.11}$$

The most straightforward way to show that ξ is a unit of $\mathcal{H}$ is to simply evaluate the right hand side of Eq. (5.11) for each of the eight possibilities $\xi' = (-1 \pm \mathrm{i} \pm \mathrm{j} \pm \mathrm{k})/2$. (What is really going on here is that the units of $\mathcal{H}$ together with $2^{-1/2}(1 \pm \mathrm{i})$ generate the binary octahedral group, which normalizes the binary tetrahedral group consisting of the units of $\mathcal{H}$.) □

6. The reflection groups

This section is the heart of the paper: we will apply the results of Section 5 to find Lorentzian lattices that are reflective. The most basic of our results is

Theorem 6.1. *Suppose Λ is a positive-definite $\mathcal{R}$-lattice which is spanned by its roots and has covering radius ≤ 1. Then $L = \Lambda \oplus II_{1,1}$ is reflective.* □

(This follows immediately from Theorem 6.2 below.) The proof of such a theorem has two parts: first that $\operatorname{Reflec} L$ acts with only finitely many orbits on the primitive null vectors in L and second that the stabilizer in $\operatorname{Reflec} L$ of one such vector, namely ρ, has finite index in its stabilizer in $\operatorname{Aut} L$. The second part has already been proven, in Theorem 5.1. The first and more interesting part of the argument proceeds by supposing v to be a primitive null vector in L and repeatedly applying reflections to decrease the height of v. Theorems 5.2 and 5.3 assure us that we can find height-decreasing reflections if v lies over a point of $\Lambda \otimes \mathbb{K}$ sufficiently close to an element of Λ or to any of various other points of $\Lambda \otimes \mathbb{K}$. In particular, if the covering radius of Λ is small enough then we may always find suitable reflections and this allows us to reduce the height of v indefinitely. Since the details of Theorems 5.2 and 5.3 are quite complicated, conditions on Λ that allow us to bring to bear the full force of these results are bound to also be complicated. The condition defined below, that Λ be well-covered, is just a generalization of the requirement of Theorem 6.1 that Λ have covering radius 1.

There are two "tracks" through this section. The first deals with well-covered lattices and the general theory, with details worked out for a few low-dimensional examples. This track yields reflective lattices in $\mathbb{C}^{n,1}$ for $n \leq 5$ and in $\mathbb{H}^{n,1}$ for $n \leq 3$, and a detailed study of the reflection groups of $I^{\mathcal{E}}_{n,1}$ for $n \leq 4$ and $I^{\mathcal{H}}_{n,1}$ for $n \leq 3$. The second track treats in detail two high-dimensional examples, $I^{\mathcal{E}}_{7,1}$ and $I^{\mathcal{H}}_{5,1}$. This track begins with Lemma 6.8 and requires none of the earlier material of the section except for Theorem 6.4. Furthermore, it does not require the use of Theorem 5.2. The reader who wishes to know why these two lattices are reflective but is not interested in further details need read only Lemmas 6.9 and 6.11, which assert that $\Lambda^{\mathcal{E}}_6$ and $\Lambda^{\mathcal{H}}_4$ are well-covered, and then apply Theorem 6.2.

Suppose Λ is a positive-definite $\mathcal{R}$-lattice. We define a family $\mathcal{C}$ (for "covering") of closed balls centered at various points of $\Lambda \otimes \mathbb{R}$. If the union of these balls is all of $\Lambda \otimes \mathbb{R}$ then we say that Λ is well-covered. The definition of $\mathcal{C}$ depends on $\mathcal{R}$, and is given in Table 6.1.

The table should be interpreted as follows. A closed ball is a member of $\mathcal{C}$ just if for one of the rows listed under $\mathcal{R}$, it has the given radius and center, where $\lambda \in \Lambda$ satisfies the condition in the 'condition' column. If Λ is well-covered then a point of $\Lambda \otimes \mathbb{R}$ is called a $\mathcal{C}$-hole if it lies in the interior of none of the balls of $\mathcal{C}$; the $\mathcal{C}$-holes of a well-covered lattice obviously form a discrete set. We say that Λ is strictly well-covered if it has no $\mathcal{C}$-holes; that is, if the interiors of the balls cover $\Lambda \otimes \mathbb{R}$.

Table 6.1. The balls used to define the notion of a lattice being well-covered

The ring $\mathcal{R}$	**(Radius)^2**	**Center**	**Condition**	**Root length(s)**
$\mathcal{G}$	$\sqrt{3}$	λ	$\lambda^2 \equiv 0(2)$	long
	1	λ	$\lambda^2 \equiv 1(2)$	short
	$1/2$	$\lambda(1+\mathrm{i})^{-1}$	$\lambda^2 \equiv 1(2)$	short
	$1/4$	$\lambda/2$	$\lambda^2 \equiv 1(4)$	short
$\mathcal{E}$	1	λ	none	long, short
	$1/3$	$\lambda\theta^{-1}$	$\lambda^2 \equiv 2(3)$	long
	$1/3$	$\lambda\theta^{-1}$	$\lambda^2 \equiv 1(3)$	short
	$1/4$	$\lambda/2$	$\lambda^2 \equiv 1(2)$	short
	$1/12$	$\lambda\theta^{-1}/2$	$\lambda^2 \equiv 1(6)$	short
$\mathcal{H}$	1	λ	none	long, short
	$1/2$	$\lambda(1+\mathrm{i})^{-1}$	$\lambda^2 \equiv 1(2)$	short
	$1/4$	$\lambda/2$	$\lambda^2 \equiv 1(2)$	short

The last column of Table 6.1 indicates whether short or long roots of $L = \Lambda \oplus II_{1,1}$ (or roots of both lengths) lie over the center of the ball. The verifications that there are such roots is straightforward. For example, if $\mathcal{R} = \mathcal{E}$ and $\lambda \in \Lambda$ we can look for short roots of the form $r = (\lambda; \theta, \nu)$, which would lie over $\lambda\theta^{-1}$. Our search will succeed if we can choose $\nu \in \mathcal{E}$ so that $r^2 = 1$. We may restate the condition $r^2 = 1$ as $\mathrm{Re}(\theta\bar{\nu}) = (1-\lambda^2)/2$. Since the set of values taken by the left hand side is $\frac{3}{2}\mathbb{Z}$, as ν varies over $\mathcal{E}$, we can solve for $\nu \in \mathcal{E}$ exactly when $1 - \lambda^2 \in 3\mathbb{Z}$, that is, when $\lambda^2 \equiv 1(3)$. This sort of computation is the source of the entries in the 'condition' column.

Theorem 6.2. *Suppose $L = \Lambda \oplus II_{1,1}$ for some well-covered positive-definite $\mathcal{R}$-lattice Λ.*
(i) If no vector of Λ is fixed by every reflection of Λ then L is reflective.
(ii) If Λ is strictly well-covered then any two primitive null sublattices of L are equivalent under Reflec L.

Proof: Suppose v is a primitive null vector of L that has minimal height in its orbit under Reflec L, is not a multiple of ρ, and lies over $\ell \in \Lambda \otimes \mathbb{R}$. If ℓ lies in the interior of any ball of $\mathcal{C}$ then the image v' of v under some reflection of L has $\mathrm{ht}\,(v') < \mathrm{ht}\,(v)$, contradicting our hypothesis on v. Here's why: if a ball of $\mathcal{C}$ whose interior contains ℓ is centered at λh^{-1} for $\lambda \in \Lambda$ then there is a root r of L lying over λh^{-1}. By applying Theorem 5.3 in the case of a short root, or Theorem 5.2 in the case of a long root, we see that there is another root r' of L and a reflection in r' which carries v to a vector of smaller height. If Λ is strictly well-covered then this shows that v cannot exist, so every primitive isotropic vector of L is equivalent to one in the span of ρ. This proves (*ii*).

If Λ is well-covered but not strictly well-covered then we conclude from the argument above that ℓ is a $\mathcal{C}$-hole of Λ, and from the conclusions of Theorems 5.2 and 5.3 that for one of finitely many pairs $h, k \in \mathbb{K}$ there is a root r' of L of height h such that $\langle r'|v\rangle = k$. Under the hypothesis of (*i*), Theorem 5.1(*iv*) shows that the stabilizer of ρ in Reflec L has finite index in the stabilizer in Aut L. Since the translations of L act with only finitely many orbits on the vectors of any given norm and (nonzero) height, we see that after applying an element of Reflec L we may take r' to be one of some fixed finite set of roots. If r' is one of these roots then the conditions $\langle r'|v\rangle = k$ and $v^2 = 0$ together with the fact that v lies over a $\mathcal{C}$-hole of Λ determine v to within finitely many possibilities. Therefore Reflec L acts with only finitely many orbits on the primitive null vectors of L. This fact, together with the finite index of the stabilizer in $\operatorname{Reflec} L$ of one such vector, ρ, in its stabilizer in $\operatorname{Aut} L$, proves (i). □

Remark: If we were to restrict the definition of the covering $\mathcal{C}$ to only involve balls with centers over which lie short roots of L (i.e., those entries in Table (6.1) with 'short' in the last column), then we would obtain new definitions of 'well-covered' and 'strictly well-covered'. The proof of (ii) shows that if Λ is strictly well-covered in this sense then $\operatorname{Reflec}_0 L$ acts transitively on the primitive null lattices in L.

Corollary 6.3. *Let Λ be any one of the $\mathcal{R}$-lattices*

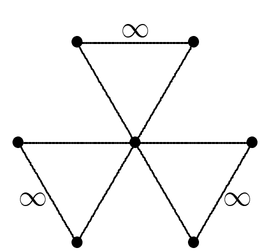
$$\begin{array}{ll} \mathcal{G},\ 2^{1/2}\mathcal{G},\ D_4^{\mathcal{G}},\ D_6^{\mathcal{G}},\ \text{or } E_8^{\mathcal{G}} & \text{if } \mathcal{R}=\mathcal{G}, \\ \mathcal{E},\ \mathcal{E}^2,\ \mathcal{E}^3,\ \text{or } D_3(\sqrt{-3}) & \text{if } \mathcal{R}=\mathcal{E}, \text{ or} \\ \mathcal{H},\ 2^{1/2}\mathcal{H},\ \mathcal{H}^2,\ \text{or } E_8^{\mathcal{H}} & \text{if } \mathcal{R}=\mathcal{H}. \end{array}$$

Then $L = \Lambda \oplus II_{1,1}^{\mathcal{R}}$ is reflective. Unless Λ is $\mathcal{E}^3$, $D_3(\sqrt{-3})$, $D_4^{\mathcal{H}}$, $\mathcal{H}^2$ or $E_8^{\mathcal{H}}$, all primitive null sublattices of L are equivalent under $\operatorname{Reflec} L$.

