

# The Moduli Space of Cubic Threefolds as a Ball Quotient

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Dedicated to Herb Clemens and Phillip Griffiths.



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## Abstract

The moduli space of cubic threefolds in  $\mathbb{C}P^4$ , with some minor birational modifications, is the Baily-Borel compactification of the quotient of the complex 10-ball by a discrete group. We describe both the birational modifications and the discrete group explicitly.

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## Introduction

One of the most basic facts in algebraic geometry is that the moduli space of elliptic curves, which can be realized as plane cubic curves, is isomorphic to the upper half plane modulo the action of linear fractional transformations with integer coefficients. In [3], we showed that there is an analogous result for cubic surfaces; the analogy is clearest when we view the upper half plane as complex hyperbolic 1-space, that is, as the unit disk. The result is that the moduli space of stable cubic surfaces is isomorphic to a quotient of complex hyperbolic 4-space by the action of a specific discrete group. This is the group of matrices which preserve the Hermitian form  $\text{diag}[-1, 1, 1, 1, 1]$  and which have entries in the ring of Eisenstein integers: the ring obtained by adjoining a primitive cube root of unity to the integers. The idea of the proof is not to use the Hodge structure of the cubic surface, which has no moduli, but rather that of the cubic threefold obtained as a triple cover of  $\mathbb{C}P^3$  branched along the cubic surface. The resulting Hodge structures have a symmetry of order three, and the moduli space of such structures is isomorphic to complex hyperbolic 4-space  $\mathbb{C}H^4$ . This is the starting point of the proof, which relies crucially on the Clemens-Griffiths Torelli Theorem for cubic threefolds [10].

The purpose of this article is to extend the analogy to cubic threefolds. The idea is to use the period map for the cubic fourfolds obtained as triple covers of  $\mathbb{C}P^4$  branched along the threefolds, using Voisin's Torelli theorem [41] in place of that of Clemens and Griffiths. In this case, however, a new phenomenon occurs. There is one distinguished point in the moduli space of cubic threefolds which is a point of indeterminacy for the period map. This point is the one represented by what we call a chordal cubic, meaning the secant variety of a rational normal quartic curve in  $\mathbb{C}P^4$ . The reason for the indeterminacy is that the limit Hodge structure depends on the direction of approach to the chordal cubic locus. In fact, the limit depends *only* on this direction, and so the period map extends to the blowup of the moduli space.

The natural period map for smooth cubic threefolds [10] embeds the moduli space in a period domain for Hodge structures of weight three,

namely, a quotient of the Siegel upper half space of genus five. For this embedding, however, the target space has dimension greater than that of the source. For the construction of this article, the dimensions of the source and target are the same.

To formulate the main result, let  $\mathcal{M}_{ss}$  be the GIT moduli space of cubic threefolds, and let  $\widehat{\mathcal{M}}_{ss}$  be its blowup at the point corresponding to the chordal cubics. Let  $\mathcal{M}_s \subseteq \mathcal{M}_{ss}$  be the moduli space of stable cubic threefolds, and let  $\widehat{\mathcal{M}}_s$  be  $\widehat{\mathcal{M}}_{ss}$  minus the proper transform of  $\mathcal{M}_{ss} - \mathcal{M}_s$ . Let  $\mathcal{M}_0$  be the moduli space of smooth cubic threefolds. Then we have the following, contained in the statement of the main result, Theorem 6.1:

**THEOREM.** *There is an arithmetic group  $P\Gamma$  acting on complex hyperbolic 10-space, such that the period map*

$$\widehat{\mathcal{M}}_s \longrightarrow P\Gamma \backslash \mathbb{C}H^{10}$$

*is an isomorphism. This map identifies  $\mathcal{M}_s$  with the image in  $P\Gamma \backslash \mathbb{C}H^{10}$  of the complement of a hyperplane arrangement  $\mathcal{H}_c$ . It also identifies the discriminant in  $\widehat{\mathcal{M}}_s$  with the image of another hyperplane arrangement  $\mathcal{H}_\Delta \subseteq \mathbb{C}H^{10}$ . In particular, it identifies  $\mathcal{M}_0$  with  $P\Gamma \backslash (\mathbb{C}H^{10} - (\mathcal{H}_c \cup \mathcal{H}_\Delta))$ . Finally, the period map extends to a morphism from  $\widehat{\mathcal{M}}_{ss}$  to the Baily-Borel compactification  $\overline{P\Gamma \backslash \mathbb{C}H^{10}}$ .*

We also provide much more detailed information about all the objects in the theorem, such as explicit descriptions of  $P\Gamma$ ,  $\mathcal{H}_c$  and  $\mathcal{H}_\Delta$ , and an analysis of the part of  $\widehat{\mathcal{M}}_{ss}$  lying over the boundary points of  $\overline{P\Gamma \backslash \mathbb{C}H^{10}}$ .

Now we will say what the group  $P\Gamma$  is. Let  $V$  be a cyclic cubic fourfold, meaning a triple cover of  $P^4$  branched over a cubic threefold. The primitive cohomology  $\Lambda(V)$  of  $V$  is naturally a module for the Eisenstein integers, where a primitive cube root of unity acts on cohomology as does the corresponding deck transformation. When  $V$  is smooth, this Eisenstein module carries a natural Hermitian form of signature  $(10, 1)$ , and  $P\Gamma$  is the projective isometry group of  $\Lambda(V)$ .

The architecture of our proof of Theorem 6.1 dates back to before [1], and follows the pattern laid out in [3] for cubic surfaces. But it is considerably more technical in its details. We therefore focus on the points where there are major differences or where substantially more work must be done as compared with the case of cubic surfaces.

In chapter 1 we establish basic facts about  $\Lambda(V)$  as an Eisenstein module endowed with a complex Hodge structure, give an inner product matrix for  $\Lambda(V)$ , and show that  $\mathcal{M}_0 \rightarrow P\Gamma \backslash \mathbb{C}H^{10}$  is an isomorphism



onto its image. The argument here follows that of [3], except that in place of the Clemens-Griffiths theorem, we use Claire Voisin's Torelli theorem for cubic fourfolds [41]. In this chapter we also establish facts about the discriminant locus for cubic threefolds near stable singularities. These facts are used in chapter 3 for extending the period map.

In chapter 2 we blow up the space  $P^{34}$  of cubic threefolds along the chordal cubic locus, and describe the proper transform of the discriminant locus. This is one of the most technical chapters in the monograph, but it is required for the extension of the period map in chapter 3. To give an idea of the main issue, consider a one-parameter family of smooth cubic threefolds degenerating to a chordal cubic. We may write it as  $F + tG = 0$ , where  $F = 0$  defines the chordal cubic. The polynomial  $G = 0$  cuts out on the rational normal curve of  $F = 0$  a set of twelve points. Thus to a tangent vector at a point of the chordal locus one associates a 12-tuple on the projective line. We show that the discriminant locus in the blown-up  $P^{34}$  has a local product structure, i.e., it is homeomorphic to the product of the discriminant locus for 12-tuples in  $P^1$ , times a disk representing the transverse direction, times a twenty-one dimensional ball corresponding to the action of the projective group  $\mathrm{PGL}(5, \mathbb{C})$ . There is also a technical variation on this result which gives an analytic model of the discriminant, not just a topological one.

In chapter 3 we extend the period map to the semistable locus of the blown-up  $P^{34}$ . This requires some geometric invariant theory to say what the semistable locus is, and here the work of Reichstein [33] is essential. Then we study the local monodromy groups at points in this semistable locus. The essential result for the extension of the period map is that these groups are all finite or virtually unipotent.

In chapter 4 we show that the extended period map sends the chordal cubic locus to a divisor in  $P\Gamma \backslash \mathbb{C}H^{10}$ . The main point here is to identify the limit Hodge structure and in so doing show that the derivative of the extended map along the blowup of the chordal cubic locus is of rank nine. We do this by establishing an isogeny between the limit Hodge structure and the sum of a 1-dimensional Hodge structure and a Hodge structure associated to a six-fold cover of the projective line branched at twelve points. These Hodge structures first arose in the work of Deligne and Mostow [13]. Our analysis shows that the image of the period map is the quotient of a totally geodesic  $\mathbb{C}H^9$  by a suitable subgroup of  $P\Gamma$ .

Chapter 5 deals with the same issues for the divisor of nodal cubic threefolds. Here the analysis is easier. We show that the Hodge structure is isomorphic to that of a special K3 surface, plus a 1-dimensional summand. This K3 surface and its Hodge structure were treated in Kondō's work [22] on moduli of genus 4 curves.

In chapter 6 we assemble the various pieces to prove the main theorem, and in chapter 7 we give some supplemental results on the monodromy group and the hyperplane arrangements.

Another proof of the main theorem has been obtained by Looijenga and Swierstra [25]. Both proofs proceed by extending the period map from the moduli space of smooth threefolds to a larger space, but the extension process is quite different in the two proofs. We use a detailed analysis of the discriminant in the space obtained by blowing up the chordal cubic locus to extend the period map. Looijenga and Swierstra use a general machinery developed earlier by them [24] to handle extensions of period mappings. We are grateful to Looijenga for sending us an early version of their argument.

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## CHAPTER 1

### Moduli of Smooth Cubic Threefolds

This chapter contains a number of foundational results, and its main theorem, Theorem 1.9, is of interest in its own right. We consider cyclic cubic fourfolds, i.e., triple covers of  $\mathbb{C}P^4$  branched along cubic threefolds. (1) The cohomology of such a fourfold is a module over  $\mathcal{E} = \mathbb{Z}[\omega = \sqrt[3]{1}]$ , equipped with a Hermitian form. (2) The monodromy on this lattice as the threefolds acquire a node is a complex reflection of order three; see Lemma 1.4. (3) To analyze the local monodromy for more complicated singularities, we give a structure theorem for the discriminant locus of the space of cubic threefolds near a cubic threefold with singularities of type  $A_n$  and  $D_4$ . See Lemma 1.5. Using this result, we give an inner product matrix for the Hermitian form; see Theorem 1.7. (4) With the previous results in hand, we define a framing of the Hodge structure of a cyclic cubic fourfold and define the period map. Finally, the main theorem of the chapter is that the period map for smooth cubic threefolds is an isomorphism onto its image; see Theorem 1.9.

Let  $\mathcal{C}$  be the space of all nonzero cubic forms in variables  $x_0, \dots, x_4$ . For such a form  $F$  let  $T$  be the cubic threefold in  $\mathbb{C}P^4$  it defines, and let  $V$  be the cubic fourfold in  $\mathbb{C}P^5$  defined by  $F(x_0, \dots, x_4) + x_5^3 = 0$ . Whenever we consider  $F \in \mathcal{C}$ ,  $T$  and  $V$  will have these meanings.  $V$  is the triple cover of  $\mathbb{C}P^4$  branched along  $T$ . We write  $\mathcal{C}_0$  for the set of  $F \in \mathcal{C}$  for which  $T$  is smooth (as a scheme) and  $\Delta$  for the discriminant  $\mathcal{C} - \mathcal{C}_0$ . We will sometimes also write  $\Delta$  for its image in  $PC$ ; context will make our meaning clear. We write  $\mathcal{C}_s$  for the set of  $F \in \mathcal{C}$  for which  $T$  is stable in the sense of geometric invariant theory. By [1] or [42], this holds if and only if  $T$  has no singularities of types other than  $A_1, \dots, A_4$ .  $\mathcal{C}_s$  will play a major role in chapters 3–6; in this chapter all we will use is the fact that  $\mathcal{C}_0$  lies within it.

Because we will vary our threefolds, we will need the universal family  $\mathcal{T} \subseteq \mathcal{C} \times \mathbb{C}P^4$  of cubic threefolds,

$$\mathcal{T} = \{ (F, [x_0 : \dots : x_4]) \in \mathcal{C} \times \mathbb{C}P^4 \mid F(x_0, \dots, x_4) = 0 \} ,$$

and the family  $\mathcal{V}$  of covers of  $\mathbb{C}P^4$  branched over them,

$$\mathcal{V} = \left\{ (F, [x_0 : \dots : x_5]) \in \mathcal{C} \times \mathbb{C}P^5 \mid F(x_0, \dots, x_4) + x_5^3 = 0 \right\} .$$

We will write  $\pi_{\mathcal{T}}$  and  $\pi_{\mathcal{V}}$  for the projections  $\mathcal{T} \rightarrow \mathcal{C}$  and  $\mathcal{V} \rightarrow \mathcal{C}$ . The total spaces of  $\mathcal{T}$  and  $\mathcal{V}$  are smooth. We write  $\mathcal{T}_0$  and  $\mathcal{V}_0$  for the topologically locally trivial fibrations which are the restrictions of  $\mathcal{T}$  and  $\mathcal{V}$  to  $\mathcal{C}_0$ . The transformation  $\sigma : \mathbb{C}^6 \rightarrow \mathbb{C}^6$  given by

$$(1.1) \quad \sigma(x_0, \dots, x_5) = (x_0, \dots, x_4, \omega x_5) ,$$

where  $\omega$  is a fixed primitive cube root of unity, plays an essential role in all that follows. We regard it as a symmetry of  $\mathcal{V}$  and of the individual  $V$ 's.