Remark: The lattices appearing here are all described in Section 3. Theorems 6.6 and 6.7 give much more precise information about $\operatorname{Reflec} L$ for $\Lambda = \mathcal{E}$, $\mathcal{E}^2$, $\mathcal{E}^3$, $\mathcal{H}$ or $\mathcal{H}^2$.

Proof: The covering radius of $\mathcal{G}$ is $2^{-1/2}$, so $\mathcal{G}$ is strictly well-covered. The remaining Gaussian lattices are all even and have covering radii 1, 1, $(3/2)^{1/2}$ and 1, respectively, so they are also strictly well-covered. The covering radii of the Eisenstein lattices are $(1/3)^{1/2}$, $(2/3)^{1/2}$, 1 and 1, respectively, so these lattices are well-covered and the first two strictly so. The Hurwitz lattices have covering radii $2^{-1/2}$, 1, 1 and 1, respectively, so all are well-covered and $\mathcal{H}$ strictly so. In each case the roots of Λ span $\Lambda \otimes \mathbb{R}$, so no element of Λ is fixed by all reflections of Λ. Our conclusions follow from Theorem 6.2. □

We can also apply the theorem when $\Lambda = \{0\}$, to deduce that $\operatorname{Reflec} II_{1,1}^{\mathcal{R}}$ acts on $\mathbb{K}H^1$ with finite covolume and that there is only one orbit of primitive isotropic lattices in $II_{1,1}^{\mathcal{R}}$. A little picture-drawing reveals that $\operatorname{Reflec} II_{1,1}^{\mathcal{E}}$ acts on the right half-plane (a copy of $\mathbb{C}H^1$) as the triangle group $(2,6,\infty)$. One can also show that $\operatorname{Aut} II_{1,1}^{\mathcal{G}}$ acts on $\mathbb{C}H^1$ as $(2,3,\infty)$ and its subgroup of index 2 consisting of elements with determinant +1 is conjugate in $\mathrm{GL}_2(\mathcal{G})$ to $\mathrm{SL}_2\,\mathbb{Z}$. The group $\operatorname{Reflec} II_{1,1}^{\mathcal{G}}$ is generated by 3 biflections, which act by rotations by π around the three finite corners of a quadrilateral in $\mathbb{C}H^1$ with corner angles $\pi/2$, $\pi/2$, $\pi/2$ and π/∞. See [16] for descriptions of the groups (p,q,r) and other information. By adapting the argument of [1, Thm. 5.3(i)], one can also show that $\operatorname{Reflec} II_{1,1}^{\mathcal{H}}$ acts on $\mathbb{H}H^1 \cong \mathbb{R}H^4$ as the rotation subgroup of the real hyperbolic reflection group with the Coxeter diagram below. Note that the 6 outer nodes generate an affine reflection group, so this strange-looking graph is just a special case of the usual procedure of "hyperbolizing" an affine reflection group by adjoining an extra node.



We will now study in more detail the reflection groups of several Lorentzian lattices over $\mathcal{E}$ and $\mathcal{H}$. The reader may follow two paths. One requires (i)–(iii) of Theorem 6.4 below but then nothing else until Lemma 6.8. This path leads to our highest-dimensional examples, Reflec $I^{\mathcal{E}}_{7,1}$ acting on $\mathbb{C}H^7$ and Reflec $I^{\mathcal{H}}_{5,1}$ acting on $\mathbb{H}H^5$. The other path requires (i')–(iii') of Theorem 6.4 and leads to detailed analyses of several low-dimensional examples, namely $I^{\mathcal{E}}_{n,1}$ for $n = 2, 3, 4$ and $I^{\mathcal{H}}_{n,1}$ for $n = 2, 3$. Theorem 6.4 is a more specialized version of Theorem 5.1. As usual, the different rings behave slightly differently.

Theorem 6.4. *Suppose $\mathcal{R} = \mathcal{E}$ or $\mathcal{H}$, that Λ is a positive-definite selfdual $\mathcal{R}$-lattice of positive dimension that is spanned by its roots, and that* Aut Λ *is generated by reflections. Let $L = \Lambda \oplus II^{\mathcal{R}}_{1,1}$. Then*

- (i) *If $\mathcal{R} = \mathcal{E}$ (resp. $\mathcal{H}$) then* Reflec L *contains all (resp. at least a quarter) of the translations of L. (If $\mathcal{R} = \mathcal{H}$ then* Reflec L *contains $T_{0,-\theta}$, and coset representatives for the translations of* Reflec L *in those of* Aut L *may be taken from $\{T_{0,0}, T_{0,\mathrm{i}}, T_{0,\mathrm{j}}, T_{0,\mathrm{k}}\}$.)*
- (ii) Reflec L *contains a transformation acting trivially on Λ and on $II^{\mathcal{R}}_{1,1}$ by left scalar multiplication by any given unit of $\mathcal{R}$.*
- (iii) *If $\mathcal{R} = \mathcal{E}$ (resp. $\mathcal{H}$) then the stabilizer of $\langle\rho\rangle$ in* Reflec L *coincides with (resp. has index at most four in) the stabilizer in* Aut L.

Furthermore, if Λ is spanned by its short roots then

- (i') *The conclusion of (i) holds with* $\mathrm{Reflec}_0\, L$ *in place of* Reflec L.
- (ii') *If $\mathcal{R} = \mathcal{H}$ then the conclusion of (ii) holds with* $\mathrm{Reflec}_0\, L$ *in place of* Reflec L. *If $\mathcal{R} = \mathcal{E}$ then* $\mathrm{Reflec}_0\, L$ *contains a transformation acting trivially on Λ and on $II^{\mathcal{E}}_{1,1}$ by the scalar ω.*
- (iii') *If $\mathcal{R} = \mathcal{H}$ then the conclusion of (iii) holds with* $\mathrm{Reflec}_0\, L$ *in place of* Reflec L. *If $\mathcal{R} = \mathcal{E}$ then the stabilizer of $\langle\rho\rangle$ in* Aut L *is generated by the stabilizer in* $\mathrm{Reflec}_0\, L$ *together with the central involution of $II^{\mathcal{E}}_{1,1}$ (or alternately that of L).*

Proof: (i),(i') The selfduality of Λ together with Theorem 5.1(ii) show that for each root r of Λ there exists $z \in \operatorname{Im}\mathbb{K}$ such that $T_{r,z} \in \operatorname{Reflec} L$. Since Λ is spanned by its roots, Eq. (4.3) shows that for all $\lambda \in \Lambda$ there exists $z \in \operatorname{Im}\mathbb{K}$ such that $T_{\lambda,z} \in \operatorname{Reflec} L$. Taking commutators of these translations and using the selfduality of Λ we find by Theorem 5.1(iii) that for each $z \in \mathcal{R}$, the central translation $T_{0,2\operatorname{Im} z}$ lies in Reflec L. (This is where we use the hypothesis $\dim \Lambda > 0$.) If $\mathcal{R} = \mathcal{E}$ then this yields all the central translations. If $\mathcal{R} = \mathcal{H}$ then this yields the central translations $T_{0,b\mathrm{i}+c\mathrm{j}+d\mathrm{k}}$ with $b, c, d \in \mathbb{Z}$ and $b \equiv c \equiv d$ (2), a subgroup of index 4 in the group of all central translations. (Coset representatives are $T_{0,0}$, $T_{0,\mathrm{i}}$, $T_{0,\mathrm{j}}$ and $T_{0,\mathrm{k}}$.) This proves (i). If the short roots of Λ span Λ then the entire argument goes through with $\mathrm{Reflec}_0\, L$ in place of Reflec L, because for a short root r of Λ, the proof of Theorem 5.1(ii) shows that there exists $z \in \operatorname{Im}\mathbb{K}$ such that $T_{r,z} \in \mathrm{Reflec}_0\, L$.

In the rest of the proof, unless otherwise specified, the term 'scalar matrix' will have a non-standard meaning: an isometry of L fixing Λ pointwise and acting on $II^{\mathcal{R}}_{1,1}$ by left-multiplication by some unit of $\mathcal{R}$. (Note that if $\mathcal{R} = \mathcal{H}$ then there may be no concept of "left-multiplication by scalars" on L.)

(ii') We have $T_{0,-\theta} \in \mathrm{Reflec}_0\, L$ by (i'). Let F be the transformation composed of $T_{0,-\theta}$ followed by $(-\omega)$-reflection in the short root $(0; 1, -\omega)$. It is obvious that F acts trivially on Λ and computation reveals that it acts on $II^{\mathcal{R}}_{1,1}$ by left multiplication by the matrix

$$\begin{pmatrix} 0 & \bar{\omega} \\ \bar{\omega} & 0 \end{pmatrix} .$$

The square of this matrix is the scalar matrix ω, which proves the claim in the case $\mathcal{R} = \mathcal{E}$. If $\mathcal{R} = \mathcal{H}$ then $\mathrm{Reflec}_0\, L$ contains the 8 matrices that are the images of F^2 under Aut $\mathcal{H}$. (This is because Aut $\mathcal{H}$, acting on $II^{\mathcal{R}}_{1,1}$, normalizes Reflec L even though it doesn't act by $\mathcal{H}$-linear transformations.) These matrices generate the group of scalar matrices, proving the claim in the case $\mathcal{R} = \mathcal{H}$.

(ii) If we no longer assume that Λ is spanned by its short roots then all of the above goes through with $\operatorname{Reflec} L$ in place of $\operatorname{Reflec}_0 L$ and the reference to (i') replaced by a reference to (i). Therefore it remains only to show that if $\mathcal{R} = \mathcal{E}$ then $\operatorname{Reflec} L$ contains all of the scalar matrices. This follows because the biflection B in $b = (0; 1, 1)$ fixes Λ pointwise and acts on $II_{1,1}^{\mathcal{E}}$ by the matrix

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} .$$

Observe that FB is the scalar matrix $-\bar{\omega}$, which generates the group of all scalar matrices.

(iii) By (ii), $\operatorname{Reflec} L$ acts transitively on the primitive vectors in $\langle \rho \rangle$. By (i), $\operatorname{Reflec} L$ contains all the translations of L (or at least a quarter, if $\mathcal{R} = \mathcal{H}$), and hence the stabilizer of ρ in $\operatorname{Reflec} L$ acts with only one orbit (or at most 4 orbits, if $\mathcal{R} = \mathcal{H}$) on the null vectors of L of height 1. The simultaneous stabilizer in $\operatorname{Aut} L$ of ρ and one such, namely $(0; 1, 0)$, is just $\operatorname{Aut} \Lambda$, which is generated by reflections by hypothesis. This proves (iii).