Our period map  $\mathcal{C}_0 \rightarrow \mathbb{C}H^{10}$  will be defined using the Hodge structure of the fourfolds and its interaction with  $\sigma$ , so we need to discuss  $H^4(V)$  for  $F \in \mathcal{C}_0$ . To compute this it suffices by the local triviality of  $\mathcal{V}_0$  to consider the single fourfold  $x_0^3 + \dots + x_5^3 = 0$ ; by thinking of it as an iterated branched cover, one finds that its Euler characteristic is 27. The Lefschetz hyperplane theorem and Poincaré duality imply that  $H^i(V; \mathbb{Z})$  is the same as  $H^i(\mathbb{C}P^5; \mathbb{Z})$  for  $i \neq 4$ , so  $H^4(V; \mathbb{Z}) \cong \mathbb{Z}^{23}$ . The class of a 3-plane in  $\mathbb{C}P^5$  pulls back to a class  $\eta(V) \in H^4(V; \mathbb{Z})$  of norm 3. The primitive cohomology  $H_0^4(V; \mathbb{Z})$  is the orthogonal complement of  $\eta(V)$  in  $H^4(V; \mathbb{Z})$ . Since  $H^4(V; \mathbb{Z})$  is a unimodular lattice,  $H_0^4(V; \mathbb{Z})$  is a 22-dimensional lattice with determinant equal to that of  $\langle \eta(V) \rangle$ , up to a sign, so  $\det H_0^4(V; \mathbb{Z}) = \pm 3$ .

$H_0^4(V; \mathbb{Z})$  is a module not only over  $\mathbb{Z}$  but over the Eisenstein integers  $\mathcal{E} = \mathbb{Z}[\omega]$  as well. To see this, observe that the isomorphism  $V/\langle \sigma \rangle \cong \mathbb{C}P^4$  implies that  $H^4(\mathbb{C}P^4; \mathbb{C})$  is the  $\sigma$ -invariant part of  $H^4(V; \mathbb{C})$ . Therefore  $\sigma$  fixes no element of  $H_0^4(V; \mathbb{C})$  except 0, hence no element of  $H_0^4(V; \mathbb{Z})$  except 0. We define  $\Lambda(V)$  to be the  $\mathcal{E}$ -module whose underlying additive group is  $H_0^4(V; \mathbb{Z})$ , with the action of  $\omega \in \mathcal{E}$  defined as  $\sigma^*$ .  $\mathcal{E}$  is a unique factorization domain, so  $\Lambda(V)$  is free of rank 11. We define a Hermitian form  $\langle | \rangle$  on  $\Lambda(V)$  by the formula

$$(1.2) \quad \langle \alpha | \beta \rangle = \frac{1}{2} \left[ 3\alpha \cdot \beta - \theta \alpha \cdot (\sigma^{*-1} \beta - \sigma^* \beta) \right] ,$$

where the dot denotes the usual pairing  $\alpha \cdot \beta = \int_V \alpha \wedge \beta$  and  $\theta = \omega - \bar{\omega} = \sqrt{-3}$ . The scale factor 1/2 is chosen so that  $\langle \alpha | \beta \rangle$  takes values in  $\mathcal{E}$ ; it is the smallest scale for which this is true.

LEMMA 1.1.  *$\langle | \rangle$  is an  $\mathcal{E}$ -valued Hermitian form, linear in its first argument and antilinear in its second. Furthermore,  $\langle \alpha | \beta \rangle \in \theta \mathcal{E}$  for all  $\alpha, \beta \in \Lambda(V)$ .*

PROOF.  $\mathbb{Z}$ -bilinearity is obvious.  $\mathcal{E}$ -linearity in its first argument holds by the following calculation. (Throughout the proof we write  $\sigma$  for  $\sigma^*$ .)

$$\begin{aligned}
\langle \theta\alpha | \beta \rangle &= \langle \sigma\alpha - \sigma^{-1}\alpha | \beta \rangle \\
&= \frac{1}{2} \left[ 3(\sigma\alpha - \sigma^{-1}\alpha) \cdot \beta - \theta(\sigma\alpha - \sigma^{-1}\alpha) \cdot (\sigma^{-1}\beta - \sigma\beta) \right] \\
&= \frac{\theta}{2} \left[ \bar{\theta}\sigma\alpha \cdot \beta - \bar{\theta}\sigma^{-1}\alpha \cdot \beta \right. \\
&\quad \left. - (\sigma\alpha \cdot \sigma^{-1}\beta - \sigma\alpha \cdot \sigma\beta - \sigma^{-1}\alpha \cdot \sigma^{-1}\beta + \sigma^{-1}\alpha \cdot \sigma\beta) \right] \\
&= \frac{\theta}{2} \left[ \bar{\theta}\alpha \cdot \sigma^{-1}\beta - \bar{\theta}\alpha \cdot \sigma\beta \right. \\
&\quad \left. - (\alpha \cdot \sigma^{-2}\beta - \alpha \cdot \beta - \alpha \cdot \beta + \alpha \cdot \sigma^2\beta) \right] \\
&= \frac{\theta}{2} \left[ 2\alpha \cdot \beta - \alpha \cdot (\sigma^{-2}\beta + \sigma^2\beta) - \theta(\alpha \cdot \sigma^{-1}\beta - \alpha \cdot \sigma\beta) \right] \\
&= \frac{\theta}{2} \left[ 3\alpha \cdot \beta - \theta\alpha \cdot (\sigma^{-1}\beta - \sigma\beta) \right] \\
&= \theta \langle \alpha | \beta \rangle .
\end{aligned}$$

In the second-to-last step we used the relation  $\sigma^{-2} + \sigma^2 = \sigma + \sigma^{-1} = -1$ . That  $\langle | \rangle$  is a  $\mathbb{C}$ -valued Hermitian form now follows from  $\langle \alpha | \beta \rangle = \overline{\langle \beta | \alpha \rangle}$ ; to prove this it suffices to check that the imaginary part of (1.2) changes sign when  $\alpha$  and  $\beta$  are exchanged, i.e.,

$$\begin{aligned}
\alpha \cdot (\sigma^{-1}\beta - \sigma\beta) &= \alpha \cdot \sigma^{-1}\beta - \alpha \cdot \sigma\beta \\
&= \sigma\alpha \cdot \beta - \sigma^{-1}\alpha \cdot \beta \\
&= -(\sigma^{-1}\alpha - \sigma\alpha) \cdot \beta \\
&= -\beta \cdot (\sigma^{-1}\alpha - \sigma\alpha) .
\end{aligned}$$

Next we check that  $\langle | \rangle$  is  $\mathcal{E}$ -valued. It is obvious that its real part takes values in  $\frac{1}{2}\mathbb{Z}$  and that its imaginary part takes values in  $\frac{\theta}{2}\mathbb{Z}$ . Since

$$\mathcal{E} = \{ a/2 + b\theta/2 \mid a, b \in \mathbb{Z} \text{ and } a \equiv b \pmod{2} \} ,$$

it suffices to prove that

$$3\alpha \cdot \beta \equiv -\alpha \cdot (\sigma^{-1}\beta - \sigma\beta) \pmod{2},$$

that is, that 2 divides  $\alpha \cdot (\beta - \sigma\beta + \sigma^{-1}\beta)$ . This follows from the relation  $1 - \omega + \bar{\omega} = -2\omega$  in  $\mathcal{E}$ . Furthermore, (1.2) shows that  $\langle \alpha | \beta \rangle$  has real part in  $\frac{3}{2}\mathbb{Z}$ ; since every element of  $\mathcal{E}$  with real part in  $\frac{3}{2}\mathbb{Z}$  is divisible by  $\theta$ , the proof is complete.  $\square$

We will describe  $\Lambda(V)$  more precisely later, after considering the Hodge structure of  $V$ . To do so, we wish to state a general version of the Griffiths residue calculus that will be used again and again, especially in chapters 4 and 5. The original reference is [18]. See [14] and [39] for the case of weighted projective spaces.

**PROPOSITION 1.2. (Griffiths Residue Calculus).** *Let  $X$  be a quasismooth hypersurface in a weighted projective space  $P^n(a) = P(a_0, a_1, \dots, a_n)$ . That is,  $X$  is given by a weighted homogeneous polynomial  $F(X_0, X_1, \dots, X_n)$  of degree  $d$  whose partial derivatives have no common zero other than the origin. Let*

$$E = \sum X_i \frac{\partial}{\partial X_i}$$

*be the Euler vector field. It is homogeneous of degree zero with respect to the  $\mathbb{C}^*$ -action defining  $P^n(a)$ . Let  $dV = dX_0 \wedge dX_1 \wedge \dots \wedge dX_n$  be the Euclidean volume form, and let  $\Omega = i(E)dV$ , where  $i$  denotes interior multiplication, be the projective volume form. It has degree equal to the sum of the weights  $a_i$ . Consider expressions of the form*

$$\Omega(A) = \frac{A\Omega}{F^{q+1}},$$

*where  $A$  is a homogeneous polynomial whose degree is such that  $\Phi$  is homogeneous of degree zero. Then the Poincaré residues of  $\Omega(A)$  span  $F^{n-q}H^n(X)_0$ , where  $F^*$  is the Hodge filtration and the subscript denotes primitive cohomology. The residue lies in  $F^{n-q+1}$  if and only if  $A$  lies in the Jacobian ideal of  $F$ , that is, the ideal generated by the first partial derivatives of  $F$ .*

From the preceding proposition we find the Hodge numbers  $h_0^{4,0} = h_0^{0,4} = 0$ ,  $h_0^{3,1} = h_0^{1,3} = 1$  and  $h_0^{2,2} = 20$ . Since  $\sigma$  is holomorphic, its eigenspace decomposition refines the Hodge decomposition. The interesting component is the  $H^{3,1}$  space, which is spanned by

$$(1.3) \quad \Omega(1) = \frac{\Omega}{(F + x_5^3)^2}.$$

The transformation multiplies the numerator by  $\omega$  and leaves the denominator invariant. Thus  $H^{3,1}$  has eigenvalue  $\omega$ . Thus  $H_\omega^4(V; \mathbb{C}) = H_\omega^{3,1} \oplus H_\omega^{2,2}$ , the summands being one- and ten-dimensional.

On  $H^4(V; \mathbb{C})$  there is the Hodge-theoretic Hermitian pairing

$$(1.4) \quad (\alpha, \beta) = 3 \int_V \alpha \wedge \bar{\beta}.$$

The Hodge-Riemann bilinear relations [19, p. 123] show that  $(,)$  is positive-definite on  $H_0^{2,2}$  and negative-definite on  $H^{3,1}$ . It follows that  $H_\omega^4(V; \mathbb{C})$  has signature  $(10, 1)$ .

If  $W$  is a complex vector space of dimension  $n+1$  with a Hermitian form of signature  $(n, 1)$  then we write  $\mathbb{C}H(W)$  for the space of lines in  $W$  on which the given form is negative definite. We call this the complex hyperbolic space of  $W$ ; it is an open subset of  $PW$  and is biholomorphic to the unit ball in  $\mathbb{C}^n$ . The previous two paragraphs may be summarized by saying that the Hodge structure of  $V$  defines a point in  $\mathbb{C}H(H_\omega^4(V; \mathbb{C}))$ .

We chose the factor 3 in (1.4) so that  $(,)$  and  $\langle | \rangle$  would agree in the sense of the next lemma. To relate the two Hermitian forms we consider the  $\mathbb{R}$ -linear map  $H_0^4(V; \mathbb{R}) \rightarrow H_\omega^4(V; \mathbb{C})$  which is the inclusion

$$H_0^4(V; \mathbb{R}) \rightarrow H^4(V; \mathbb{R}) \rightarrow H^4(V; \mathbb{C})$$

followed by projection to  $\sigma$ 's  $\omega$ -eigenspace. This is an isomorphism of real vector spaces. Since the  $\mathbb{Z}$ -lattice underlying  $\Lambda(V)$  is  $H_0^4(V; \mathbb{Z}) \subseteq H_0^4(V; \mathbb{R})$ , we get a map  $Z : \Lambda(V) \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow H_\omega^4(V; \mathbb{C})$ . Since the complex structure on  $\Lambda(V) \otimes_{\mathbb{Z}} \mathbb{R}$  is defined by taking  $\omega$  to act as  $\sigma^*$ , and since  $H_\omega^4$  is defined as a space on which  $\sigma^*$  acts by multiplication by  $\omega$ ,  $Z$  is complex-linear. Since  $\Lambda(V) \otimes_{\mathbb{Z}} \mathbb{R} = \Lambda(V) \otimes_{\mathcal{E}} \mathbb{C}$ , we may regard  $Z$  as an isomorphism  $\Lambda(V) \otimes_{\mathcal{E}} \mathbb{C} \rightarrow H_\omega^4(V; \mathbb{C})$  of complex vector spaces.