(iii') Let G be the group generated by $\operatorname{Reflec}_0 L$ and the central involution of L; it is obvious that G is normal in $\operatorname{Aut} L$. Since $\operatorname{Reflec}_0 L$ contains the central involution of Λ, we see that G contains the central involution J of $II_{1,1}^{\mathcal{R}}$. By the proof of (ii), $J = WFB$ where B is the biflection in $b = (0, \dots, 0; 1, 1)$ and W denotes the scalar matrix ω. Since $WF \in \operatorname{Reflec}_0 L$, we see that G is generated by $\operatorname{Reflec}_0 L$ and B, so $G \subseteq \operatorname{Reflec} L$. We claim that G contains $\operatorname{Aut} \Lambda$. Since Λ is spanned by its short roots, $\Lambda \cong \mathcal{R}^n$ for some n, and we may introduce an orthonormal basis for Λ. If $n = 1$ then $\operatorname{Aut} \Lambda$ is generated by reflections in its short roots, proving the claim. If $n > 1$ then it suffices to prove that G contains the coordinate permutations with respect to the chosen basis of Λ. That is, we must show that G contains the biflections in vectors like $x = (1, -1, 0, \dots, 0; 0, 0)$. Since G is normal in $\operatorname{Aut} L$ and contains B, it suffices to show that x and b are equivalent under $\operatorname{Aut} L$. To see this, observe that $T_{(1,0,\dots,0),\theta/2}$ followed by F, followed by the scalar matrix $-\omega$, followed by $T_{(-1,1,0,\dots,0),0}$, carries x to b.

Repeating the argument of (iii), replacing $\operatorname{Reflec} L$ by $G \subseteq \operatorname{Reflec} L$ and the references to (i) and (ii) by references to (i') and (ii'), proves that the stabilizers of $\langle \rho \rangle$ in G and $\operatorname{Aut} L$ coincide. If $\mathcal{R} = \mathcal{H}$ then by (ii') we have $J \in \operatorname{Reflec}_0 L$, so $\operatorname{Reflec}_0 L = G$, proving ($iii'$). If $\mathcal{R} = \mathcal{E}$ then (iii') follows from the several descriptions of G given above. □

Remarks: The condition $\dim \Lambda > 0$ is necessary; one can show that $\operatorname{Reflec} II_{1,1}^{\mathcal{E}}$ contains no scalars except the identity. It is interesting to note that when $\mathcal{R} = \mathcal{H}$ and Λ (of dimension > 0) is spanned by its short roots then $\operatorname{Reflec}_0 L$ contains biflections such as B in long roots of L. This follows from the fact that as in the proof of (ii), B may be written as a product of F and a scalar matrix. I suspect that in some cases, such as those studied in Theorem 6.7 below, $\operatorname{Reflec}_0 L$ contains all the reflections of L.

Lemma 6.5. *If r and r' are short roots in a lattice over $\mathcal{R} = \mathcal{E}$ or $\mathcal{H}$ and $\langle r | r' \rangle = 1$ then r and r' are equivalent under the group generated by the reflections in them.*

Proof: One checks that the $(-\omega)$-reflections R and R' in r and r' satisfy the braid relation $RR'R = R'RR'$. (Because the Hermitian form is degenerate on the span of r and r', one must check that this relation holds by using Eq. (2.1), not by just multiplying matrices for the actions of R and R' on the span of r and r'.) Rewriting this as $R'^{-1} R R' = R R' R^{-1}$ we see that R and R' are conjugate in the group they generate, which implies the lemma. □

Remark: The proof suggests connections between the braid groups and complex reflection groups. This connection was first observed in [15], and the braid groups play a central role in the work of Deligne and Mostow [17], Mostow [21] and Thurston [23]. They are also important in work of the author, J. Carlson and D. Toledo [2] on moduli of cubic surfaces.

A useful tool for studying Eisenstein lattices $L = \Lambda \oplus II_{1,1}^{\mathcal{E}}$ with Λ selfdual is the reduction of L modulo θ. We write q for the natural map $q : \mathcal{E} \to \mathcal{E}/\theta\mathcal{E} \cong \mathbb{F}_3$, and also for the natural map $q : L \to L/L\theta$; we write V for the $\mathbb{F}_3$-vector space $L/L\theta$. The Hermitian inner product on L gives rise to a symmetric bilinear form on V, given by $\langle q(v)|q(w)\rangle = q(\langle v|w\rangle)$. Since L is selfdual this pairing is a nondegenerate bilinear form on V and yields a natural homomorphism from $\operatorname{Aut} L$ to $H = \operatorname{Isom} V$, an orthogonal group over $\mathbb{F}_3$. There is a homomorphism from H to the set $\{\pm 1\}$ of nonzero square classes of $\mathbb{F}_3$, called the spinor norm map and characterized by the following property: if $v \in V$ has nonzero norm then the spinor norm of the reflection in v is the square class of the norm of v. We define the spinor norm of an element of $\operatorname{Aut} L$ to be the spinor norm of its image in H. If v is a long (resp. short) root of L then the biflection (resp. either 6-fold reflection) in r acts on V as the reflection in the image of r, and thus has spinor norm -1 (resp. $+1$). By choosing an orthogonal basis for L we obtain an orthogonal basis for V and then it is clear that the central involution has spinor norm -1. (There is an orthogonal basis for L by Theorem 7.1.)

Theorem 6.6. *Let $\mathcal{R} = \mathcal{E}$, $\Lambda = \mathcal{E}^n$ and $L = \Lambda \oplus II_{1,1}^{\mathcal{E}}$.*
(i) If $n = 1$ then $\operatorname{Reflec}_0 L$ acts with exactly 2 orbits of primitive null vectors, represented by $\pm\rho$. If $n = 2$ or 3 then $\operatorname{Reflec}_0 L$ acts transitively on the primitive null vectors of L.
(ii) If $n = 1$, 2 or 3 then $\operatorname{Aut} L = \operatorname{Reflec} L = \operatorname{Reflec}_0 L \times \{\pm I\}$.

Proof: For $n = 1$ or 2, $\mathcal{E}^n$ is strictly well-covered in the sense of the remark following Theorem 6.2 and so $\operatorname{Reflec}_0 L$ acts transitively on the primitive null lattices in L. Now suppose $n = 3$. Suppose $v \in L$ is a primitive null vector not proportional to ρ and of smallest height in its orbit under $\operatorname{Reflec}_0 L$. Since the covering radius of $\mathcal{E}^3$ is 1, Theorem 5.3(ii) implies that v is orthogonal to a short root of height 1. By applying a translation we may suppose that this root is $r_1 = (0, 0, 0; 1, -\omega)$. Taking $r_2 = (0, 0, 1; 0, 1)$ and $r_3 = (0, 0, 1; 0, 0)$ we see that $\langle r_1|r_2\rangle = \langle r_2|r_3\rangle = 1$, so by Lemma 6.5, r_1 is equivalent to r_3 under $\operatorname{Reflec}_0 L$. Thus v is equivalent to an element of $r_3^{\perp}$, which is a copy of $\mathcal{E}^2 \oplus II_{1,1}^{\mathcal{E}}$. Applying the $n = 2$ case, we see that for $n = 3$, $\operatorname{Reflec}_0 L$ acts transitively on the primitive null sublattices of L.

By considering the spinor norm, we see that $-I \notin \operatorname{Reflec}_0 L$. The transitivity above together with the equality (Theorem 6.4(iii')) of the stabilizers of $\langle\rho\rangle$ in $\operatorname{Aut} L$ and $\operatorname{Reflec}_0 L \times \{\pm I\}$ proves that $\operatorname{Aut} L = \operatorname{Reflec}_0 L \times \{\pm I\}$ and that $\operatorname{Reflec}_0 L$ is the kernel of the spinor norm map. By considerations of the spinor norm, no biflection in a long root of L lies in $\operatorname{Reflec}_0 L$. This proves ($ii$)

From the above, we conclude that any primitive null vector of L is equivalent under $\operatorname{Reflec}_0 L$ to one of $\pm\rho$. If $n > 1$ then $\pm\rho$ are equivalent, because the central involution followed by biflection in a long root of Λ exchanges them and has spinor norm 1. If $n = 1$ then $\pm\rho$ are not equivalent, or else the fact that the stabilizers of ρ in $\operatorname{Reflec}_0 L$ and $\operatorname{Aut} L$ coincide would prove $\operatorname{Reflec}_0 L = \operatorname{Aut} L$. □

Remark: The quotient by $\operatorname{Reflec}_0 I_{4,1}^{\mathcal{E}}$ of $\mathbb{C}H^4$ minus the union of the hyperplanes orthogonal to short roots of $I_{4,1}^{\mathcal{E}}$ may be identified with the moduli space of smooth cubic surfaces over $\mathbb{C}$. The analogous construction with $I_{3,1}^{\mathcal{E}}$ in place of $I_{4,1}^{\mathcal{E}}$ yields the moduli space of smooth genus 2 curves over $\mathbb{C}$. One can also construct the (fine) moduli space of marked smooth cubic surfaces by taking the quotient of $\mathbb{C}H^4$ (minus the same hyperplanes as above) by the congruence subgroup of $\operatorname{Reflec}_0 I_{4,1}^{\mathcal{E}}$ associated to the prime $\theta \in \mathcal{E}$. The quotient of $\operatorname{Reflec}_0 I_{4,1}^{\mathcal{E}}$ by this normal subgroup is the E_6 Weyl group, also known as "the group of the 27 lines on a cubic surface". See [2] for details.

Theorem 6.7. *Let $\mathcal{R} = \mathcal{H}$, $\Lambda = \mathcal{H}^n$ for $n = 1$ or 2, and $L = \Lambda \oplus II_{1,1}^{\mathcal{H}}$. Then $\operatorname{Reflec}_0 L$ acts transitively on the primitive null vectors of L and has index at most 4 in $\operatorname{Aut} L$.*

Proof: We observe that $\mathrm{Reflec}_0 L$ acts transitively on primitive null lattices in L. For $n = 1$ this follows from the fact that $\Lambda = \mathcal{H}$ is strictly well-covered in the sense of the remark following the proof of Theorem 6.2. For $n = 2$ it follows from an argument similar to the $n = 3$ case of Theorem 6.6. That is, the covering radius of $\Lambda = \mathcal{H}^2$ is 1, so if $v \in L$ is primitive, isotropic and of smallest height in its orbit under $\mathrm{Reflec}_0 L$ then by Theorem 5.3(ii) we see that v is either proportional to ρ or orthogonal to a short root r_1 of height 1. In the latter case, after applying a translation of $\mathrm{Reflec}_0 L$, courtesy of Theorem 6.4(i'), we may take $r = (0, 0; 1, x - \omega)$ for $x = 0$, i, j or k. In any of these cases, upon taking $r_2 = (0, 1; 0, 1)$ and $r_3 = (0, 1; 0, 0)$ we have $\langle r_1 | r_2 \rangle = \langle r_2 | r_3 \rangle = 1$. By Lemma 6.5, v is equivalent under $\mathrm{Reflec}_0 L$ to an element of $r_3^\perp$. Since $r_3^\perp$ is a copy of $\mathcal{H}^1 \oplus II_{1,1}^{\mathcal{H}}$, the transitivity follows from the case $n = 1$.