LEMMA 1.3. *For all  $\alpha, \beta \in \Lambda(V)$ ,  $(Z(\alpha), Z(\beta)) = \langle \alpha | \beta \rangle$ .*

PROOF. Since both  $(,)$  and  $\langle | \rangle$  are Hermitian forms, it suffices to check that  $(Z\alpha, Z\alpha) = \langle \alpha | \alpha \rangle$  for all  $\alpha$ . We write  $\alpha_\omega$  and  $\alpha_{\bar{\omega}}$  for the projections of  $\alpha \in H_0^4(V; \mathbb{R})$  to  $H_\omega^4(V; \mathbb{C})$  and  $H_{\bar{\omega}}^4(V; \mathbb{C})$ . By definition,

$$(Z\alpha, Z\alpha) = 3 \int_V \alpha_\omega \wedge \bar{\alpha}_{\bar{\omega}} = 3 \int_V \alpha_\omega \wedge \alpha_{\bar{\omega}}.$$

We will write  $\sigma$  for  $\sigma^*$  throughout the proof. Since  $\sigma\alpha = \omega\alpha_\omega + \bar{\omega}\alpha_{\bar{\omega}}$  and  $\sigma^{-1}\alpha = \bar{\omega}\alpha_\omega + \omega\alpha_{\bar{\omega}}$ , we deduce  $\alpha_\omega = -\frac{1}{\theta}(\omega\sigma\alpha - \bar{\omega}\sigma^{-1}\alpha)$  and  $\alpha_{\bar{\omega}} = -\frac{1}{\theta}(-\bar{\omega}\sigma\alpha + \omega\sigma^{-1}\alpha)$ . Therefore

$$\begin{aligned} (Z\alpha, Z\alpha) &= 3 \int \frac{1}{\theta^2} \left[ \omega\sigma\alpha \wedge (-\bar{\omega}\sigma\alpha) + \omega\sigma\alpha \wedge \omega\sigma^{-1}\alpha \right. \\ &\quad \left. - \bar{\omega}\sigma^{-1}\alpha \wedge (-\bar{\omega}\sigma\alpha) - \bar{\omega}\sigma^{-1}\alpha \wedge \omega\sigma^{-1}\alpha \right] \\ &= - \int \left[ -\sigma\alpha \wedge \sigma\alpha + \bar{\omega}\sigma\alpha \wedge \sigma^{-1}\alpha \right. \\ &\quad \left. + \omega\sigma^{-1}\alpha \wedge \sigma\alpha - \sigma^{-1}\alpha \wedge \sigma^{-1}\alpha \right] \end{aligned}$$

$$\begin{aligned}
&= - \int \left[ -2\alpha \wedge \alpha - \sigma\alpha \wedge \sigma^{-1}\alpha \right] \\
&= 2\alpha \cdot \alpha + \int \sigma\alpha \wedge \sigma^{-1}\alpha .
\end{aligned}$$

To evaluate the second term, we use

$$(1.5) \quad \int \sigma\alpha \wedge \alpha + \int \sigma\alpha \wedge \sigma\alpha + \int \sigma\alpha \wedge \sigma^{-1}\alpha = 0 ,$$

which follows from  $1 + \sigma + \sigma^{-1} = 0$ . Since  $\sigma$  is an isomorphism and  $\wedge$  is symmetric, the first and last terms of (1.5) are equal, so we get

$$\int \sigma\alpha \wedge \sigma^{-1}\alpha = -\frac{1}{2} \int \sigma\alpha \wedge \sigma\alpha = -\frac{1}{2}\alpha \cdot \alpha .$$

This yields

$$(Z\alpha, Z\alpha) = \frac{3}{2}\alpha \cdot \alpha = \langle \alpha | \alpha \rangle$$

as desired.  $\square$

From the lemma it follows that  $\Lambda(V)$  has signature  $(10, 1)$  and that  $\mathbb{C}H(H_\omega^4(V; \mathbb{C}))$  is naturally identified with  $\mathbb{C}H(\Lambda(V) \otimes_{\mathcal{E}} \mathbb{C})$ . We write  $\mathbb{C}H(V)$  for either of these complex hyperbolic spaces.

In order to study the variation of the Hodge structure of  $V$  we must realize our constructions in local systems over  $\mathcal{C}_0$ . To do this we use the fact that the family  $\mathcal{V}_0$  over  $\mathcal{C}_0$  gives rise to a sheaf  $R^4\pi_*(\mathbb{Z})$  over  $\mathcal{C}_0$ . Recall that this is the sheaf associated to the presheaf  $U \mapsto H^4(\pi_V^{-1}(U); \mathbb{Z})$ . Since  $\mathcal{V}_0$  is topologically locally trivial,  $R^4\pi_*(\mathbb{Z})$  is a local system of 23-dimensional  $\mathbb{Z}$ -lattices isomorphic to  $H^4(V; \mathbb{Z})$ . The map  $\eta : F \mapsto \eta(V)$ , for  $F \in \mathcal{C}_0$ , is a section over  $\mathcal{C}_0$ , and the subsheaf  $(R^4\pi_*(\mathbb{Z}))_0$ , whose local sections are the local sections of  $R^4\pi_*(\mathbb{Z})$  orthogonal to  $\eta$ , is a local system of 22-dimensional  $\mathbb{Z}$ -lattices isomorphic to  $H_0^4(V; \mathbb{Z})$ . Since  $\sigma$  acts on  $\mathcal{V}_0$ , it acts on  $R^4\pi_*(\mathbb{Z})$ ; since it preserves  $\eta$  it acts on  $(R^4\pi_*(\mathbb{Z}))_0$ , giving the sheaf the structure of a local system of  $\mathcal{E}$ -modules isomorphic to  $\Lambda(V)$ . We call this local system  $\Lambda(\mathcal{V}_0)$ . The formula (1.2) endows  $\Lambda(\mathcal{V}_0)$  with the structure of a local system of Hermitian  $\mathcal{E}$ -modules. We write  $\mathbb{C}H(\mathcal{V}_0)$  for the corresponding local system of hyperbolic spaces.

We can also consider the sheaf  $R^4\pi_*(\mathbb{C})$  over  $\mathcal{C}_0$ ; it is a local system because it is the complexification of  $R^4\pi_*(\mathbb{Z})$ . Now,  $\sigma$  acts on  $R^4\pi_*(\mathbb{C})$  and we consider its  $\omega$ -eigensheaf  $(R^4\pi_*(\mathbb{C}))_\omega$ , which is a local system of Hermitian vector spaces isometric to  $H_\omega^4(V; \mathbb{C})$ , hence of signature  $(10, 1)$ , with a corresponding local system of complex hyperbolic spaces. The map  $Z$  identifies the local systems  $\Lambda(\mathcal{V}_0) \otimes_{\mathcal{E}} \mathbb{C}$  and  $(R^4\pi_*(\mathbb{C}))_\omega$ ,



and therefore identifies the two local systems of complex hyperbolic spaces. Therefore we may regard the inclusion  $H^{3,1}(V) \rightarrow H_\omega^4(V)$  as defining a section

$$(1.6) \quad g : \mathcal{C}_0 \rightarrow \mathbb{C}H(\mathcal{V}_0).$$

It is holomorphic since the Hodge filtration varies holomorphically. This is the period map; all our results refer to various formulations of it.

Next we obtain a concrete description of the  $\mathcal{E}$ -lattice  $\Lambda(V)$ , by investigating the monodromy of  $\Lambda(\mathcal{V}_0)$ . Fix a basepoint  $F \in \mathcal{C}_0$ , let  $\Gamma(V)$  be the isometry group of  $\Lambda(V)$ , and let  $\rho$  be the monodromy representation  $\pi_1(\mathcal{C}_0, F) \rightarrow \Gamma(V)$ . By a meridian around a divisor, such as  $\Delta$ , we mean the boundary circle of a small disk transverse to the divisor at a generic point of it, traversed once positively.

If  $W$  is a complex vector space then a complex reflection of  $W$  is a linear transformation that fixes a hyperplane pointwise and has finite order  $> 1$ . If this order is 3 or 6 then it is called a triflection or hexafflection. If  $W$  has a Hermitian form  $\langle | \rangle$ ,  $r \in W$  has nonzero norm and  $\zeta$  is a primitive  $n$ th root of unity, then the transformation

$$(1.7) \quad x \mapsto x - (1 - \zeta) \frac{\langle x|r \rangle}{\langle r|r \rangle} r$$

is a complex reflection of order  $n$  and preserves  $\langle | \rangle$ . It fixes  $r^\perp$  pointwise and sends  $r$  to  $\zeta r$ ; we call it the  $\zeta$ -reflection in  $r$ . If  $r$  has norm 3 and lies in an  $\mathcal{E}$ -lattice in which  $\theta$  divides all inner products, such as  $\Lambda(V)$ , then (1.7) shows that  $\omega$ -reflection in  $r$  also preserves the lattice. In this case we call  $r$  a root; this is meant to bring out the analogy with roots in the usual  $A_n, D_n, E_n$  root lattices, which also define reflections.

LEMMA 1.4. *The image of a meridian under the monodromy representation  $\rho : \pi_1(\mathcal{C}_0, F) \rightarrow \Gamma(V) = \text{Aut } \Lambda(V)$  is the  $\omega$ -reflection in a root of  $\Lambda(V)$ .*

PROOF. The argument is much the same as for Lemma 5.4 of [3]. Let  $D$  be a small disk in  $\mathcal{C}$ , meeting  $\Delta$  only at its center, and transversally there. We write  $F_0$  for the form at the center of  $D$ . Suppose without loss of generality that the basepoint  $F$  of  $\mathcal{C}_0$  is on  $\partial D$ , and let  $\gamma$  be the element of  $\pi_1(\mathcal{C}_0, F)$  that traverses  $\partial D$  once positively. The essential ingredients of the proof are the following. First,  $T_0$  has an  $A_1$  singularity, so we can choose local analytic coordinates  $x_1, \dots, x_4$  in which it is defined by  $x_1^2 + \dots + x_4^2$ . Then  $V_0$  has local analytic description  $x_1^2 + \dots + x_4^2 + x_5^3 = 0$ , which is to say that it is an  $A_2$

singularity. Second, the vanishing cohomology for this singularity, i.e., the Poincaré dual of the kernel of

$$H_4(V; \mathbb{Z}) \rightarrow H_4(\mathcal{V}|_D; \mathbb{Z}) \cong H_4(V_0; \mathbb{Z}) ,$$

is a (positive-definite) copy of the  $A_2$  root lattice. (An  $A_2$  surface singularity has vanishing cohomology a negative-definite copy of this lattice, and the sign changes when the dimension increases by 2.) Third, following Sebastiani-Thom [35], this lattice may be described as

$$V(2) \otimes V(2) \otimes V(2) \otimes V(2) \otimes V(3) ,$$

where  $V(k)$  is the  $\mathbb{Z}$ -module spanned by the differences of the  $k$ th roots of unity,  $\gamma$  acts by

$$(-1) \otimes (-1) \otimes (-1) \otimes (-1) \otimes \omega .$$

Here,  $-1$  (resp.  $\omega$ ) indicates the action on  $V(2)$  (resp.  $V(3)$ ) given by sending each square root (resp. cube root) of unity to itself times  $-1$  (resp.  $\omega$ ). This shows that the vanishing cohomology is a 1-dimensional  $\mathcal{E}$ -lattice; we write  $r$  for a generator. Fourth, we can compare the action of  $\gamma$  on the vanishing cycles with the action of  $\sigma$ . The Sebastiani-Thom argument proceeds by describing the Milnor fiber, up to homotopy, as the join of 2 points with 2 points with 2 points with 2 points with 3 points, corresponding to the exponents in the local defining equation. Therefore  $\sigma : x_5 \rightarrow \omega x_5$  acts on the vanishing cycles as  $1 \otimes 1 \otimes 1 \otimes 1 \otimes \omega$ , in the notation just introduced. This shows that  $\gamma$  acts on the vanishing cycles in the same way that  $\sigma$  does. Since  $\omega$ 's action on  $\Lambda(V)$  is defined to be  $\sigma^*$ ,  $\gamma$  acts on  $\langle r \rangle$  by  $\omega$ . Fifth,  $\gamma$  acts trivially on the orthogonal complement of the vanishing cohomology in  $H^4(V; \mathbb{Z})$ ; this implies that  $\gamma$  is the  $\omega$ -reflection in  $r$ . Finally, since the roots of the  $A_2$  lattice have norm 2, we see by (1.2) that  $\langle r|r \rangle = 3$ .  $\square$

The following two lemmas play only a small role in this chapter, at one point in the proof of Theorem 1.7, to which the reader could skip right away. However, they will be very important in chapter 3, where we extend the domain of the period map. Their content is that the discriminant has nice local models, which make many homomorphisms from braid groups into  $\pi_1(\mathcal{C}_0)$  visible. We also show that distinct braid generators have distinct monodromy actions.

We recall that the fundamental group of the discriminant complement of an  $A_n$  singularity is the braid group  $B_{n+1}$ , also known as the Artin group  $\mathcal{A}(A_n)$  of type  $A_n$ . More generally, the Artin group of an  $A_n$ ,  $D_n$  or  $E_n$  Dynkin diagram has one generator for each node, with two of the generators braiding ( $aba = bab$ ) or commuting, corresponding to whether the corresponding nodes are joined or not. It is

the fundamental group of the discriminant complement of that corresponding singularity [8]. Only  $\mathcal{A}(A_n)$  and  $\mathcal{A}(D_4)$  will be relevant to this monograph.

LEMMA 1.5. *Suppose  $F \in \mathcal{C}$  defines a cubic threefold with singularities  $s_1, \dots, s_m$ , each having one of the types  $A_n$  or  $D_4$ , and no other singularities. Let  $K_{i=1, \dots, m}$  be the base of a miniversal deformation of a singularity having the type of  $s_i$ , with discriminant locus  $\Delta_i \subseteq K_i$ . Then there is a neighborhood  $U$  of  $F$  in  $\mathcal{C}$  biholomorphic to  $K_1 \times \dots \times K_m \times B^N$ , where  $N = 35 - \sum \dim K_i$ , such that  $U - \Delta$  corresponds to*

$$(K_1 - \Delta_1) \times \dots \times (K_m - \Delta_m) \times B^N .$$

*In particular,  $\pi_1(U - \Delta)$  is the direct product of  $m$  Artin groups, the  $i$ th factor having the type of the singularity  $s_i$ .*

PROOF. This is essentially the assertion that  $\mathcal{C}$  contains a simultaneous versal deformation of all the singularities of  $T$ . By Theorem 1.1 of [32], it suffices to show that the sum of the Tjurina numbers of  $s_1, \dots, s_m$  is less than 16. Because the singularities of  $T$  are quasihomogeneous, their Tjurina numbers coincide with their Milnor numbers.

We will write  $\mu_i$  for the Milnor number of  $T$ 's singularity at  $s_i$ , and  $Z_i \subseteq H_0^4(V)$  for the vanishing cohomology of the corresponding singularity of  $V$ . If  $T$  has a  $D_4$  singularity at  $s_i$ , then  $\mu_i = 4$ , and  $V$  has an  $\tilde{E}_6$  singularity there, with  $\dim Z_i = 8$  and  $\dim(Z_i \cap Z_i^\perp) = 2$ . When  $T$  has an  $A_n$  singularity at  $s_i$ , we have  $\mu_i = n$ , and  $V$  has a singularity locally modeled on  $x_0^2 + x_1^2 + x_2^2 + y^3 + z^{n+1} = 0$ . By [4, p. 77],  $Z_i$  has a basis  $a_1, \dots, a_n, b_1, \dots, b_n$  with  $a_i^2 = b_i^2 = 2$ ,  $a_i \cdot a_{i\pm 1} = -1$ ,  $b_i \cdot b_{i\pm 1} = -1$ ,  $a_i \cdot b_i = -1$ ,  $a_i \cdot b_{i-1} = 1$ , and all other inner products zero. For  $n > 11$  this quadratic form has a negative-definite subspace of dimension  $\geq 4$ , so it cannot lie in  $H_0^4(V)$ , which has signature  $(20, 2)$ . Therefore cubic threefolds cannot have  $A_{n>11}$  singularities. For  $n < 12$ ,  $Z_i$  is nondegenerate except for  $n = 5$  and  $11$ , when  $\dim(Z_i \cap Z_i^\perp) = 2$ . We made these calculations using PARI/GP [31].

In every case, we have  $\mu_i \leq \frac{2}{3} \dim(Z_i / (Z_i \cap Z_i^\perp))$ . Since  $i \neq j$  implies  $Z_i \perp Z_j$ , we have

$$\sum \dim(Z_i / (Z_i \cap Z_i^\perp)) \leq \dim H_0^4(V) = 22 .$$

Putting these inequalities together yields  $\sum \mu_i \leq \frac{2}{3} \cdot 22 < 16$ , so that [32] applies. This gives the claimed description of  $\Delta$  near  $F$ , and the description of  $\pi_1(U - \Delta)$  follows immediately.  $\square$

LEMMA 1.6. *In the situation of the previous lemma, suppose  $g$  and  $g'$  are two of the standard generators for  $\pi_1(U - \Delta, F')$ , where  $F'$  is a basepoint in  $U - \Delta$ . Then  $\rho(g), \rho(g') \in \Gamma(V')$  are the  $\omega$ -reflections in linearly independent roots  $r, r' \in \Lambda(V')$ .*

PROOF. If  $T$  has exactly two singularities, both of type  $A_1$ , then near  $F$ ,  $\Delta$  has two components, and  $g$  and  $g'$  are meridians around them. As we saw in the proof of Lemma 1.4,  $V$  has two  $A_2$  singularities, and the vanishing cohomology of each of them is a  $\sigma$ -invariant sublattice of  $H_0^4(V')$ . In fact, they are the  $\mathcal{E}$ -spans of  $r$  and  $r'$ . With respect to the cup product, vanishing cocycles for distinct singularities are orthogonal, and it follows from (1.2) that they are also orthogonal under  $\langle | \rangle$ . Therefore  $r \perp r'$ . Since  $r$  and  $r'$  have nonzero norm, they must be linearly independent.

Now allow  $T$  to have  $A_n$  and/or  $D_4$  singularities, and suppose  $g$  and  $g'$  commute. Then there exists  $F_0 \in U$  and a neighborhood  $U_0$  of  $F_0$  in  $U$  with the following properties.  $T_0$  has just two singularities, both nodes,  $U_0$  contains  $F'$ ,  $\pi_1(U_0 - \Delta, F') \cong \mathbb{Z}^2$ , and  $g, g' \in \pi_1(U - \Delta, F')$  are represented by loops in  $U_0$  that are meridians around the two components of  $\Delta$  at  $F_0$ . These properties follow from Brieskorn's description [7] of the versal deformations of simple singularities in terms of the corresponding Coxeter groups. By the previous paragraph,  $r$  and  $r'$  are linearly independent.

Before treating the braiding case, we discuss  $A_2$ -threefolds, meaning those having a single singularity, of type  $A_2$ . Call such a threefold bad if its local monodromy group is cyclic, and good otherwise. Because the discriminant near an  $A_2$ -threefold is locally a product, badness and goodness are both open properties. One can show that the set of  $A_2$ -threefolds is irreducible. (See the remark following Lemma 3.9.) Therefore either every  $A_2$ -threefold is bad or every one is good.

Now suppose  $T$  is a cubic threefold with only  $A_n$  and  $D_4$  singularities, and suppose  $g$  and  $g'$  braid. The same reference to [7] shows that there exists  $F_0 \in U$  and a neighborhood  $U_0$  of  $F_0$  in  $U$  with the following properties.  $T_0$  has just one singularity, of type  $A_2$ ,  $F' \in U_0$ ,  $\pi_1(U_0 - \Delta, F') \cong B_3$ , and  $g, g' \in \pi_1(U - \Delta, F')$  are represented by the standard generators of  $B_3$ .

Now we claim that every  $A_2$ -threefold is good. As explained above, it suffices to exhibit a single good  $A_2$ -threefold, and for this purpose we take  $T$  to be an  $A_3$ -threefold, with  $g$  and  $g'$  the first two standard generators of  $\pi_1(U - \Delta, F') \cong B_4$ . Adjoining the relation  $g = g'$  would reduce  $B_4$  to  $\mathbb{Z}$ . If  $\rho(g) = \rho(g')$ , then  $\rho|_{B_4}$  factors through  $\mathbb{Z}$ , which

implies that all three generators of  $B_4$  have the same  $\rho$ -image. This is impossible by the commuting case, because the first and last generators commute. Therefore the threefold  $T_0$  near  $T$  is a good  $A_2$ -threefold.

Finally, suppose  $T$  is any cubic threefold with only  $A_n$  and  $D_4$  singularities, and  $g, g'$  are standard generators for  $\pi_1(U - \Delta, F')$  that braid. If  $\rho(g) = \rho(g')$  then some threefold  $T_0$  near  $T$  would be a bad  $A_2$ -threefold, which is impossible. Therefore  $r$  and  $r'$  are not proportional.

Remark: the hypothesis that  $T$  have only  $A_n$  and  $D_4$  singularities is needed for the statement of the lemma to make sense, since it presupposes the description of  $\pi_1(U - \Delta)$  from the previous lemma. But this hypothesis is probably inessential for both lemmas.  $\square$

**THEOREM 1.7.** *For  $F \in \mathcal{C}_0$ ,  $\Lambda(V)$  is isometric to the  $\mathcal{E}$ -lattice with inner product matrix*

$$(1.8) \quad \Lambda := (3) \oplus \begin{pmatrix} 3 & \theta & 0 & 0 \\ \bar{\theta} & 3 & \theta & 0 \\ 0 & \bar{\theta} & 3 & \theta \\ 0 & 0 & \bar{\theta} & 3 \end{pmatrix} \oplus \begin{pmatrix} 3 & \theta & 0 & 0 \\ \bar{\theta} & 3 & \theta & 0 \\ 0 & \bar{\theta} & 3 & \theta \\ 0 & 0 & \bar{\theta} & 3 \end{pmatrix} \oplus \begin{pmatrix} 0 & \theta \\ \bar{\theta} & 0 \end{pmatrix}.$$

REMARKS. Regarding (1.8) as an  $11 \times 11$  matrix  $(\lambda_{ij})$ , this means that  $\Lambda = \mathcal{E}^{11}$ , with

$$\langle (x_1, \dots, x_{11}) | (y_1, \dots, y_{11}) \rangle = \sum_{i,j} \lambda_{ij} x_i \bar{y}_j.$$

The four-dimensional lattice appearing twice among the summands is called  $E_8^{\mathcal{E}}$ , because its underlying  $\mathbb{Z}$ -lattice is a scaled copy of the  $E_8$  root lattice.

PROOF. By section 2 of [1], the cubic threefold  $T_0$  defined by

$$F_0 = x_2^3 + x_0 x_3^2 + x_1^2 x_4 - x_0 x_2 x_4 - 2x_1 x_2 x_3 + x_4^3$$

has an  $A_{11}$  singularity at  $[1:0:\dots:0] \in \mathbb{C}P^4$  and no other singularities. By Lemma 1.5, we may choose a neighborhood  $U$  of  $F_0$  and  $F \in U - \Delta$  such that  $\pi_1(U - \Delta, F) \cong B_{12}$ . By Lemma 1.4, the standard generators of  $B_{12}$  act on  $\Lambda(V)$  by the  $\omega$ -reflections  $R_1, \dots, R_{11}$  in roots  $r_1, \dots, r_{11} \in \Lambda(V)$ . By Lemma 1.6, these roots are pairwise linearly independent. The commutation relations imply  $r_i \perp r_j$  if  $j \neq i \pm 1$ . By the argument of [2, §5], the relation  $R_i R_{i+1} R_i = R_{i+1} R_i R_{i+1}$  implies that  $|\langle r_i | r_{i+1} \rangle| = \sqrt{3}$ , so after multiplying some of the  $r_i$  by scalars, we may take  $\langle r_i | r_{i+1} \rangle = \theta$ .

The rank of the inner product matrix of the  $r_i$  is 10. Therefore, if they were linearly independent then they would span  $\Lambda$  up to finite

index, and the Hermitian form on  $\Lambda$  would be degenerate. It is not, so the span must be only 10-dimensional. By the argument of [2, §5], the  $r_i$  span a sublattice of  $\Lambda(V)$  isometric to the direct sum (call it  $\Lambda_{10}$ ) of the last three summands of (1.8). In [2] we used a form of signature (1, 9) rather than (9, 1); this difference is unimportant. One can check directly that  $\theta\Lambda_{10}^* = \Lambda_{10}$ ; the underlying reason is that the real forms of  $E_8^\mathcal{E}$  and  $\begin{pmatrix} 0 & \theta \\ \theta & 0 \end{pmatrix}$  are scaled copies of even unimodular  $\mathbb{Z}$ -lattices. Since  $\Lambda(V) \subseteq \theta\Lambda(V)^*$ ,  $\Lambda_{10}$  is a direct summand of  $\Lambda(V)$ , so  $\Lambda(V) \cong (n) \oplus \Lambda_{10}$  for some  $n \in \mathbb{Z}$ . We have  $n > 0$  because  $\Lambda(V)$  has signature (10, 1).

For an  $\mathcal{E}$ -lattice  $M$  we define  $M^\mathbb{Z}$  to be the  $\mathbb{Z}$ -module underlying  $M$ , equipped with the  $\mathbb{Z}$ -bilinear pairing  $\alpha \cdot \beta = \frac{2}{3} \operatorname{Re} \langle \alpha | \beta \rangle$ . Computation shows that  $(n)^\mathbb{Z}$  has inner product matrix  $\begin{pmatrix} 2n/3 & -n/3 \\ -n/3 & 2n/3 \end{pmatrix}$ ,  $(E_8^\mathcal{E})^\mathbb{Z}$  is the even unimodular  $\mathbb{Z}$ -lattice  $E_8$ , and  $\begin{pmatrix} 0 & \theta \\ \theta & 0 \end{pmatrix}^\mathbb{Z}$  is the even unimodular  $\mathbb{Z}$ -lattice  $II_{2,2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . Since  $(\Lambda(V))^\mathbb{Z} = H_0^4(V; \mathbb{Z})$  has determinant  $\pm 3$ , we must have  $n = 3$ .  $\square$

We define a framing of a form  $F \in \mathcal{C}_0$  to be an equivalence class  $[\phi]$  of isometries  $\phi : \Lambda(V) \rightarrow \Lambda$ , two isometries being equivalent if they differ by multiplication by a scalar. Sometimes we write  $\phi$  rather than  $[\phi]$  and leave it to the reader to check that the construction at hand depends only on  $[\phi]$ . We define  $\mathcal{F}_0$  to be the set of all framings of all smooth cubic forms. Since the stalk of  $\Lambda(\mathcal{V}_0)$  at  $F \in \mathcal{C}_0$  is canonically isomorphic to  $\Lambda(V)$ , the set  $\mathcal{F}_0$  is in natural bijection with the subsheaf of  $P\operatorname{Hom}(\Lambda(\mathcal{V}_0), \mathcal{C}_0 \times \Lambda)$  consisting of projective equivalence classes of homomorphisms which are isometries on each stalk. This bijection gives  $\mathcal{F}_0$  the structure of a complex manifold. We refer to an element  $(F, [\phi])$  of  $\mathcal{F}_0$  as a framed smooth cubic form.

We write  $\Gamma$  for  $\operatorname{Aut} \Lambda$  and  $P\Gamma$  for  $P\operatorname{Aut} \Lambda$ . On  $\mathcal{F}_0$  are defined commuting actions of  $P\Gamma$  and  $G = \operatorname{GL}_5\mathbb{C}/D$ , where  $D$  is the group  $\{I, \omega I, \bar{\omega} I\}$ . An element  $\gamma$  of  $P\Gamma$  acts on the left by

$$\gamma.(F, [\phi]) = (F, [\gamma \circ \phi]) .$$

This action realizes  $P\Gamma$  as the group of deck transformations of the covering space  $\mathcal{F}_0 \rightarrow \mathcal{C}_0$ . An element  $g$  of  $\operatorname{GL}_5\mathbb{C}$  acts on the right by

$$(1.9) \quad (F, [\phi]).g = (F \circ g, [\phi \circ g^{*-1}]) .$$

Here,  $\operatorname{GL}_5\mathbb{C}$  acts on  $\mathbb{C}^5$  on the left, hence acts on  $\mathcal{C}$  on the right by  $(F.g)(x) = F(g.x)$ , i.e.,  $F.g = F \circ g$ . We extend  $\operatorname{GL}_5\mathbb{C}$ 's action on  $\mathbb{C}^5$  to  $\mathbb{C}^6 = \mathbb{C}^5 \oplus \mathbb{C}$  by the trivial action on the  $\mathbb{C}$  summand. This induces a right action on  $\mathcal{V}$  by  $(F, x).g = (F.g, g^{-1}x)$ . That is,  $g$  carries the zero-locus of  $(F + x_5^3).g$  to the zero-locus of  $F + x_5^3$ . The  $g^{*-1}$

appearing in (1.9) is the inverse of the induced map on cohomology, which respects the  $\mathcal{E}$ -module structure since  $g$  commutes with  $\sigma$ . The subgroup  $D \subseteq \mathrm{GL}_5\mathbb{C}$  acts trivially on  $\mathcal{F}_0$  because it acts trivially on  $\mathcal{C}_0$  and by scalars on every  $\Lambda(V)$ .