Using Theorem 6.4(ii'), the transitivity on primitive null vectors follows. The bound on the index of $\mathrm{Reflec}_0 L$ in $\mathrm{Aut}\, L$ follows from Theorem 6.4(iii'). □

Now we move on to higher-dimensional examples—we will construct a group acting on $\mathbb{C}H^7$ and another acting on $\mathbb{H}H^5$. These arise from our basic construction by taking $\Lambda = \Lambda_6^{\mathcal{E}}$ or $\Lambda_4^{\mathcal{H}}$.

Lemma 6.8. *Suppose $v, r_1, \ldots, r_m \in \mathbb{K}^n \oplus \mathbb{K}^{1,1}$ lie over $\ell, \lambda_1, \ldots, \lambda_m \in \mathbb{K}^n$ respectively. Suppose $v^2 = 0$, that $\langle r_i | v \rangle = 0$ for all i, and that the vectors $\lambda_i - \ell$ are linearly independent in $\mathbb{K}^n$. Then the images of the r_i in $v^\perp / \langle v \rangle$ are linearly independent.*

Proof: We may obviously replace v and the r_i by any scalar multiples of themselves and so suppose that they have height 1. Thus $v = (\ell; 1, ?)$ and $r_i = (\lambda_i; 1, ?)$ where the question marks denote irrelevant and possibly distinct elements of $\mathbb{K}$. Let T be the translation carrying v to $(0; 1, 0)$, so $T(r_i) = (\lambda_i - \ell; 1, 0)$. (The last coordinate vanishes because $\langle T(r_i) | Tv \rangle = 0$.) Since the image of $T(r_i)$ in $(Tv)^\perp / \langle Tv \rangle$ may be identified with its first coordinate, namely $\lambda_i - \ell$, the images of the $T(r_i)$ in $(Tv)^\perp / \langle Tv \rangle$ are linearly independent. The theorem follows immediately. □

Lemma 6.9. *$\Lambda_6^{\mathcal{E}} \otimes \mathbb{R}$ is covered by the closed balls of radius 1 centered at points of $\Lambda_6^{\mathcal{E}}$, together with those of radius $3^{-1/2}$ centered at points $\lambda\theta^{-1}$ with $\lambda \in \Lambda_6^{\mathcal{E}}$ and $\lambda^2 \equiv 1(3)$. In particular, $\Lambda_6^{\mathcal{E}}$ is well-covered.*

Proof: Section 7 of [13] defines a linear "gluing map" $g : \frac{1}{\theta}\Lambda_6^{\mathcal{E}}/\Lambda_6^{\mathcal{E}} \to \frac{1}{\theta}\Lambda_6^{\mathcal{E}}/\Lambda_6^{\mathcal{E}}$ with the property that the Leech lattice Λ_{24}, scaled down by $2^{1/2}$, is the real form of the lattice of vectors $(x_1, x_2) \in \left(\frac{1}{\theta}\Lambda_6^{\mathcal{E}}\right)^2$ satisfying $g(x_1 + \Lambda_6^{\mathcal{E}}) = x_2 + \Lambda_6^{\mathcal{E}}$. Identifying $\Lambda_6^{\mathcal{E}}$ with the set of such (x_1, x_2) with $x_2 = 0$, we see that the only points of $2^{-1/2}\Lambda_{24}$ at distance < 1 from $\Lambda_6^{\mathcal{E}} \otimes \mathbb{R}$ are those in $\Lambda_6^{\mathcal{E}}$ and those of the form $(x_1\theta^{-1}, x_2\theta^{-1})$ with x_2 a minimal vector of $\Lambda_6^{\mathcal{E}}$ (a long root). The definition of g (see [13]) shows that $x_1^2 \equiv 1(3)$ if and only if there is a long root x_2 of $\Lambda_6^{\mathcal{E}}$ such that $(x_1\theta^{-1}, x_2\theta^{-1}) \in 2^{-1/2}\Lambda_{24}$. By [10], the covering radius of $2^{-1/2}\Lambda_{24}$ is 1, so the intersections of the balls of radius 1 centered at the points of $\Lambda_6^{\mathcal{E}}$ and at the points $(x_1\theta^{-1}, x_2\theta^{-1})$ with $x_1^2 \equiv 1(3)$ and $x_2^2 = 2$ cover $\Lambda_6^{\mathcal{E}} \otimes \mathbb{R}$. Computing the radius of the intersection of a ball of the second family with $\Lambda_6^{\mathcal{E}} \otimes \mathbb{R}$, we obtain the lemma. □

Theorem 6.10. *Let $\Lambda = \Lambda_6^{\mathcal{E}}$ and $L = \Lambda \oplus II_{1,1}^{\mathcal{E}} \cong I_{7,1}^{\mathcal{E}}$.*

(i) *If $v \in L$ is primitive, isotropic and not equivalent to a multiple of ρ under $\mathrm{Reflec}_0 L$, then $v^\perp / \langle v \rangle \cong \mathcal{E}^6$.*

(ii) $\mathrm{Aut}\, L = \mathrm{Reflec}\, L$. *In particular, L is reflective.*

Proof: (i) Suppose that v is a primitive isotropic vector of smallest height in its orbit under $\mathrm{Reflec}_0 L$. Suppose v is not a multiple of ρ, so that it lies over some $\ell \in \Lambda \otimes \mathbb{R}$. By the minimality of its height and Theorem 5.3, ℓ lies at distance ≥ 1 from each lattice point $\lambda \in \Lambda$ and at distance $\geq 3^{-1/2}$ from each $\lambda\theta^{-1}$ with $\lambda \in \Lambda$ and $\lambda^2 \equiv 1(3)$. By Lemma 6.9 the set S of such points in $\Lambda \otimes \mathbb{R}$ is discrete. Let $\mu_1, \ldots, \mu_n$ be the elements of Λ with $(\ell - \mu_i)^2 = 1$ and let $\nu_1, \ldots, \nu_m$ be those vectors of the form $\lambda\theta^{-1}$ with $\lambda \in \Lambda$ and $\lambda^2 \equiv 1(3)$ such that $(\ell - \nu_i)^2 = 1/3$. Over each μ_i (resp. ν_i) there is a short root of L, say r_i (resp. s_i), of height 1 (resp. θ). By Theorem 5.3(*ii*), we may suppose that the r_i and s_i are orthogonal to v. Because S is discrete the vectors $\mu_i - \ell$ and $\nu_i - \ell$ span $\Lambda \otimes \mathbb{R}$. By Lemma 6.8 this implies that among the images of the vectors r_i and s_i in $v^\perp/\langle v\rangle$ are 6 short roots that are linearly independent over $\mathcal{E}$. Since $v^\perp/\langle v\rangle$ is positive-definite, it follows that $v^\perp/\langle v\rangle \cong \mathcal{E}^6$.

(*ii*) Follows from the transitivity on primitive null sublattices that are orthogonal to no short roots (such as $\langle\rho\rangle$) and Theorem 6.4(*iii*). □

We now study the quaternionic lattice $I^{\mathcal{H}}_{5,1}$. The analysis is surprisingly similar to our study of $I^{\mathcal{E}}_{7,1}$. In particular, Lemma 6.11 below is very similar to Lemma 6.9. Section 5 of [11] describes an embedding of the real form BW_{16} of $2^{1/2}\Lambda^{\mathcal{H}}_4$ into the Leech lattice Λ_{24}. (Up to isometry of Λ_{24}, there is only one embedding.) When we refer to concepts involving Λ_{24} while discussing $\Lambda^{\mathcal{H}}_4$, we implicitly refer to this embedding.

Lemma 6.11. *$\Lambda^{\mathcal{H}}_4 \otimes \mathbb{R}$ is covered by the closed balls of radius 1 centered at points of $\Lambda^{\mathcal{H}}_4$ together with those of radius $2^{-1/2}$ centered at points $\lambda(1+\mathrm{i})^{-1}$ with $\lambda \in \Lambda^{\mathcal{H}}_4$ of odd norm. In particular, $\Lambda^{\mathcal{H}}_4$ is well-covered. Any point of $\Lambda^{\mathcal{H}}_4 \otimes \mathbb{R}$ not in the interior of one of these balls is a deep hole of $2^{-1/2}\Lambda_{24}$.*

Proof: The orthogonal complement of $\Lambda^{\mathcal{H}}_4$ in $2^{-1/2}\Lambda_{24}$ is a copy of the E_8 lattice. Properties of the embedding are described in [11] and include the following. If $(x, y) \in (\Lambda^{\mathcal{H}}_4 \otimes \mathbb{R}) \times (E_8 \otimes \mathbb{R})$ lies in $2^{-1/2}\Lambda_{24}$, then $y \in \frac{1}{2}E_8$ and hence has norm $n/2$ for some nonnegative integer n. We write $B(x, y)$ for the ball of radius 1 with center $(x, y) \in 2^{-1/2}\Lambda_{24}$. Only if the norm of y is 0 or 1/2 does the interior of $B(x, y)$ meet $\Lambda^{\mathcal{H}}_4 \otimes \mathbb{R}$. Those x for which $(x, y) \in 2^{-1/2}\Lambda_{24}$ for some y of norm 1/2 are exactly the deep holes of $\Lambda^{\mathcal{H}}_4$. By Theorem 3.1, the set of such x coincides exactly with $\{\, \lambda(1+\mathrm{i})^{-1} \mid \lambda \in \Lambda^{\mathcal{H}}_4, \lambda^2 \equiv 1(2) \,\}$. For such (x, y), the ball $B(x, y)$ meets $\Lambda^{\mathcal{H}}_4 \otimes \mathbb{R}$ in a ball of radius $2^{-1/2}$. The theorem follows from the fact [10] that the covering radius of $2^{-1/2}\Lambda_{24}$ is 1. □

The deep holes of $2^{-1/2}\Lambda_{24}$ played a role in our study of $L^{\mathcal{E}} = \Lambda^{\mathcal{E}}_6 \oplus II^{\mathcal{E}}_{1,1}$ in Theorem 6.10. We showed there that if v is a primitive null vector of $L^{\mathcal{E}}$ lying over such a hole in $\Lambda^{\mathcal{E}}_6 \otimes \mathbb{R}$, then there are short roots of $v^\perp/\langle v\rangle$ associated to certain elements of $\Lambda^{\mathcal{E}}_6$ and $\Lambda^{\mathcal{E}}_6\theta^{-1}$ near the hole. Given the similarity between Lemmas 6.9 and 6.11, one should expect something similar to happen in the study of $L^{\mathcal{H}} = \Lambda^{\mathcal{H}}_4 \oplus II^{\mathcal{H}}_{1,1}$. Indeed it does, but because the conclusions of Theorem 5.3 for short roots of height $1 + \mathrm{i}$ in Hurwitz lattices are slightly weaker than those for short roots of heights 1 and θ in Eisenstein lattices, the arguments are more complicated. Lemma 6.13 provides the necessary technical information about the deep holes of Λ_{24}. The language of affine Coxeter-Dynkin diagrams is the natural way to discuss these deep holes; see [10] for background. Since the real form BW_{16} of $2^{1/2}\,\Lambda^{\mathcal{H}}_4$ is the sublattice of Λ_{24} fixed by an involution, we will study the actions of involutions on affine diagrams.