We now introduce the moduli spaces  $\mathcal{M}_0$  and  $\mathcal{M}_0^f$  of smooth and framed smooth cubic threefolds. Since  $\mathcal{C}_0 \subseteq \mathcal{C}_s$ ,  $G$  acts properly on  $\mathcal{C}_0$ , with the quotient  $\mathcal{M}_0 = \mathcal{C}_0/G$  a complex analytic orbifold and a quasiprojective variety. The properness on  $\mathcal{C}_0$  implies properness on  $\mathcal{F}_0$ , so  $\mathcal{M}_0^f = \mathcal{F}_0/G$  is a complex analytic orbifold and an analytic space. We will see in Lemma 1.8 that  $\mathcal{M}_0^f$  is a complex manifold. Since the  $G$ -stabilizer of a point  $(F, [\phi])$  of  $\mathcal{F}_0$  is a subgroup of the  $G$ -stabilizer of  $F \in \mathcal{C}_0$ , the covering map  $\mathcal{F}_0 \rightarrow P\Gamma \backslash \mathcal{F}_0 = \mathcal{C}_0$  descends to an orbifold covering map  $\mathcal{M}_0^f = \mathcal{F}_0/G \rightarrow \mathcal{C}_0/G = \mathcal{M}_0$ . Counting dimensions shows that  $\mathcal{M}_0$  and  $\mathcal{M}_0^f$  are 10-dimensional.

We write  $\mathbb{C}H^{10}$  for  $\mathbb{C}H(\Lambda \otimes_{\mathcal{E}} \mathbb{C})$ . Recall that  $H^{3,1}(V; \mathbb{C})$  is a negative line in the Hermitian vector space  $H_{\omega}^4(V; \mathbb{C})$  and that  $Z$  is an isometry  $\Lambda(V) \otimes_{\mathcal{E}} \mathbb{C} \rightarrow H_{\omega}^4(V; \mathbb{C})$ . We reformulate the period map (1.6) as the holomorphic map  $g : \mathcal{F}_0 \rightarrow \mathbb{C}H^{10}$  given by

$$(1.10) \quad g(F, [\phi]) = \phi(Z^{-1}(H^{3,1}(V; \mathbb{C}))) .$$

On the right we have written just  $\phi$  for  $\phi$ 's  $\mathbb{C}$ -linear extension  $\Lambda(V) \otimes_{\mathcal{E}} \mathbb{C} \rightarrow \Lambda \otimes_{\mathcal{E}} \mathbb{C}$ . Since  $\mathbb{C}H^{10}$  is a 10-ball and bounded holomorphic functions on  $\mathrm{GL}_5\mathbb{C}$  are constant,  $g$  is constant along  $\mathrm{GL}_5\mathbb{C}$ -orbits, so it descends to a holomorphic map  $g : \mathcal{M}_0^f \rightarrow \mathbb{C}H^{10}$ , also called the period map. This map is equivariant with respect to the action of  $P\Gamma$ , so it in turn descends to a map

$$(1.11) \quad g : \mathcal{M}_0 = P\Gamma \backslash \mathcal{M}_0^f = P\Gamma \backslash \mathcal{F}_0 / \mathrm{GL}_5\mathbb{C} \rightarrow P\Gamma \backslash \mathbb{C}H^{10} ,$$

again called the period map.

**LEMMA 1.8.**  *$G$  acts freely on  $\mathcal{F}_0$ , so that  $\mathcal{M}_0^f$  is a complex manifold, not just an orbifold. The period map  $g : \mathcal{M}_0^f \rightarrow \mathbb{C}H^{10}$  has rank 10 at every point of  $\mathcal{M}_0^f$ .*

**PROOF.** We prove the second assertion first. Let  $F \in \mathcal{C}_0$ , let  $F' \in \mathcal{C}$  be different from  $F$ , and let  $\varepsilon > 0$  be small enough that the disk  $D = \{F + tF' \mid t \in \mathbb{C} \text{ and } |t| \leq \varepsilon\}$  lies in  $\mathcal{C}_0$ . Writing  $F_t$  for  $F + tF'$ , we know from the discussion surrounding (1.3) that  $H^{3,1}(V_t)$  is spanned by the residue of  $\Omega/(F_t + x_5^3)^2$ . Since  $\mathcal{V}$  trivializes over  $D$ , we may unambiguously translate this class into  $H^4(V; \mathbb{C})$ ; this gives a map  $h : D \rightarrow H^4(V; \mathbb{C})$ . For sufficiently small  $t$ ,  $h(F_t)$  is the element of

$\text{Hom}(H_4(V; \mathbb{Z}), \mathbb{C})$  given by

$$(\text{an integral 4-cycle } C) \mapsto \int_{\partial N} \frac{\Omega}{(F_t + x_5^3)^2}$$

where  $N$  is the part of the boundary of a tubular neighborhood of  $V$  in  $\mathbb{C}P^5$  that lies over a submanifold of  $V$  representing  $C$ . Therefore we may differentiate with respect to  $t$  under the integral sign, so the derivative of  $h$  at the center of  $D$  is the element of  $\text{Hom}(H_4(V; \mathbb{Z}), \mathbb{C})$  given by

$$C \mapsto \int_{\partial N} \frac{\Omega}{(F_t + x_5^3)^3} \cdot (-2) \frac{\partial}{\partial t} (F_t + x_5^3) \Big|_{t=0} = -2 \int_{\partial N} \frac{\Omega F'(x_0, \dots, x_4)}{(F + x_5^3)^3}.$$

This lies in  $H_\omega^4(V; \mathbb{C})$ , and it lies in  $H^{3,1}$  if and only if  $F'$  lies in the Jacobian ideal of  $F + x_5^3$ , i.e., if and only if  $F'$  lies in the Jacobian ideal of  $F$ , i.e., if and only if the pencil  $\langle F, F' \rangle$  in  $\mathcal{C}$  is tangent to the  $G$ -orbit of  $F$ .

Upon choosing a framing  $\phi$  for  $F$  and lifting  $D$  to a disk  $\tilde{D} = \{(F_t, [\phi_t])\}$  in  $\mathcal{F}_0$  passing through  $(F, [\phi])$ , it follows that the derivative of  $g : \mathcal{F}_0 \rightarrow \mathbb{C}H^{10}$  along  $\tilde{D}$  at  $(F, [\phi])$  is zero if and only if  $\tilde{D}$  is tangent to the  $G$ -orbit of  $(F, [\phi])$ . Since the orbit has codimension 10,  $g$  has rank 10.

To prove the first assertion, recall that an orbifold chart about the image of  $(F, [\phi])$  in  $\mathcal{M}_0^f$  is  $U \rightarrow U/H \subseteq \mathcal{M}_0^f$ , where  $H$  is the  $G$ -stabilizer of  $(F, [\phi])$  and  $U$  is a small  $H$ -invariant ball transverse to the  $G$ -orbit of  $(F, [\phi])$ . We have just seen that the composition  $U \rightarrow U/H \subseteq \mathcal{M}_0^f \rightarrow \mathbb{C}H^{10}$  has rank 10 and is hence a biholomorphism onto its image. It follows that  $H = \{1\}$ .  $\square$

**THEOREM 1.9.** *The period map  $g : \mathcal{M}_0 \rightarrow P\Gamma \backslash \mathbb{C}H^{10}$  is an isomorphism onto its image.*

**PROOF.** We begin by proving that if  $F$  and  $F'$  are generic elements of  $\mathcal{C}_0$  with the same image under  $g$  then they are  $G$ -equivalent, i.e.,  $T$  and  $T'$  are projectively equivalent. By hypothesis there exists an isometry  $b : \Lambda(T) \rightarrow \Lambda(T')$  which carries  $Z^{-1}(H^{3,1}(V)) \in \Lambda(V) \otimes_{\mathcal{E}} \mathbb{C}$  to  $Z^{-1}(H^{3,1}(V')) \in \Lambda(V') \otimes_{\mathcal{E}} \mathbb{C}$ . Passing to the underlying integer lattices,  $b$  is an isometry  $H_0^4(V; \mathbb{Z}) \rightarrow H_0^4(V'; \mathbb{Z})$  carrying  $H^{3,1}(V; \mathbb{C})$  to  $H^{3,1}(V'; \mathbb{C})$ . By complex conjugation it also identifies  $H^{1,3}(V; \mathbb{C})$  with  $H^{1,3}(V'; \mathbb{C})$ , and by considering the orthogonal complement of  $H^{3,1} \oplus H^{1,3}$  we see that it identifies  $H_0^{2,2}(V; \mathbb{C})$  with  $H_0^{2,2}(V'; \mathbb{C})$ . That is, it induces an isomorphism of Hodge structures.



Next: one of  $\pm b$  extends to an isometry  $H^4(V; \mathbb{Z}) \rightarrow H^4(V'; \mathbb{Z})$  carrying  $\eta(V)$  to  $\eta(V')$ . This follows from some lattice-theoretic considerations: if  $L$  is a primitive sublattice of a unimodular lattice  $M$  (i.e.,  $L = (L \otimes \mathbb{Q}) \cap M$ ), and is nondegenerate, then the projections of  $M$  into  $L \otimes \mathbb{Q}$  and  $L^\perp \otimes \mathbb{Q}$  define an isomorphism of  $L^*/L$  with  $(L^\perp)^*/L^\perp$ . Here the asterisk denotes the dual lattice. It is easy to check that an isometry of  $L$  and an isometry of  $L^\perp$  together give an isometry of  $M$  if and only if their actions on  $L^*/L$  and  $(L^\perp)^*/L^\perp$  coincide under this identification. Since  $\langle \eta(V) \rangle^*/\langle \eta(V) \rangle \cong \mathbb{Z}/3$ , it follows that exactly one of the isometries

$$\langle \eta(V) \rangle \oplus H_0^4(V; \mathbb{Z}) \rightarrow \langle \eta(V') \rangle \oplus H_0^4(V'; \mathbb{Z}),$$

given on the first summand by  $\eta(V) \mapsto \eta(V')$ , and on the second by  $\pm b$ , extends to an isometry  $H^4(V; \mathbb{Z}) \rightarrow H^4(V'; \mathbb{Z})$ .

From Voisin's theorem [41] we deduce that there is a projective transformation  $\beta$  carrying  $V$  to  $V'$ . The variety  $S$  of smooth cubic fourfolds admitting a triflection is irreducible, so that one can speak of a generic such fourfold. To see this irreducibility, decompose the space of cubic forms into eigenspaces under a triflection; it turns out that only one of these eigenspaces contains smooth cubics. Furthermore, a generic such fourfold admits only one triflection (and its inverse). Since  $V$  and  $V'$  admit the triflections  $\sigma^{\pm 1}$  and are generic points of  $S$ ,  $\beta$  carries the fixed-point set  $T$  of  $\sigma$  in  $V$  to the fixed-point set  $T'$  of  $\sigma$  in  $V'$ . That is,  $T$  and  $T'$  are projectively equivalent.

We have proven that the period map from  $\mathcal{M}_0$  to  $P\Gamma \backslash \mathbb{C}H^{10}$  is generically injective, and the previous lemma shows that it is a local isomorphism. It follows that it is an isomorphism onto its image.  $\square$

In [40], van Geemen and Izadi discuss the notion of a positive half-twist  $V_{1/2}$  for a Hodge structure  $V$  on which a CM field  $k$  acts. In our case  $V$  is the primitive  $H^4$  of the cyclic cubic fourfold and  $k = \mathbb{Q}(\omega)$ . Its Tate twist  $V_{1/2}(1)$  is a weight one structure of genus eleven, and so corresponds to an abelian variety. These special abelian varieties are parametrized by an orbit of  $U(1, 10) \subset Sp(11, \mathbb{R})$ . In our approach we use  $V$  directly to find the ball quotient via an imbedding  $U(1, 10) \subset SO(2, 20)$ . In both approaches the Griffiths residue calculus is used to prove a Torelli theorem.

## CHAPTER 2

### The Discriminant near a Chordal Cubic

In the next chapter we will enlarge the domain of the period map  $\mathcal{C}_0 \rightarrow P\Gamma \backslash \mathcal{C}H^{10}$ , in order to obtain a map from a compactification of  $\mathcal{M}_0$  to the Baily-Borel compactification  $\overline{P\Gamma \backslash \mathcal{C}H^{10}}$ . In order to do this we will need to understand the local structure of the discriminant  $\Delta \subseteq \mathcal{C}_0$ , at least near the threefolds to which we will extend  $g$ . In [1] (see also [42]), the GIT-stability of cubic threefolds is completely worked out. There is one distinguished type of threefold, which we call a chordal cubic, which is the secant variety of the rational normal quartic curve. Except for the chordal cubics and those cubics that are GIT-equivalent to them, a cubic threefold is semistable if and only if it has singularities only of types  $A_1, \dots, A_5$  and  $D_4$ . At such a threefold the local structure of  $\Delta$  is given by Lemma 1.5.

The rest of this chapter addresses the nature of  $\Delta$  near the chordal cubic locus. It turns out (see the remark following Theorem 4.1) that the period map  $PC_0 \rightarrow P\Gamma \backslash \mathcal{C}H^{10}$  does not extend to a regular map  $PC_{ss} \rightarrow \overline{P\Gamma \backslash \mathcal{C}H^{10}}$ . The problem is that it does not extend to the chordal cubic locus. Therefore it is natural to try to enlarge the domain of the period map not to  $PC_{ss}$  but rather to  $(\widehat{PC})_{ss}$ , where  $\widehat{PC}$  is the blowup of  $PC$  along the closure of the chordal cubic locus. The details concerning the GIT analysis and the extension of the period map appear in chapter 3; at this point we are only motivating the study of the local structure of the proper transform  $\widehat{\Delta}$  of  $\Delta$  along the exceptional divisor  $E$ . Recall that we defined  $\Delta$  as a subset of  $\mathcal{C}$ , but will also write  $\Delta$  for its image in  $PC$ .

If  $T \in PC$  is a chordal cubic then we write  $E_T$  for  $\pi^{-1}(T) \subseteq E$ , where  $\pi$  is the natural projection  $\widehat{PC} \rightarrow PC$ . (There are a number of projection maps in this monograph, such as  $\pi_{\mathcal{T}}$  and  $\pi_{\mathcal{V}}$  in chapter 1, and some others introduced later. To keep them straight, we will use a subscript to indicate the domain for all of them except this one.)  $E_T$  may be described as the set of unordered 12-tuples in the rational normal curve  $R_T$  which is the singular locus of  $T$ . To see this, one counts dimensions to find that the chordal cubic locus has codimension 13 in

$PC$ , so  $E_T$  is a copy of  $P^{12}$ . To identify  $E_T$  with the set of unordered 12-tuples in  $R_T$ , consider a pencil of cubic threefolds degenerating to  $T$ . The 12-tuple may be obtained as the intersection of  $R_T$  with a generic member of the pencil; since  $R_T$  has degree 4, this intersection consists of 12 points, counted with multiplicity. (If every member of the pencil vanishes identically on  $R_T$  then the pencil is not transverse to the chordal cubic locus.) We will indicate an element of  $E_T$  by a pair  $(T, \tau)$ , where  $\tau$

We will describe  $\widehat{\Delta} \subseteq \widehat{PC}$  by proving the following two theorems, which are similar to but weaker than Lemma 1.5. The first is weaker because it asserts a homeomorphism with a standard model of the discriminant, rather than a biholomorphism. The second gives a complex-analytic isomorphism, but refers to a finite cover of (an open set in)  $\widehat{PC}$ , branched over  $E$ . But we don't know any reason that the homeomorphism in Theorem 2.1 couldn't be promoted to a biholomorphism.

**THEOREM 2.1.** *Suppose  $T$  is a chordal cubic and  $\tau$  is a 12-tuple in  $R_T$ , with  $m$  singularities, of types  $A_{n_1}, \dots, A_{n_m}$ , where an  $A_n$  singularity means a point of multiplicity  $n + 1$ . Let  $K_{i=1, \dots, m}$  be the base of a miniversal deformation of an  $A_{n_i}$  singularity, with discriminant locus  $\Delta_i \subseteq K_i$ . Then there is a neighborhood  $U$  of  $(T, \tau)$  in  $\widehat{PC}$  homeomorphic to  $B^1 \times K_1 \times \dots \times K_m \times B^N$ , where  $N = 33 - \sum \dim K_i$ , such that  $E$  corresponds to*

$$\{0\} \times K_1 \times \dots \times K_m \times B^N$$

and  $U - \widehat{\Delta}$  to

$$B^1 \times (K_1 - \Delta_1) \times \dots \times (K_m - \Delta_m) \times B^N.$$

In particular,

$$\pi_1(U - (\widehat{\Delta} \cup E)) \cong \mathbb{Z} \times B_{n_1+1} \times \dots \times B_{n_m+1},$$

where the  $\mathbb{Z}$  factor is generated by a meridian of  $E$  and the standard generators of the braid group factors are meridians of  $\widehat{\Delta}$ .

**THEOREM 2.2.** *Suppose  $(T, \tau)$ ,  $m$ ,  $n_1, \dots, n_m$ ,  $K_1, \dots, K_m$ ,  $\Delta_1, \dots, \Delta_m$  and  $N$  are as in Theorem 2.1. Then there exists a neighborhood  $U$  of  $(T, \tau)$  in  $\widehat{PC}$  biholomorphic to  $B^1 \times B^{33}$ , with  $U \cap E$  corresponding to  $\{0\} \times B^{33}$ , such that the following holds. We write  $\beta : \tilde{U} \rightarrow U$  for the 6-fold cover of  $U$  branched over  $U \cap E$ , meaning that  $\tilde{U}$  is obtained from  $U \cong B^1 \times B^{33}$  by the base extension  $t \mapsto t^6$  of the  $B^1$  factor. We write  $(T, \tau)^\sim$  for the point  $\beta^{-1}(T, \tau)$ . There is a neighborhood  $V$  of*

$(T, \tau) \sim$  in  $\tilde{U}$  biholomorphic to  $B^1 \times K_1 \times \cdots \times K_m \times B^N$ , such that

$$(2.1) \quad V \cap \beta^{-1}(E) = \{0\} \times K_1 \times \cdots \times K_m \times B^N$$

and

$$(2.2) \quad V - \beta^{-1}(\widehat{\Delta}) = B^1 \times (K_1 - \Delta_1) \times \cdots \times (K_m - \Delta_m) \times B^N .$$

The rest of this chapter is devoted to proving Theorems 2.1 and 2.2. It is rather technical, especially Lemma 2.11 and beyond; although these theorems are analogues of Lemma 1.5, the proofs are much more complicated.

LEMMA 2.3. *Suppose  $T$  is a chordal cubic and  $(T, \tau) \in E_T$ .*

- (i)  *$(T, \tau)$  lies in  $\widehat{\Delta}$  if and only if  $\tau$  has a multiple point.*
- (ii) *If  $\tau$  has a point of multiplicity  $n + 1$ , then  $(T, \tau)$  is a limit of points of  $\widehat{PC}$  representing cubic threefolds with  $A_n$  singularities.*

Nowhere else in the monograph do we refer to any result or notation introduced from here to the end of this chapter.

When we refer to the “standard chordal cubic”, we mean the one defined by

$$(2.3) \quad F(x_0, \dots, x_4) = \det \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix} = 0 ,$$

which is the secant variety of the rational normal curve parameterized by

$$(2.4) \quad s \mapsto [1, s, s^2, s^3, s^4] \quad (s \in P^1).$$

As always,  $T$  is the threefold defined by  $F$ . We will abbreviate  $R_T$  to  $R$ . We write  $P$  for the point  $[1, 0, 0, 0, 0]$ .

Because  $\mathrm{PGL}_5\mathbb{C}$  acts transitively on the chordal locus, many questions regarding local structure can be reduced to questions concerning a suitable linear space transverse to it. The one we use is the following:

$$(2.5) \quad \begin{aligned} G_{u_1, \dots, u_{12}} = F + & x_4^3 + u_1 x_4^2 x_3 + u_2 x_4 x_3^2 + u_3 x_3^3 + u_4 x_3^2 x_2 \\ & + u_5 x_3 x_2^2 + u_6 x_2^3 + u_7 x_2^2 x_1 + u_8 x_2 x_1^2 \\ & + u_9 x_1^3 + u_{10} x_1^2 x_0 + u_{11} x_1 x_0^2 + u_{12} x_0^3 ; \end{aligned}$$

see below for the claimed transversality. The first key property of  $G_{u_1, \dots, u_{12}}$  is that it restricts to  $R$  as the polynomial

$$(2.6) \quad s^{12} + u_1 s^{11} + u_2 s^{10} + \cdots + u_{11} s + u_{12} .$$

The second key property is that the singularities at  $P$  of the members of  $\langle F, G_{u_1, \dots, u_{12}} \rangle$  correspond to the degree of vanishing of (2.6). This is the reason we chose this particular family.

LEMMA 2.4. *For  $\lambda \neq 0$ ,  $F + \lambda(G_{u_1, \dots, u_{12}} - F)$  has an  $A_n$  singularity at  $P$  if and only if  $G_{u_1, \dots, u_{12}}|_R$  has an  $A_n$  singularity there—i.e., if and only if (2.6) vanishes to order exactly  $n + 1$  at  $s = 0$ . Furthermore,  $\{G_{0, u_2, \dots, u_{12}}\}$  provides a versal deformation of the  $A_{11}$  singularity of  $G_{0, \dots, 0}$ .*

PROOF. We verified this by writing down the local defining equation at  $P$  and recognizing the singularities using the method of [1, sec. 2]. This amounted to applying a succession of local coordinate changes until the singularity was in a standard form (semiquasihomogeneous with lowest-weight terms defining an isolated singularity). Conceptually, what is going on is that the Jacobian scheme of  $G_{u_1, \dots, u_{12}}$  contains the “infinitesimal segment of  $R$  of length  $n$ ” whenever the last  $n + 1$  subscripts are zero. Formally, we mean the 0-dimensional subscheme  $X_n$  of  $R$ , of length  $n$ , supported at  $P$ . A geometric way to phrase the recognition method [9, Lemma 1 and corollary] for the singularities is that a generic hypersurface whose Jacobian scheme contains  $X_n$  has an  $A_n$  singularity there. So the calculations served merely to verify something we expected to be true. The particular case of  $G_{0, \dots, 0}$  is an  $A_{11}$  singularity, and verifying this case will give the reader a full understanding of the method.

Our picture also makes visible the fact that  $\{G_{0, u_2, \dots, u_{12}}\}$  provides a versal deformation of the  $A_{11}$  singularity of  $G_{0, \dots, 0}$ . Formally, this is the assertion that their restrictions to the Jacobian scheme  $X_{11}$  span the coordinate ring of  $X_{11}$  as a vector space. This is visible because  $G_{0, 1, 0, \dots, 0}$  vanishes along  $X_{11}$  to order exactly 10,  $G_{0, 0, 1, 0, \dots, 0}$  to order exactly 9, etc.  $\square$

LEMMA 2.5. *Each pencil  $\langle F, G_{u_1, \dots, u_{12}} \rangle$  is transverse to the chordal locus at  $T \in PC$ .*

PROOF. The key point is that a pencil of cubics through  $T$  is transverse to the chordal locus if and only if some member of it is nonzero at some point of  $R$ . To see this, use the easily verified fact that  $\mathrm{GL}_2\mathbb{C}$  acts on  $\mathcal{C}$  as

$$\mathrm{Sym}^3\mathrm{Sym}^4(\mathbb{C}^2) \cong \mathrm{Sym}^{12}\mathbb{C}^2 \oplus \mathrm{Sym}^8\mathbb{C}^2 \oplus \mathrm{Sym}^6\mathbb{C}^2 \oplus \mathrm{Sym}^4\mathbb{C}^2 \oplus \mathbb{C} ,$$

with the summands having dimensions 13, 9, 7, 5 and 1, and the  $\mathbb{C}$  summand spanned by  $F$ . Because the tangent space of the chordal locus at  $T \in PC$  has codimension 13, it must be the sum of the 9-, 7- and

5-dimensional summands. Also, restriction of a cubic form to  $R$  gives a  $\mathrm{PGL}_2\mathbb{C}$ -equivariant map from  $\mathbb{C}$  to  $\mathrm{Sym}^{12}\mathbb{C}^2$ . Therefore the kernel of this restriction map consists exactly of the 9-, 7- and 5-dimensional summands. We have just seen that this is the tangent space to the chordal locus, completing the proof.  $\square$

**PROOF OF LEMMA 2.3:** To prove (i), we may restrict attention to the  $\mathrm{PGL}_2\mathbb{C}$ -invariant linear space  $H \subseteq PC$  complementary to the chordal locus at  $T$ . (The complement (2.5) is not invariant, so this is a different complement.) We must show that the tangent cone of  $\Delta \cap H$  at  $T$  is the classical discriminant of binary forms of degree 12. We use the main result of [34], which describes the tangent cone to the discriminant in any family of hypersurfaces. The family we apply it to is the restriction to  $H$  of the universal family over  $PC$ . The result is that there is a rational map  $\phi$  from the singular locus  $R$  of  $T$  to the projectivized cotangent space to  $H$  at  $T$ , and subvarieties  $\{Z_\alpha\}$  of  $R$ , such that the projective tangent cone to  $\Delta \cap H$  at  $T$  is the set-theoretic union of the dual varieties of the  $\phi(Z_\alpha)$ . The  $\mathrm{PGL}_2\mathbb{C}$  symmetry makes this easy to apply. First, the locus where  $\phi$  is regular is invariant, so it is all of  $R$ , so  $\phi$  is regular. The same argument shows that there is only one  $Z_\alpha$ , namely all of  $R$ , so the projective tangent cone to  $\Delta \cap H$  is the variety dual to  $\phi(R)$ . Also, we saw in the proof of Lemma 2.5 that  $\mathrm{PGL}_2\mathbb{C}$  acts on  $T_T H$  as  $\mathrm{Sym}^{12}\mathbb{C}^2$ . There is a unique 1-dimensional orbit in  $P\mathrm{Sym}^{12}\mathbb{C}^2$ , namely the rational normal curve of degree 12, so the  $\mathrm{PGL}_2\mathbb{C}$ -equivariance forces  $\phi(R)$  to be this curve. Now, the dual variety of a rational normal curve of degree  $d$  in  $P^d$  is the discriminant of binary forms of degree  $d$ , so we have successfully identified the tangent cone of  $\Delta \cap H$ .