If Δ is an affine Coxeter-Dynkin diagram each of whose components has type A_n, D_n or E_n, then an affine simple root system of type Δ is a set of vectors v_i of norm 2 in real Euclidean space, one for each node of Δ, satisfying $\langle v_i|v_j\rangle = 0$, -1 or -2 according as the corresponding nodes of Δ are disjoined, joined or doubly joined. (The last case occurs only for the affine diagram A_1, where the doubly joined vectors are each others negatives.) We usually identify the nodes of Δ with the vectors v_i, so we may speak of vectors being disjoined or of nodes being linearly independent. Each component X_n has $n+1$ nodes and spans an n-dimensional space; any n of its vectors are linearly independent. The spaces spanned by distinct components are orthogonal.

Lemma 6.12. *Suppose Δ is an affine simple root system spanning a Euclidean space E and that ϕ is an isometry of E permuting the vectors of Δ. Let A (resp. B) be the number of orbits of ϕ*

on the vertices (resp. components) of Δ. Then $A - B$ is the dimension of the space F of points fixed by ϕ. Furthermore, if ϕ has prime order p then

$$p \dim F - \dim E = (P_V - P_C)(p-1)\ , \tag{6.1}$$

where P_V (resp. P_C) is the number of vertices (resp. components) of Δ preserved by ϕ.

Proof: Since the vectors v_i of Δ span E, ϕ is determined by its action on Δ. Let M be the order of ϕ and let π denote the projection $x \mapsto \frac{1}{M}\sum_{j=1}^{M} \phi^j(x)$ of E to F. Naturally, F is spanned by the vectors $\pi(v_i)$; there are A distinct such vectors. Let the minimal affine subdiagrams of Δ preserved by ϕ be denoted $\Delta_1, \ldots, \Delta_B$. (Such a diagram is just a union of components of Δ permuted cyclically by ϕ.) Let F_j (for $1 \leq j \leq B$) be the intersection of the span of Δ_j with F. If $j \neq k$ then Δ_j is orthogonal to Δ_k and thus F_j and F_k are also orthogonal. Since each vector $\pi(v_i)$ lies in some F_j, we see that $F = \oplus_{j=1}^{B} F_j$. Therefore the lemma holds if it holds with Δ replaced by each Δ_j in turn.

It now suffices to prove the lemma under the hypothesis that ϕ acts transitively on the components of Δ. Each of the vectors $\pi(v_i)$ has the form $\frac{1}{|J_i|}\sum_{j\in J_i} v_i$ for some ϕ-orbit J_i of nodes of Δ that meets each component of Δ. If v_i and v_j are not ϕ-equivalent then the sets J_i and J_j are disjoint. In order for some subset of the $\pi(v_i)$ to be linearly independent, the union of the corresponding J_i must contain an affine diagram. Since such a diagram would be a component of Δ and ϕ permutes the components transitively, the union of the J_i would have to be all of Δ. Therefore there is at most one linear dependence among the $\pi(v_i)$. There is at least one linear dependence because the vectors of any given component of Δ are dependent and therefore their images under π are also. (Indeed, some of the $\pi(v_i)$ might vanish, as happens when Δ is an A_1 diagram and ϕ is the nontrivial automorphism.) Therefore F has dimension $A - 1 = A - B$.

Now suppose ϕ has prime order p. If Δ has C components then it has $C + \dim E$ vertices. Then $B = P_C + (C - P_C)/p$ and $A = P_V + (C + \dim E - P_V)/p$. Arithmetic proves Eq. (6.1). □

Lemma 6.13. *Let ℓ be a deep hole of $2^{-1/2}\Lambda_{24}$ that lies in $\Lambda_4^{\mathcal{H}} \otimes \mathbb{R}$. Then there are at least 5 vertices v_i of ℓ that lie in $\Lambda_4^{\mathcal{H}}$ such that the vectors $v_i - \ell$ are linearly independent over $\mathbb{R}$.*

Proof: In this proof we'll scale everything up by $2^{1/2}$, so ℓ is a deep hole of Λ_{24} that lies in $BW_{16} \otimes \mathbb{R}$. By [10], if we take coordinates centered at ℓ then the vertices in Λ_{24} of the hole form an affine simple root system spanning $\mathbb{R}^{24}$. Let Δ be the associated Coxeter-Dynkin diagram, which is a union of affine diagrams of types A_n, D_n and E_n. The space $BW_{16} \otimes \mathbb{R}$ is the fixed space of an involution ϕ of Λ_{24}. Since ϕ fixes ℓ, it acts on Δ. Let F, P_V and P_C be as in Lemma 6.12. Then we have $2 \cdot 16 = 24 + P_V - P_C$, so $P_V - P_C = 8$. This shows $P_V \geq 8$; since $P_V > 0$ we have $P_C > 0$ and so we actually have $P_V > 8$.

Therefore there are $P_V > 8$ vertices of Δ fixed by ϕ, which is to say, lying in BW_{16}. The number of independent linear conditions satisfied by a set S of vertices of Δ equals the number of components of Δ all of whose vertices lie in S. Since each affine diagram has at least 2 vertices, there are at most $P_V/2$ conditions, so among the vertices of ℓ lying in BW_{16} there are at least 5 that are linearly independent over $\mathbb{R}$. The lemma follows because we took ℓ as the origin. □

Remark: The proof shows that any deep hole of Λ_{24} that lies in in $BW_{16} \otimes \mathbb{R}$ has at least 9 vertices in BW_{16}. This is the best possible result, because there are holes of type E_8^3 in BW_{16}; the involution ϕ fixes one E_8 diagram and exchanges the other two.

Theorem 6.14. *Let $\Lambda = \Lambda_4^{\mathcal{H}}$ and $L = \Lambda \oplus II_{1,1}^{\mathcal{H}} \cong I_{5,1}^{\mathcal{H}}$.*

(i) If $v \in L$ is a primitive isotropic vector not equivalent under $\mathrm{Reflec}_0\, L$ *to a multiple of ρ then $v^{\perp}/\langle v \rangle$ contains at least two linearly independent short roots.*

(ii) The index of $\operatorname{Reflec} L$ *in* $\operatorname{Aut} L$ *is at most* 4*. In particular,* L *is reflective.*

Proof: (i) This is very similar to the proof of Theorem 6.10(i). Suppose that v is a primitive isotropic vector of L of smallest height in its orbit under $\operatorname{Reflec} L$. Suppose v is not a multiple of ρ, so that v lies over some $\ell \in \Lambda \otimes \mathbb{R}$. By Theorem 5.3, ℓ lies at distance ≥ 1 from each lattice point $\lambda \in \Lambda$ and at distance $\geq 2^{-1/2}$ from each point $\lambda(1+\mathrm{i})^{-1}$ with $\lambda \in \Lambda$ and $\lambda^2 \equiv 1(2)$. By Lemma 6.11, ℓ must be a deep hole of $2^{-1/2}\Lambda_{24}$. By Lemma 6.13 there are 5 vertices $w_1, \ldots, w_5$ of the hole that lie in $\Lambda_4^{\mathcal{H}}$ such that the vectors $w_i - \ell$ are linearly independent over $\mathbb{R}$. Since there are 5 of them, we may suppose that $w_1 - \ell$ and $w_2 - \ell$ are linearly independent over $\mathbb{H}$. Because $w_1, w_2 \in \Lambda_4^{\mathcal{H}}$, There are short roots r_1 and r_2 of L of height 1 lying over them. Since their heights are 1, Theorem 5.3(ii) assures us that r_1 and r_2 may be chosen orthogonal to v. By Lemma 6.8, the images of r_1 and r_2 in $v^\perp/\langle v\rangle$ are linearly independent, proving (i).

(ii) Follows from Theorem 6.4(iii) and the fact (i) that $\operatorname{Reflec} L$ acts transitively on the primitive null lattices (such as $\langle\rho\rangle$) in L that are orthogonal to no short roots. □

Remark: In light of the fact that $\operatorname{Reflec}_0 I_{n,1}^{\mathcal{H}}$ contains the biflections in some long roots (see the remark following the proof of Theorem 6.4), it is possible that $\operatorname{Reflec} L = \operatorname{Reflec}_0 L$.

7. Enumeration of selfdual lattices

As we explain below, the orbits of primitive isotropic lattices in the Lorentzian lattice $I_{n+1,1}^{\mathcal{R}}$ are in natural 1-1 correspondence with the equivalence classes of positive-definite selfdual lattices of dimension n over $\mathcal{R}$. This means that one may classify such lattices by studying $\operatorname{Aut} I_{n+1,1}^{\mathcal{R}}$. Since we did just that in Section 6, for various $\mathcal{R}$ and n, we can now provide geometric proofs of such classifications. We begin with an analogue of a result well-known for lattices over $\mathbb{Z}$.