Now we prove (ii). We take  $T$  to be the standard chordal cubic, and place the multiple point of  $\tau$  at  $P \in R$ . We suppose without loss of generality that  $[0, 0, 0, 0, 1] \in R$  is not one of the points of  $\tau$ . Since  $\tau$  has no point at  $s = \infty$ , there is a choice of  $u_1, \dots, u_{12}$  such that  $\tau$  is the limiting direction of the pencil  $\langle F, G_{u_1, \dots, u_{12}} \rangle$ . Since  $\tau$  has a point of multiplicity  $n + 1$  at  $P$ , we have  $u_{12} = \dots = u_{12-n} = 0$  and  $u_{11-n} \neq 0$ . Then Lemma 2.4 shows that a generic member of the pencil has an  $A_n$  singularity at  $P$ . This proves (ii).  $\square$

Because of the action of  $PG$ , proving Theorems 2.1 and 2.2 reduces to a similar but lower-dimensional problem. Let  $T$  be the standard chordal cubic. In our arguments, the 12-tuple consisting of 12 points all concentrated at  $P$  will play a special role; we call it  $\tau_0$ . Distinguishing  $\tau_0$  breaks the  $\mathrm{PGL}_2\mathbb{C}$  symmetry, leaving only the subgroup  $\mathbb{C} \times \mathbb{C}^*$

stabilizing a point, of codimension 22 in  $PG$ . We will describe a neighborhood of  $(T, \tau_0) \in \widehat{PC}$  in terms of a decomposition  $\widehat{B} \times B^{22}$ , where  $\widehat{B}$  is  $P^{12}$  blown up at a point and  $B^{22}$  is the 22-ball. Namely, let  $A$  be the affine 11-space in  $PC$  consisting of the  $G_{0, u_2, \dots, u_{12}}$  of (2.5). Let  $B$  be the projective space spanned by  $A$  and  $T$ , and  $\widehat{B}$  be its proper transform. Lemma 2.5 shows that  $B$  is transverse to the chordal cubic locus at  $T$ , so over a neighborhood of  $T$ ,  $\widehat{B}$  is just  $P^{12}$  blown up at a point.  $\widehat{B}$  is where most of our work will take place. Choose a ball  $B^{22} \subseteq PG$  transverse to  $\mathbb{C} \times \mathbb{C}^*$  at 1; we will show that, after shrinking  $B^{22}$ ,  $(T, \tau_0)$  has a neighborhood  $\widehat{B} \times B^{22}$ , the second factor representing 22 “trivial directions”.

LEMMA 2.6. *The map  $\widehat{B} \times B^{22} \rightarrow \widehat{PC}$  given by  $(b, g) \mapsto b.g$  is a local biholomorphism at  $((T, \tau_0), 1)$ .*

PROOF. We write  $Y$  for the  $PG$ -orbit of  $(T, \tau_0)$ . We must prove that  $Y$  and  $\widehat{B}$  are transverse in  $\widehat{PC}$  at  $(T, \tau_0)$ . Since  $B$  is transverse to the chordal cubic locus at  $T$  (Lemma 2.5), it suffices to prove that  $Y \cap E_T$  and  $\widehat{B} \cap E_T$  are transverse in  $E_T$  at  $(T, \tau_0)$ . We use  $u_1, \dots, u_{12}$  as coordinates around  $(T, \tau_0)$  in  $E_T$  as in the proof of Lemma 2.3. Then  $\widehat{B} \cap E_T$  has equation  $u_1 = 0$ . And  $Y \cap E_T$  is the curve consisting of binary 12-tuples

$$(s - \lambda)^{12} = s^{12} + 12\lambda s^{11} + \dots + \lambda^{12},$$

which passes through  $(T, \tau_0)$  when  $\lambda = 0$  and is transverse to  $\widehat{B}$  there because the  $s^{11}$  coefficient (i.e., the  $u_1$ -coordinate) is linear in  $\lambda$ .  $\square$

REMARK. It doesn't matter for us, but we note that  $\widehat{B}$  and  $Y$  are not transverse everywhere. Since  $Y \cap E_T$  is a rational normal curve of degree 12 and  $\widehat{B} \cap E_T$  is a hyperplane in  $E_T$ , they intersect in 12 points, counted with multiplicity. Besides  $(T, \tau_0)$ , the only place they intersect is at  $\lambda = \infty$ , so they make 11th-order contact there.

The analogues of Theorems 2.1 and 2.2 in this lower-dimensional setting are the following. It turns out (see the proof of Theorem 2.1) that restricting attention to  $\tau_0$ , rather than treating general  $\tau$ , is sufficient.

THEOREM 2.7. *There exists a neighborhood  $U'$  of  $(T, \tau_0)$  in  $\widehat{B}$  which is homeomorphic to  $B^1 \times (U' \cap E)$ , such that  $U' \cap E$  corresponds to  $\{0\} \times (U' \cap E)$  and  $U' \cap \widehat{\Delta}$  to  $B^1 \times (U' \cap E \cap \widehat{\Delta})$ .*

THEOREM 2.8. *There exists a neighborhood  $U'$  of  $(T, \tau_0)$  in  $\widehat{B}$  which is biholomorphic to  $B^1 \times B^{11}$ , with  $U' \cap E$  corresponding to  $\{0\} \times B^{11}$ ,*

such that the following holds. As in Theorem 2.2, write  $\beta : \tilde{U}' \rightarrow U'$  for the 6-fold cover of  $U'$  branched over  $U' \cap E$ , meaning that  $\tilde{U}'$  is got from  $U \cong B^1 \times B^{11}$  by the base extension  $t \mapsto t^6$  of the  $B^1$  factor. And write  $(T, \tau_0)^\sim$  for the preimage in  $\tilde{U}'$  of  $(T, \tau_0)$ . There is a neighborhood  $\tilde{V}'$  of  $(T, \tau_0)^\sim$  in  $\tilde{U}'$ , and a neighborhood  $W'$  of  $(T, \tau_0)$  in  $\widehat{B} \cap E$ , such that  $\tilde{V}'$  is biholomorphic to  $B^1 \times W'$ , such that

$$\tilde{V}' \cap \beta^{-1}(E) = \{0\} \times W'$$

and

$$\tilde{V}' \cap \beta^{-1}(\widehat{\Delta}) = B^1 \times (W' \cap \widehat{\Delta}) .$$

These theorems describe  $\widehat{\Delta}$  in a neighborhood of  $(T, \tau_0)$  in  $\widehat{B} \cap E$  in terms of its intersection with  $E$ . Therefore we need to understand  $\widehat{B} \cap E \cap \widehat{\Delta}$ :

LEMMA 2.9. *Suppose  $(T, \tau) \in \widehat{B} \cap E$ , and that none of the points of  $\tau$  is  $[0, 0, 0, 0, 1]$ . Let  $m, n_1, \dots, n_m, K_1, \dots, K_m$  and  $\Delta_1, \dots, \Delta_m$  be as in Theorem 2.1. Let  $N' = 11 - \sum \dim K_i$ . Then there is a neighborhood  $Z$  of  $(T, \tau)$  in  $\widehat{B} \cap E$  biholomorphic to  $K_1 \times \dots \times K_m \times B^{N'}$ , such that  $Z - \widehat{\Delta}$  corresponds to*

$$(K_1 - \Delta_1) \times \dots \times (K_m - \Delta_m) \times B^{N'} .$$

PROOF. Using coordinates  $u_2, \dots, u_{12}$  on  $E \cap \widehat{B}$  around  $(T, \tau_0)$  as in the proof of Lemma 2.3, and parameterizing  $R$  by  $s$  as in (2.4), the  $\tau$ 's treated in this lemma are parameterized by the functions

$$f(s) = s^{12} + u_2 s^{10} + \dots + u_{11} s + u_{12} ,$$

i.e., by the monic polynomials with root sum equal to zero. The lemma amounts to the assertion that any singular function in this family admits a simultaneous versal deformation of all its singularities, within the family. The family of functions

$$(2.7) \quad (s - s_0)^n + c_2(s - s_0)^{n-2} + \dots + c_{n-1}(s - s_0) + c_n$$

provides a versal deformation of  $(s - s_0)^n$ , with every member of the family having the same root sum, namely  $ns_0$ . Given  $f$  as above, we may take a product of terms like (2.7), one for each singularity of  $f$ . This obviously provides a simultaneous versal deformation, and the root sum of any member of the family is that of  $f$ , namely 0. Therefore the versal deformation lies in  $\widehat{B} \cap E$ .  $\square$

PROOF OF THEOREM 2.1, GIVEN THEOREM 2.7; First, we claim that  $(T, \tau_0)$  has a neighborhood  $Z$  in  $\widehat{B} \cap E$  and a neighborhood  $U$



in  $\widehat{PC}$ , such that  $U$  is homeomorphic to  $B^1 \times Z \times B^{22}$ , with  $U \cap E$  corresponding to  $\{0\} \times Z \times B^{22}$  and  $U \cap \widehat{\Delta}$  to  $B^1 \times (Z \cap \widehat{\Delta}) \times B^{22}$ . To get this, apply Theorem 2.7 to obtain  $U' \subseteq \widehat{B}$  with the properties stated there, and set  $Z$  equal to  $U' \cap E$ . By shrinking  $U'$  and  $B^{22} \subseteq PG$  if necessary, we may suppose by Lemma 2.6 that  $U' \times B^{22} \rightarrow \widehat{PC}$  is a biholomorphism onto a neighborhood of  $(T, \tau_0)$  in  $\widehat{PC}$ , which we take to be  $U$ . Then

$$U \cong U' \times B^{22} \cong B^1 \times (U' \cap E) \times B^{22} = B^1 \times Z \times B^{22}.$$

Here, the first ‘ $\cong$ ’ is a biholomorphism and the second is a homeomorphism. Also,  $U \cap E$  corresponds to  $(U' \cap E) \times B^{22} = \{0\} \times Z \times B^{22}$ , and  $U \cap \widehat{\Delta}$  to  $(U' \cap \widehat{\Delta}) \times B^{22} = B^1 \times (Z \cap \widehat{\Delta}) \times B^{22}$ .

Observe that by the nature of the claim, the same conclusions apply when  $\tau_0$  is replaced by any  $\tau \in Z$ . The point is that  $Z$  is a neighborhood of  $\tau$  as well as of  $\tau_0$ , so we can use the same  $Z$  and  $U$ .

Now we use Lemma 2.9, which describes  $Z \cap \widehat{\Delta}$  (possibly after shrinking  $Z$  to a smaller neighborhood of  $(T, \tau)$ , which shrinks  $U$ ). The result is that  $U$  is homeomorphic to

$$B^1 \times K_1 \times \cdots \times K_m \times B^{N'} \times B^{22},$$

such that

$$U \cap E = \{0\} \times K_1 \times \cdots \times K_m \times B^{N'} \times B^{22}$$

and

$$U \cap \widehat{\Delta} = B^1 \times (K_1 - \Delta_1) \times \cdots \times (K_m - \Delta_m) \times B^{N'} \times B^{22}.$$

Therefore Theorem 2.1 holds for  $(T, \tau)$ .

We have proven the theorem for every  $(T, \tau) \in Z$ , and we will complete to proof by observing that every  $(T, \tau') \in E_T$  is  $\text{PGL}_2\mathbb{C}$ -equivalent to some element of  $Z$ . This is because one can use a 1-parameter group to push the 12 points of  $\tau'$  together, so that  $\tau'$  approaches  $\tau_0$ .  $\square$

The proof of Theorem 2.2, given Theorem 2.8, is essentially the same. Therefore it remains only to prove Theorems 2.7 and 2.8. We will treat Theorem 2.8 first.

LEMMA 2.10. *In a neighborhood of  $(T, \tau_0)$ ,  $\widehat{B \cap \Delta} = \widehat{B} \cap \widehat{\Delta}$ .*

PROOF. By Lemma 2.6, we may choose a neighborhood  $U'$  of  $(T, \tau_0)$  in  $\widehat{B}$ , and shrink  $B^{22} \subseteq PG$  if necessary, so that  $U' \times B^{22} \rightarrow \widehat{PC}$  is a biholomorphism onto a neighborhood of  $(T, \tau_0)$  in  $\widehat{PC}$ . Under this

identification,  $\widehat{\Delta} - E$  corresponds to  $((\widehat{\Delta} - E) \cap U') \times B^{22}$ . To get  $\widehat{B} \cap \widehat{\Delta}$ , we take the closure and then intersect with  $U' \times \{\text{point}\}$ , and to get  $\widehat{B \cap \Delta}$ , we intersect with  $U' \times \{\text{point}\}$  and then take the closure. Clearly, both give the same result.  $\square$

The following technical lemma is impossible to motivate without seeing its use in the proof of Theorem 2.8; the reader should skip it and refer back when needed.

LEMMA 2.11. *Suppose  $\delta(u_2, \dots, u_{12})$  is a quasihomogeneous polynomial of weight 132, where  $\text{wt}(u_i) = i$ . Suppose also that the  $u_{12}^{11}$  and  $u_{12}^{11-i}u_{11}^i u_i$  ( $i = 11, \dots, 2$ ) terms of  $\delta$  have nonzero coefficients. Suppose  $g : (\mathbb{C}^{11}, 0) \rightarrow (\mathbb{C}^{11}, 0)$  is the germ of a diffeomorphism such that  $\delta \circ g$  has no terms of weight  $< 132$ . Then  $g$  preserves the weight filtration, in the sense that*

$$(2.8) \quad u_i \circ g = c_i u_i + p_i(u_2, \dots, u_{12}) + q_i(u_2, \dots, u_{12})$$

for each  $i$ , where  $c_i$  is a nonzero constant,  $p_i$  is quasihomogeneous of weight  $i$  with no linear terms, and  $q_i$  is an analytic function whose power series expansion has only terms of weight  $> i$ .