Theorem 7.1. *An indefinite selfdual lattice* L *over* $\mathcal{R} = \mathcal{E}$ *or* $\mathcal{H}$ *is characterized up to isometry by its dimension and signature. An indefinite selfdual lattice* L *over* $\mathcal{R} = \mathcal{G}$ *is characterized up to isometry by its dimension, signature, and whether it is even; if* L *is even then its signature is divisible by* 4*.*

Proof: First we show that L contains an isotropic vector. If $\mathcal{R} = \mathcal{H}$, or if $\dim L > 2$, then the real form of $L \otimes \mathbb{Q}$ is an indefinite rational bilinear form of rank > 4, so Meyer's theorem [19, Chap. 2] asserts the existence of an isotropic vector. If $\dim L = 2$ and $\mathcal{R} = \mathcal{G}$ or $\mathcal{E}$, then we consider the 2×2 matrix of inner products of the elements of a basis for L. This may be diagonalized by row and column operations over $\mathcal{R} \otimes \mathbb{Q}$ to a diagonal matrix $[a, -a^{-1}]$ with $a \in \mathbb{Q}$. (Each term is real because the matrix is Hermitian, and each determines the other because the determinant is -1.) Then the vector $(1, a)$ is isotropic. Having obtained an isotropic vector in $L \otimes \mathbb{Q}$, we may multiply by a scalar to obtain an isotropic vector of L.

If L is odd then the proof of Theorem 4.3 in [19, Chap. 2] applies, and $L \cong I_{n,m}^{\mathcal{R}}$ for some n and m. This completes the proof of the first claim, since any selfdual lattice over $\mathcal{E}$ or $\mathcal{H}$ is odd: if $v, w \in L$ satisfy $\langle v|w\rangle = \omega$ then v^2, w^2 and $(v+w)^2$ cannot all have the same parity. Existence of such v and w is assured by the selfduality of L. This also proves that an odd indefinite selfdual Gaussian lattice is characterized by its dimension and signature.

One may construct lattices N from an odd Gaussian lattice M by considering the sublattice M^e consisting of the elements of M of even norm, and considering the 3 lattices N such that $M_e' \subseteq N \subseteq M_e$. When M is $I_{1,1}^{\mathcal{G}}$, then N may be chosen to be $II_{1,1}^{\mathcal{G}}$. Now consider an indefinite even selfdual $\mathcal{G}$-lattice L. We know that L contains an isotropic vector, and as in [19] there is a decomposition $L = \Lambda \oplus II_{1,1}^{\mathcal{G}}$. We see that L arises from applying the construction above to the odd selfdual lattice $\Lambda \oplus I_{1,1}^{\mathcal{G}}$. Since $\Lambda \oplus I_{1,1}^{\mathcal{G}}$ is isomorphic to $I_{n,m}^{\mathcal{G}}$, with n and m determined by the dimension and signature of L, is is clear that L can be constructed by applying our construction to $I^{\mathcal{G}}_{n,m}$. No even lattices arise unless $n-m \equiv 0(4)$, when two isometric ones do. □

Special cases of Theorem 7.1 are $I^{\mathcal{E}}_{7,1} \cong \Lambda^{\mathcal{E}}_6 \oplus II^{\mathcal{E}}_{1,1}$ and $I^{\mathcal{E}}_{6,1} \cong \Lambda^{\mathcal{H}}_4 \oplus II^{\mathcal{H}}_{1,1}$, which are the lattices studied in Theorems 6.10 and 6.14. Theorem 7.1 also provides the correspondence mentioned above: if V is a primitive isotropic lattice in $I^{\mathcal{R}}_{n+1,1}$ then it is easy to check that $V^{\perp}/V$ is an n-dimensional positive-definite selfdual lattice. Every such lattice Λ arises this way, because we may write $I^{\mathcal{R}}_{n+1,1} \cong \Lambda \oplus I^{\mathcal{R}}_{1,1}$, and $I^{\mathcal{R}}_{1,1}$ contains isotropic lattices. Furthermore, if V and V' are primitive isotropic lattices in $I^{\mathcal{R}}_{n+1,1}$ and $V^{\perp}/V \cong V'^{\perp}/V'$ then one may find an isometry of $I^{\mathcal{R}}_{n+1,1}$ carrying V to V'. This provides a one-to-one correspondence between orbits of primitive isotropic lattices of $I^{\mathcal{R}}_{n+1,1}$ and selfdual positive-definite lattices in dimension n. Similarly, the orbits of primitive isotropic lattices of $II^{\mathcal{G}}_{n+1,1}$ correspond to even positive-definite selfdual Gaussian lattices of dimension n.

Theorem 7.2. *There are exactly two positive-definite selfdual $\mathcal{E}$-lattices in dimension* 6, *namely $\Lambda^{\mathcal{E}}_6$ and $\mathcal{E}^6$. There are exactly two positive-definite selfdual $\mathcal{H}$-lattices of dimension* 4, *namely $\Lambda^{\mathcal{H}}_4$ and $\mathcal{H}^4$.*

Proof: In light of the correspondence between selfdual lattices over $\mathcal{E}$ and orbits of primitive isotropic lattices in $I^{\mathcal{E}}_{n,1}$, to prove the first claim it suffices to show that if V is such a lattice then $V^{\perp}/V$ is isomorphic to one of $\Lambda^{\mathcal{E}}_6$ and $\mathcal{E}^6$. This follows from Theorem 6.10(i).

Before proving the second claim, we show that the only positive-definite selfdual $\mathcal{H}$-lattice of dimension 2 is $\mathcal{H}^2$. This follows from the correspondence between such lattices and the primitive null sublattices of $I^{\mathcal{H}}_{3,1}$ and the fact (Theorem 6.7) that all such sublattices are equivalent. Now suppose that Λ is a positive-definite selfdual $\mathcal{H}$-lattice of dimension 4. By the correspondence between such lattices and primitive null sublattices of $I^{\mathcal{H}}_{5,1}$ we know that $\Lambda \cong V^{\perp}/V$ for some primitive null lattice V in $I^{\mathcal{H}}_{5,1} \cong \Lambda^{\mathcal{H}}_4 \oplus II^{\mathcal{H}}_{1,1}$. By Theorem 6.14($i$), either $\Lambda \cong \Lambda^{\mathcal{H}}_4$ or else Λ contains two linearly independent short roots r_1 and r_2. In the latter case we observe that in a positive-definite integral lattice any two short roots are either proportional or orthogonal, so $r_1 \perp r_2$. Thus their span S in Λ is a copy of $\mathcal{H}^2$, and their orthogonal complement $S^{\perp}$ is a selfdual $\mathcal{H}$-lattice of dimension 2. By the above, $S^{\perp} \cong \mathcal{H}^2$ so $\Lambda = S \oplus S^{\perp} \cong \mathcal{H}^4$. □

The theorem implies that the only positive-definite selfdual $\mathcal{E}$-lattices of dimension $n \leq 6$ are $\mathcal{E}^n$ and $\Lambda^{\mathcal{E}}_6$ and that the only such $\mathcal{H}$-lattices of dimension $n \leq 4$ are $\Lambda^{\mathcal{H}}_4$ and $\mathcal{H}^n$. Similarly, Theorem 6.3 shows that $II^{\mathcal{G}}_{5,1} \cong E^{\mathcal{G}}_8 \oplus II^{\mathcal{G}}_{1,1}$ contains only one orbit of primitive isotropic lattices, so $E^{\mathcal{G}}_8$ is the only 4-dimensional even positive-definite selfdual $\mathcal{G}$-lattice.

These results have been obtained before but only by computational means. Feit [18] found examples of many positive-definite selfdual $\mathcal{E}$-lattices. He derived and used a version of the mass formula to verify that his list was complete for dimensions $n \leq 12$. Conway and Sloane [13, Thm. 3] provide a nice proof of this classification in dimensions $n \leq 6$ based on theta series and modular forms. (Their proof does not apply for $6 < n < 12$: in the second-to-last sentence of the proof, "12" should be replaced by "7".) Although selfdual $\mathcal{G}$-lattices have not been tabulated, it would be easy (and boring) to enumerate them through dimension 12 by using the fact that the real form of a selfdual $\mathcal{G}$-lattice is selfdual over $\mathbb{Z}$. Selfdual lattices over $\mathbb{Z}$ have been extensively tabulated; see [14, Chap. 16] and [5]. An enumeration of positive-definite selfdual $\mathcal{H}$-lattices for dimensions $n \leq 7$ has recently been completed by Bachoc [3] and for $n = 8$ by Bachoc and Nebe [4]. These enumerations are based on a generalization of Kneser's notion of "neighboring" lattices, together with a suitable version of the mass formula.

8. Comparison with the lists of Mostow and Thurston

In this section we justify the word "new" in our title, by showing that our "largest" three reflection groups do not appear on the lists of Mostow [22] and Thurston [23]. Deligne and Mostow [17] and Mostow [21] constructed 94 reflection groups acting on $\mathbb{C}H^n$ for various $n = 2, \ldots, 9$ by considering the monodromy of hypergeometric functions. Thurston [23] constructed the same set of groups in terms of moduli of flat metrics (with specified sorts of singularities) on the sphere S^2. We will sometimes refer to these groups as "the monodromy groups". We show here that none of the groups $\operatorname{Reflec} I_{n,1}^{\mathcal{E}}$ $(n \geq 4)$ or $\operatorname{Reflec} II_{4n+1,1}^{\mathcal{G}}$ $(n \geq 1)$ appear on their lists. In particular, our groups $\operatorname{Reflec} I_{7,1}^{\mathcal{E}}$, $\operatorname{Reflec} I_{4,1}^{\mathcal{E}}$ and $\operatorname{Reflec} II_{5,1}^{\mathcal{G}}$ are new. According to Conjecture 9.1 and the commentary following it, the lattices $I_{n,1}^{\mathcal{E}}$ $(n \leq 13)$ and $II_{4n+1,1}^{\mathcal{G}}$ $(n \leq 2)$ are reflective; if this is correct then the reflection groups of these lattices are also new. We will also identify $\operatorname{Reflec} I_{3,1}^{\mathcal{E}}$ with one of the monodromy groups. We leave open the question of whether our other groups appear on their lists and also the question of the commensurability of our groups and theirs.

We write G for $\operatorname{Reflec} L$ where L is one of the lattices $I_{n,1}^{\mathcal{E}}$ $(n \geq 4)$ or $II_{4n+1,1}^{\mathcal{G}}$ $(n \geq 1)$. We will show that each monodromy group that acts on $\mathbb{C}H^n$ for $n \geq 4$ is either cocompact or contains a primitive reflection of order 3 or 4. We will also show that G contains no such reflections, so G cannot be a monodromy group. (A reflection is primitive if it is not a power of a reflection of larger order.) It is easy to identify all the reflections of L (Lemma 8.2), but to compare G with the monodromy groups we must also consider elements R of G that are not reflections of L but still act on $\mathbb{C}H^n$ as reflections—such an R differs from a reflection of L by a complex scalar of norm one. To see that this is a nontrivial issue, consider $\operatorname{Aut} II_{1,1}^{\mathcal{G}}$. The subgroup of elements of determinant one is conjugate to $\mathrm{SL}_2 \mathbb{Z}$ and hence contains an element acting on $\mathbb{C}H^1$ as a triflection. This happens despite the fact that (by Lemma 8.2) the only reflections of $II_{1,1}^{\mathcal{G}}$ are biflections. We deal with this issue by calling a reflection of $\mathbb{C}^n$ an honest reflection and a reflection of $\mathbb{C}P^{n-1}$ a projective reflection. This distinction requires a more careful definition of a primitive reflection. If H is a group acting on $\mathbb{C}^n$ (resp. $\mathbb{C}P^{n-1}$) and $R \in H$ is an honest (resp. projective) reflection, then we say that R is primitive in H if it is not a power of an honest (resp. projective) reflection of H of larger order. Our first lemma assures that the behavior of $II_{1,1}^{\mathcal{G}}$ discussed above is a low-dimensional phenomenon, and gives a condition for a projective reflection in G to "come from" an honest reflection.

Lemma 8.1. *Suppose M is an n-dimensional lattice over $\mathcal{R} = \mathcal{E}$ or $\mathcal{G}$ and that R is an element of* $\operatorname{Aut} M$ *whose action on $\mathbb{C}P^{n-1}$ is a projective reflection of order $m < n$. Then R differs by a unit of $\mathcal{R}$ from an honest reflection of M.*

Proof: Since R acts on $\mathbb{C}P^{n-1}$ as a projective reflection, it has two distinct eigenvalues λ and λ', with one (say λ) having multiplicity $n-1$. Furthermore, since R^m preserves M and acts trivially on $\mathbb{C}P^{n-1}$, we see that there is a unit α of $\mathcal{R}$ such that $\lambda^m = \lambda'^m = \alpha$. The characteristic polynomial of R is $(x-\lambda)^{n-1}(x-\lambda')$; since $R \in \mathrm{GL}_n \mathcal{R}$, the coefficient of each power of x must lie in $\mathcal{R}$. Considering the coefficients of x^{n-1} and x^{n-m-1} we find that

$$\begin{aligned} \binom{n-1}{1}\lambda \quad &+ \binom{n-1}{0}\lambda' \quad = y \qquad \text{and} \\ \binom{n-1}{m+1}\lambda^{m+1} &+ \binom{n-1}{m}\lambda'\lambda^m = z \end{aligned}$$

for some $y, z \in \mathcal{R}$. Because $\lambda^m = \alpha \in \mathcal{R}$, the second equation reduces to a linear equation in λ and λ'. For $n > m$ this is a nonsingular system of equations, so $\lambda, \lambda' \in \mathcal{R} \otimes \mathbb{Q}$. Since λ, λ' are roots of unity they must actually lie in $\mathcal{R}$. Then $\lambda^{-1}R \in \operatorname{Aut} M$ has eigenvalues 1 (with multiplicity $n-1$) and $\lambda^{-1}\lambda'$, completing the proof. $\square$

Now we show that the only honest reflections of a selfdual lattice are the obvious ones; the result is well-known for lattices over $\mathbb{Z}$.

Lemma 8.2. *Any honest reflection R of a selfdual lattice M over $\mathcal{R} = \mathcal{E}$ or $\mathcal{G}$ is either a reflection in a lattice vector of norm ± 1 or a biflection in a lattice vector of norm ± 2.*

Proof: By considering the determinant of R we discover that its only nontrivial eigenvalue is a unit of $\mathcal{R}$, so M contains an element of the corresponding eigenspace, so R is a reflection in some lattice vector v. Taking v to be primitive, every vector in the complex span of v lies in the $\mathcal{R}$-span of v. (This uses the fact that $\mathcal{R}$ is a principal ideal domain.) Furthermore, by the selfduality of M, there exists $w \in M$ satisfying $\langle v|w\rangle = 1$. Then $R(w) = w - v(1-\alpha)/v^2$ and so $w - R(w) = v(1-\alpha)/v^2$ lies in M. Therefore $(1-\alpha)/v^2 \in \mathcal{R}$. Unless $\alpha = -1$ this requires $v^2 = \pm 1$ and if $\alpha = -1$ then it requires that v^2 divide 2. $\square$

Theorem 8.3. *The image of $\operatorname{Aut} L$ in the isometry group of complex hyperbolic space contains no primitive projective reflection of order 3 or 4.*

Proof: If $R \in \operatorname{Aut} L$ acted on hyperbolic space as a projective reflection of order $m = 3$ or 4, then because $\dim L > m$ we see by Lemmas 8.1 and 8.2 that R differs by a unit from a reflection in a root r of L. Since R acts as a reflection on hyperbolic space, r^2 must be positive. Since $m \neq 2$, r^2 must be 1. If $L = II^{\mathcal{G}}_{4n+1,1}$ then this is impossible because even lattices contain no short roots. If $L = I^{\mathcal{E}}_{n,1}$ then we must have $m = 3$, and then there is a 6-fold reflection in r whose square acts on hyperbolic space as R, showing that R is not primitive as a projective reflection. $\square$

Now we will return to the monodromy groups and find primitive reflections in them. We will discuss them in Thurston's terms; here is a sketch of his construction. Let $n \geq 4$ and let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be an n-tuple of numbers in the interval $(0, 2\pi)$ that sum to 4π. Let $P(\alpha)$ be the moduli space of pairs (p, g) where p is an injective map from $\{1, \ldots, n\}$ to an oriented sphere S^2 and g is a singular Riemannian metric on S^2 which is flat except on the image of p, with $p(i)$ being a "cone point" of curvature α_i. (We denote $p(i)$ by p_i.) Two such pairs are considered equivalent if they differ by an orientation-preserving similarity that identifies the corresponding points p_i with each other. This moduli space is a manifold of real dimension $2(n-3)$ and admits a metric which is locally isometric to $\mathbb{C}H^{n-3}$. Let H be the group of elements σ of the symmetric group S_n satisfying $\alpha_{\sigma(i)} = \alpha_i$ for all $i = 1, \ldots, n$. Then H acts by isometries of $P(\alpha)$, by permuting the points p_i. We denote the quotient orbifold by $C(\alpha)$ and its metric completion by $\bar{C}(\alpha)$. The fundamental group of $P(\alpha)$ is the pure (spherical) braid group on n strands, and the orbifold fundamental group of $C(\alpha)$ is the subgroup of the full (spherical) braid group the maps to H under the usual map from the braid group to the symmetric group.

If the α_i satisfy certain combinatorial identities then $\bar{C}(\alpha)$ turns out to be the quotient of $\mathbb{C}H^{n-3}$ by a reflection group $\Gamma(\alpha)$. There are only 94 choices for α (with $n \geq 5$) satisfying these conditions, and the corresponding $\Gamma(\alpha)$ are the monodromy groups. The points of $\bar{C}(\alpha) \smallsetminus C(\alpha)$ are the images of the mirrors of certain reflections of $\Gamma(\alpha)$. One can figure out the orders of the primitive reflections associated to these mirrors by finding the "cone angle" at each generic point of $\bar{C}(\alpha) \smallsetminus C(\alpha)$: if the cone angle is $2\pi/m$ then the corresponding (primitive) projective reflections have order m. (This cone angle should not be confused with the cone angles at the points $p_i \in S^2$.) The generic points of $\bar{C}(\alpha) \smallsetminus C(\alpha)$ are associated to "collisions" between pairs of points p_i and p_j on S^2 for which $\alpha_i + \alpha_j < 2\pi$. We quote Thurston's Proposition 3.5, which provides a way to compute the cone angles at these points of $\bar{C}(\alpha) \smallsetminus C(\alpha)$.

Proposition 8.4. *Let S be the stratum of $\bar{C}(\alpha_1,\ldots,\alpha_n)$ where two cone points of curvature α_i and α_j collide. If $\alpha_i=\alpha_j$ then the cone angle around S is $\pi-\alpha_i$; otherwise it is $2\pi-\alpha_i-\alpha_j$.* □

For example, take α to be the 10-tuple $(\frac{2\pi}{3},\frac{2\pi}{3},\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3})$, which is number 13 on Thurston's list and number 4 on Mostow's. Then at the singular strata of $\bar{C}(\alpha)$ where two cone points of curvature $\frac{2\pi}{3}$ (resp. two of curvature $\frac{\pi}{3}$, resp. one of each curvature) collide, the cone angle is $\pi-\frac{2\pi}{3}=\frac{\pi}{3}$ (resp. $\pi-\frac{\pi}{3}=\frac{2\pi}{3}$, resp. $2\pi-\frac{2\pi}{3}-\frac{\pi}{3}=\pi$). We deduce that $\Gamma(\alpha)$ contains primitive projective reflections of orders 6, 3 and 2.

Theorem 8.5. *Each of the monodromy groups that acts on $\mathbb{C}H^{n-3}$ for $n\geq 7$ is either cocompact or contains a primitive projective reflection of order* 3 *or* 4.

Proof: Using Proposition 8.4 and the list of n-tuples α provided in [22] or [23], it is easy to compute the cone angles at all the generic points of $\bar{C}(\alpha)\smallsetminus C(\alpha)$. (The author wrote a short computer program to do this, and also performed the computation by hand.) The only one with $n\geq 7$ for which none of the cone angles are $2\pi/4$ or $2\pi/3$ is number 50 on Thurston's list (number 21 on Mostow's). According to Thurston's table, $\Gamma(\alpha)$ is cocompact for this choice of α. □

From Theorems 8.3 and 8.5 we immediately deduce

Theorem 8.6. *If L is $I^{\mathcal{E}}_{n,1}$ $(n\geq 4)$ or $II^{\mathcal{G}}_{4n+1,1}$ $(n\geq 1)$ then* Reflec L *does not appear among the* 94 *monodromy groups.* □

We close this section by sketching a proof that Reflec $I^{\mathcal{E}}_{3,1}$ is one of the monodromy groups—it is the group $\Gamma(\alpha)$ with $\alpha=(\frac{2\pi}{3},\frac{2\pi}{3},\frac{2\pi}{3},\frac{2\pi}{3},\frac{2\pi}{3},\frac{2\pi}{3})$, which is number 1 on Thurston's list and number 23 on Mostow's. Because all the α_i are equal, the orbifold fundamental group of $C(\alpha)$ is the spherical braid group B_6 on six strands. A standard generator for B_6, braiding two points p_i and p_{i+1}, corresponds to a loop in $C(\alpha)$ encircling the singular stratum S of $\bar{C}(\alpha)$ associated to a collision between p_i and p_{i+1}. Since the cone angle at S is $\pi/3$ we find that the standard generators map to 6-fold reflections. This fact, together with the braid relations and the fact that the image of B_6 is not finite, specifies the representation uniquely up to complex conjugation. The five standard generators may be taken to map to $(-\omega)$-reflections in short roots of $I^{\mathcal{E}}_{3,1}$, two of these being orthogonal if the corresponding braid generators commute and having inner product $+1$ otherwise. One may then use the techniques of Sections 5 and 6 to show that the image of B_6 is all of $\text{Reflec}_0\, I^{\mathcal{E}}_{3,1}$, which has the same projective action as Reflec $I^{\mathcal{E}}_{3,1}$. The arguments we have sketched here concerning the braid group representation are carried out in detail in [2].

9. Comments

The Leech lattice plays an interesting role in hyperbolic geometry. Conway [9] showed that its special properties allow a remarkably simple description of the symmetry group of the real Lorentzian lattice $II^{\mathbb{Z}}_{25,1}$. In Lemmas 6.9 and 6.11, we showed that it also "explains" our best examples, through embeddings of the Coxeter-Todd and Barnes-Wall lattices. We note that Borcherds [7, Sect. 3] used the same embeddings to produce interesting *real* reflection groups acting on $\mathbb{R}H^{13}$ and $\mathbb{R}H^{17}$.

There are a number of interesting Eisenstein lattices besides $\mathcal{E}^n$ and $\Lambda^{\mathcal{E}}_6$, but unfortunately most seem unsuitable for our purposes. The D_4, E_6, and E_8 root lattices are $\mathcal{E}$-lattices, but at the smallest scale at which they are integral, they have minimal norm 3 and covering radii $(3/2)^{1/2}$, $2^{1/2}$, and $(3/2)^{1/2}$, so Theorem 6.1 doesn't apply. There are also complex and quaternionic forms of the Leech lattice (see [27] and [28]), but they also have large covering radii. I do not know whether any of these lattices are well-covered. (Note also that the groups of the complex and quaternionic Leech lattices contains no reflections.)

It would be nice to understand the groups G generated by reflections in the short roots of $L = I^{\mathcal{E}}_{7,1}$ or $I^{\mathcal{H}}_{5,1}$. The transitivity of G on primitive null lattices in L that are orthogonal to no short roots of L proves that $(\operatorname{Aut} L)/G$ is the image of the stabilizer of $\langle\rho\rangle$, where ρ is a primitive null vector corresponding to $\Lambda^{\mathcal{E}}_6$ or $\Lambda^{\mathcal{H}}_4$. One can show that the central translations (relative to the decomposition $\Lambda^{\mathcal{E}}_6 \oplus II^{\mathcal{E}}_{1,1}$ or $\Lambda^{\mathcal{H}}_4 \oplus II^{\mathcal{H}}_{1,1}$ of L) of G have finite index in those of $\operatorname{Aut} L$. This implies that $(\operatorname{Aut} L)/G$ is virtually free abelian (possibly finite). We note the similar behavior of the integer lattices $I^{\mathbb{Z}}_{9,1}$ and $II^{\mathbb{Z}}_{25,1}$. That is, $\operatorname{Reflec}_0 I^{\mathbb{Z}}_{9,1}$ acts transitively on the primitive null sublattices orthogonal to no short roots, and

$$\operatorname{Aut} I^{\mathbb{Z}}_{9,1}/(\operatorname{Reflec}_0 I^{\mathbb{Z}}_{9,1} \times \{\pm 1\}) \cong E_8 \colon \operatorname{Aut} E_8 \ ,$$

where the right hand side indicates a semidirect product of the additive group of E_8 by $\operatorname{Aut} E_8$. Similarly, $\operatorname{Reflec} II^{\mathbb{Z}}_{25,1}$ acts transitively on the primitive null sublattices orthogonal to no roots at all, and

$$\operatorname{Aut} II^{\mathbb{Z}}_{25,1}/(\operatorname{Reflec} II^{\mathbb{Z}}_{25,1} \times \{\pm 1\}) \cong \Lambda_{24} \colon \operatorname{Aut} \Lambda_{24} \ .$$

It is natural to wonder whether $I^{\mathcal{E}}_{7,1}$ and $I^{\mathcal{H}}_{5,1}$ behave similarly.

We close with a conjecture that there are arithmetic complex and quaternionic hyperbolic reflection groups in dimensions considerably higher than we have so far considered:

Conjecture 9.1. *The lattice $I^{\mathcal{R}}_{n+1,1}$ is reflective if and only if each positive-definite selfdual $\mathcal{R}$-lattice Λ of dimension n has finite index in the span of its roots.*

Inspiration for the conjecture comes from the real case, which has been exhaustively studied (see for example [24], [26], [12], [6]). Namely, the groups of the Lorentzian lattices $I^{\mathbb{Z}}_{n+1,1}$ contain reflection groups of finite index for $n \leq 18$, and the failure of this in higher dimension is associated with the existence of selfdual $\mathbb{Z}$-lattices of dimension ≥ 19 that are not virtually spanned by their roots.

If the conjecture is true, then it provides reflection groups acting with finite covolume on $\mathbb{C}H^n$ for all $n \leq 13$ and on $\mathbb{H}H^n$ for all $n \leq 9$. The condition on the lattices Λ appearing in the conjecture has been verified in the Eisenstein case for dimensions ≤ 12 by Feit [18] and in the Hurwitz case for dimensions ≤ 8 by Bachoc [3] and Bachoc and Nebe [4]. Furthermore, since enumerations of selfdual lattices over $\mathcal{E}$ and $\mathcal{H}$ have not been accomplished in higher dimensions, it might be that $I^{\mathcal{E}}_{n+1,1}$ (resp. $I^{\mathcal{H}}_{n+1,1}$) is reflective for n even larger than 12 (resp. 8). The lowest-dimensional selfdual lattice over $\mathcal{E}$ or $\mathcal{H}$ of which the author is aware and whose roots fail to virtually span it is 19-dimensional, the tensor product of $\mathcal{E}$ or $\mathcal{H}$ with a 19-dimensional $\mathbb{Z}$-lattice having this property.

A similar conjecture could be made for the even Gaussian lattices $II^{\mathcal{G}}_{4n+1,1}$. If it were true then it would imply that this lattice is reflective exactly for $n = 0, 1$ or 2. The failure to be reflective in higher dimensions would follow from the fact that the Leech lattice may be regarded as a 12-dimensional selfdual $\mathcal{G}$-lattice with no roots.

References

[1] Allcock, D.J.: Reflection groups on the octave hyperbolic plane (preprint)

[2] Allcock, D.J., Carlson, J., Toledo, D.: (in preparation)

[3] Bachoc, C.: Voisinage au sens de Kneser pour les réseaux quaternioniens, Comm. Math. Helv. **70**, 350–374 (1995)

[4] Bachoc, C., Nebe, G.: Classification of two genera of 32-dimensional lattices of rank 8 over the Hurwitz order. To appear in Experimental Math.

[5] Borcherds, R.E.: The Leech Lattice and Other Latttices, PhD thesis, Cambridge University 1984

[6] Borcherds, R.E.: Automorphism groups of Lorentzian lattices, J. Alg. **111**, 133–53 (1987)
[7] Borcherds, R.E.: Lattices like the Leech lattice, J. Alg. **130**, 219–34 (1990)
[8] Borel, A., Harish-Chandra: Arithmetic subgroups of algebraic groups, Ann. Math. **75**, 485–535 (1962)
[9] Conway, J.H.: The automorphism group of the 26-dimensional even unimodular Lorentzian lattice, J. Alg. **80**, 159–163 (1983). Reprinted in [14]
[10] Conway, J.H., Parker, R.A., Sloane, N.J.A.: The covering radius of the Leech lattice, Proc. Roy. Soc. Lond. **A380**, 261–90 (1982). Reprinted in [14]
[11] Conway, J.H., Sloane, N.J.A.: Laminated lattices, Ann. Math. **116**, 593–620 (1982). Reprinted in [14]
[12] Conway, J.H., Sloane, N.J.A.: Leech roots and Vinberg groups, Proc. Roy. Soc. Lond. **A384**, 233–58 (1982). Reprinted in [14]
[13] Conway, J.H., Sloane, N.J.A.: the Coxeter-Todd lattice, the Mitchell group, and related sphere packings, Proc. Camb. Phil. Soc. **93**, 421–40 (1983)
[14] Conway, J.H., Sloane, N.J.A.: Sphere Packings, Lattices, and Groups, Springer 1988
[15] Coxeter, H.S.M.: Factor groups of the braid groups, Proceedings of the Fourth Canadian Mathematical Congress, Banff, Toronto University Press 1959
[16] Coxeter, H.S.M., Moser, W.O.J.: Generators and Relations for Discrete Groups, Springer 1984
[17] Deligne, P., Mostow, G.D.: Monodromy of hypergeometric functions and non-lattice integral monodromy, Pub. I.H.E.S. **63**, 5–90 (1986)
[18] Feit, W.: Some lattices over $Q(\sqrt{-3})$, J. Alg. **52**, 248–263 (1978)
[19] Milnor, J., Husemoller, D.: Symmetric Bilinear Forms, Springer 1973
[20] Mostow, G.D.: A remarkable class of polyhedra in complex hyperbolic space, Pac. J. Math. **86**, 171–276 (1980)
[21] Mostow, G.D.: Generalized Picard lattices arising from half-integral conditions, Pub. I.H.E.S. **63**, 91–106 (1986)
[22] Mostow, G.D.: On discontinuous action of monodromy groups on the complex n-ball, Jour. A.M.S. **1**, 555–586 (1988)
[23] Thurston, W.P.: Shapes of polyhedra (preprint)
[24] Vinberg, E.B.: On the groups of units of certain quadratic forms, Math. USSR—Sb. **16(1)**, 17–35 (1972)
[25] Vinberg, E.B.: On unimodular integral quadratic forms, Funct. Anal. Appl. **6**, 105–11 (1972)
[26] Vinberg, E.B., Kaplinskaja, I.M.: On the groups $O_{18,1}(Z)$ and $O_{19,1}(Z)$, Sov. Math.—Doklady **19(1)**, 194–197 (1978)
[27] Wilson, R.A.: The quaternionic lattice for $2G_2(4)$ and its maximal subgroups, J. Alg. **77**, 449–466 (1982)
[28] Wilson, R.A.: The complex Leech lattice and maximal subgroups of the Suzuki group, J. Alg. **84**, 151–188 (1983)