PROOF. We write  $v_i$  for  $u_i \circ g$ , regarded as a function of  $u_2, \dots, u_{12}$ . One obtains  $(\delta \circ g)(u_2, \dots, u_{12})$  by beginning with  $\delta(u_2, \dots, u_{12})$  and replacing each  $u_i$  by  $v_i(u_2, \dots, u_{12})$ . This leads to a big mess, with the coefficients of  $\delta \circ g$  depending on those of  $\delta$  and the  $v_i$  in a complicated way. Nevertheless, there are some coefficients of  $\delta \circ g$  to which only one term of  $g$  can contribute, and this will allow us to deduce that various coefficients of the  $v_i$  vanish. To be able to speak precisely, we make the following definitions. When we refer to a term or monomial of  $\delta$  (resp.  $v_i$ ), we mean a monomial whose coefficient in  $\delta$  (resp.  $v_i$ ) is nonzero. If  $m$  is a monomial  $u_{i_1} \cdots u_{i_n}$ , and  $\mu_j(u_2, \dots, u_{12})$  is a monomial of  $v_{i_j}$  for each  $j$ , then we say that  $m$  produces the monomial

$$\mu = \mu_1(u_2, \dots, u_{12}) \cdots \mu_n(u_2, \dots, u_{12}) .$$

For example, if  $v_{12} = u_{12} + u_2^2$  and  $v_{11} = u_{11} + u_2$ , then  $m = u_{12}^2 u_{11}$  produces the monomials  $u_{12}^2 u_{11}$ ,  $u_{12} u_2^2 u_{11}$ ,  $u_2^4 u_{11}$ ,  $u_{12}^2 u_2$ ,  $u_{12} u_2^3$  and  $u_2^5$ . When we wish to be more specific, we say that  $m$  produces  $\mu$  by substituting  $\mu_j$  for each  $u_{i_j}$ . Continuing the example, we would say that  $u_{12}^2 u_{11}$  produces  $u_{12} u_2^3$  by substituting  $u_{12}$  for one factor  $u_{12}$  of  $m$ ,  $u_2^2$  for the other, and  $u_2$  for the factor  $u_{11}$ . We note that even if a monomial  $m$  of  $\delta$  produces a monomial  $\mu$ , the coefficient of  $\mu$  in  $\delta \circ g$  (or even in  $m \circ g$ ) may still be zero, because of possible cancellation.

We will prove the lemma by proving the following assertions  $\mathcal{A}_d$  by increasing induction on  $d$ , and we will prove each  $\mathcal{A}_d$  by proving the assertions  $\mathcal{B}_{d,i}$  by decreasing induction on  $i$ . The  $\mathcal{B}_{1,i}$  require special treatment, particularly  $\mathcal{B}_{1,12}$  and  $\mathcal{B}_{1,11}$ .

Assertion  $\mathcal{A}_d$ : No  $v_i$  has any term of degree  $\leq d$  and weight  $< i$ .

Assertion  $\mathcal{B}_{d,i}$ :  $v_i$  has no term of degree  $d$  and weight  $< i$ .

Proof of  $\mathcal{B}_{1,12}$ : Consider the monomial  $\mu = u_j^{11}$  for some  $j < 12$ . The only monomial of  $\delta$  of degree 11 is  $m = u_{12}^{11}$ , so  $m$  is the only monomial of  $\delta$  that could produce  $\mu$ . If  $v_{12}$  had a term  $u_j$ , then  $\mu$  would be a term of  $\delta \circ g$ . This is impossible because  $\text{wt } \mu < 132$ .

Proof of  $\mathcal{B}_{1,11}$ : Consider the monomial  $\mu = u_j^{12}$  for  $j < 11$ . We study a monomial  $m$  of  $\delta$  and a monomial  $m'$  it produces, and investigate the conditions under which  $m' = \mu$ . First note that  $m = u_{12}^{11}$  does not produce  $\mu$ , because  $\deg m' = 12$  requires a linear substitution for 10 of the factors, and then  $\mathcal{B}_{1,12}$  implies that  $m'$  is divisible by  $u_{12}^{10}$ . Similarly, if  $m$  has degree 12 then the only substitutions yielding  $m'$  with degree 12 are linear substitutions. If  $u_{12} | m$  then  $\mathcal{B}_{1,12}$  implies  $u_{12} | m'$ , so  $m' \neq \mu$ . The only degree 12 monomial of weight 132 not divisible by  $u_{12}$  is  $m = u_{11}^{12}$ , so this  $m$  is the only monomial of  $\delta$  that can produce  $\mu$ . If  $v_{11}$  had a term  $u_j$ , then  $\delta \circ g$  would have a term  $\mu$ , which is impossible since  $\text{wt } \mu < 132$ .

Having proven  $\mathcal{B}_{1,12}$  and  $\mathcal{B}_{1,11}$ , we observe that  $v_{12}$  has a linear term because  $g$  is a diffeomorphism, and since  $v_{12}$  has no term  $u_{11}, \dots, u_2$ , it does have a term  $u_{12}$ . Similarly, using  $\mathcal{B}_{1,11}$  and the fact that  $v_{11}$  and  $v_{12}$  have linearly independent linear parts, we see that  $v_{11}$  has a  $u_{11}$  term. We will use these facts in the rest of the proof.

Proof that  $\mathcal{B}_{1,12}, \dots, \mathcal{B}_{1,i+1}$  imply  $\mathcal{B}_{1,i}$  ( $i = 10, \dots, 2$ ): Consider  $\mu = u_{12}^{11-i} u_{11}^i u_j$  for some  $j < i$ , and let  $m$  and  $m'$  be as before. If  $m = u_{12}^{11}$  then  $\deg m' = 12$  requires that 10 of the 11 substitutions be linear. Then  $\mathcal{B}_{1,12}$  implies  $u_{12}^{10} | m'$ , so  $m' \neq \mu$ . If  $\deg m = 12$  then  $\deg m' = 12$  requires that all substitutions be linear. If all the factors of  $m$  are among  $u_{12}, \dots, u_{i+1}$ , then  $\mathcal{B}_{1,12}, \dots, \mathcal{B}_{1,i+1}$  imply  $\text{wt } m' \geq \text{wt } m = 132$ , so  $m' \neq \mu$ . So suppose some factor is  $u_{k \leq i}$ . Now we use  $\mathcal{B}_{1,12}$  again, to conclude that  $m'$  is divisible by at least as many powers of  $u_{12}$  as  $m$  is. Therefore we can only have  $m' = \mu$  if number number (say  $p$ ) of  $u_{12}$ 's in  $m$  satisfies  $p \leq 11 - i$ . We have shown

$$m = u_{12}^{p \leq 11-i} \cdot u_{k \leq i} \cdot (11 - p \text{ other factors, none of which is } u_{12}).$$

The average weight of the  $11 - p$  other factors is

$$\frac{132 - 12p - k}{11 - p} \geq 11 ,$$

with equality only if  $p = 11$  and  $k = i$ . This completely determines  $m$ , forcing  $m = u_{12}^{11-i} u_{11}^i u_i$ . If  $v_i$  had a term  $u_{j < i}$ , then  $m \circ g$  and hence  $\delta \circ g$  would have a term  $u_{12}^{11-i} u_{11}^i u_j$ , of weight  $< 132$ , contrary to hypothesis. (This uses the fact that  $v_{12}$  does have a term  $u_{12}$  and  $v_{11}$  does have a term  $u_{11}$ .)

$\mathcal{A}_0$  and all the  $\mathcal{B}_{d,1}$  are vacuously true.

Proof that  $\mathcal{A}_{d-1}$  and  $\mathcal{B}_{d,12}, \dots, \mathcal{B}_{d,1}$  imply  $\mathcal{A}_d$  ( $d \geq 1$ ): trivial.

Proof that  $\mathcal{A}_d$  implies  $\mathcal{B}_{d+1,12}$  ( $d \geq 1$ ): We follow the strategy and notation of the previous steps. Take  $\mu = u_{12}^{10} t$  where  $t$  is a monomial of degree  $d + 1$  and weight  $< 12$ . If  $\deg m \geq 12$ , then the only way for  $\deg m'$  to equal  $\deg \mu = 11 + d$  is for the substitution to replace every factor of  $m$  with a monomial of degree  $\leq d$ . Then  $\mathcal{A}_d$  implies that  $\text{wt } m' \geq \text{wt } m = 132$ , so  $m' \neq \mu$ . So  $m = u_{12}^{11}$  is the only monomial of  $\delta$  that can produce  $\mu$ . If  $v_{12}$  had a term  $t$ , then  $\delta \circ g$  would have a term  $u_{12}^{10} t$ , of weight  $< 132$ , contrary to hypothesis.

Proof that  $\mathcal{A}_d$  and  $\mathcal{B}_{d+1,12}, \dots, \mathcal{B}_{d+1,i+1}$  imply  $\mathcal{B}_{d+1,i}$  ( $d \geq 1$  and  $i = 11, \dots, 2$ ): Take  $\mu = u_{12}^{11-i} u_{11}^i t$ , where  $t$  is a monomial of degree  $d + 1$  and weight  $< i$ . If  $m = u_{12}^{11}$ , then  $\deg m' = \deg \mu (= 12 + d)$  requires that all substitutions are of degree  $\leq d + 1$ , or one has degree  $d + 2$  and the rest are linear. In the first case,  $\mathcal{A}_d$  and  $\mathcal{B}_{d+1,12}$  imply  $\text{wt } m' \geq \text{wt } m = 132$ , so  $m' \neq \mu$ . In the second case,  $\mathcal{B}_{1,12}$  implies  $u_{12}^{10} | m'$ , so again  $m' \neq \mu$ .

If  $\deg m > 12$  then  $\deg m' = \deg \mu$  requires that every factor of  $m$  be substituted by a monomial of degree  $\leq d$ . Then  $\mathcal{A}_d$  implies  $\text{wt } m' \geq \text{wt } m = 132$ , so  $m' \neq \mu$ . The same argument works in the case  $\deg m = 12$  unless one the substitutions has degree  $d + 1$  and the rest are linear. If the factor for which the degree  $d + 1$  substitution is made is one of  $u_{12}, \dots, u_{i+1}$ , then  $\mathcal{A}_1$  and  $\mathcal{B}_{d+1,12}, \dots, \mathcal{B}_{d+1,i+1}$  imply  $\text{wt } m' \geq \text{wt } m = 132$ , so  $m' \neq \mu$ . So the exceptional substitution is in a factor  $u_{k \leq i}$  of  $m$ ; in particular,  $m$  has such a factor. Then one follows the proof of  $\mathcal{B}_{1,i}$  to show  $m = u_{12}^{11-i} u_{11}^i u_i$ . If  $v_i$  had a term  $t$ , then  $m \circ g$ , hence  $\delta \circ g$ , would have a term  $\mu$ , of weight  $< 132$ , contrary to hypothesis.

The induction proves  $\mathcal{A}_1, \dots, \mathcal{A}_5$  successively, and  $\mathcal{A}_5$  implies that the  $v_i$  have no terms other than those in (2.8). To see that  $c_i \neq 0$  for all  $i$ , apply the argument used to prove that  $v_{12}$  (resp.  $v_{11}$ ) has a term

$$\begin{array}{ccc}
\tilde{V}' \subseteq B^1 \times W \subseteq \mathbb{C} \times A & & \\
\downarrow \alpha: (\lambda, w) \mapsto (\lambda, \eta_{\lambda-1} \rho_{\lambda}(w)) & & \\
\tilde{U}' \subseteq \mathbb{C} \times A & & \\
\downarrow \beta: (\lambda, a) \mapsto (\lambda^6, a) & & \\
U' \subseteq \mathbb{C} \times A & \xlongequal{\quad} & \mathbb{C} \times A \\
\downarrow \gamma: (\lambda, a) \mapsto F + \lambda(a - F) & & \downarrow \\
\widehat{B} & \xrightarrow{\quad \pi \quad} & B \\
\downarrow & & \downarrow \\
\widehat{PC} & \xrightarrow{\quad \pi \quad} & PC
\end{array}$$

FIGURE 1. Maps defined in the proof of Theorem 2.8.

$u_{12}$  (resp.  $u_{11}$ ). Namely,  $\mathcal{B}_{1,12}, \dots, \mathcal{B}_{1,i}$  plus the linear independence of the linear parts of  $v_{12}, \dots, v_i$  imply that  $v_i$  has a term  $u_i$ .  $\square$

**PROOF OF THEOREM 2.8:** We will introduce a number of sets and maps and we hope figure 1 will help the reader keep them organized. We begin by defining  $\gamma : \mathbb{C} \times A \rightarrow B$  by  $\gamma(\lambda, a) = F + \lambda(a - F)$ . For fixed  $a$ , this pencil defines a direction at  $T$ , transverse to the chordal locus (Lemma 2.5), hence a point of the exceptional fiber  $E_T$  over  $T$ . Therefore we regard  $\gamma$  as defining a map  $\mathbb{C} \times A \rightarrow \widehat{B}$ , also denoted  $\gamma$ , with  $\gamma(0, a)$  being this point of  $E_T$ . In particular,  $\gamma(0, G_{0,\dots,0}) = (T, \tau_0)$ . We are being slightly sloppy, because we should check that  $F + \lambda(a - F)$  is in the chordal locus only when  $\lambda = 0$ . (Otherwise, we'd have to blow up  $\mathbb{C} \times A$  to get a map to  $\widehat{B}$ .) But  $\gamma$  is well-defined as a map to  $\widehat{B}$  on a neighborhood  $U'$  of  $(0, G_{0,\dots,0})$  in  $\mathbb{C} \times A$ , biholomorphic to  $B^1 \times B^{11}$ , with  $\{0\} \times B^{11}$  being the locus  $\lambda = 0$ . This uses the transversality established in Lemma 2.5. Strictly speaking, we shouldn't refer to  $\gamma$  as a map  $\mathbb{C} \times A \rightarrow \widehat{B}$  but rather  $U' \rightarrow \widehat{B}$ . But the reader will lose nothing by thinking of  $U'$  as being all of  $\mathbb{C} \times A$ , so we will not be this picky.

We realize the base extension by defining  $\beta : \mathbb{C} \times A \rightarrow \mathbb{C} \times A$  by  $\beta(\lambda, a) = (\lambda^6, a)$ , and we define  $\tilde{U}'$  as  $\beta^{-1}(U')$ . As for  $U'$ , the reader will lose nothing by thinking of  $\tilde{U}'$  as all of  $\mathbb{C} \times A$ .

The first idea (of five) is to use a 1-parameter group to work out a defining equation for  $(\gamma \circ \beta)^{-1}(\widehat{\Delta})$  in terms of a defining equation for  $\Delta \cap A$ . We write  $\delta'$  for the defining equation for  $\Delta \cap A$  with respect to

the coordinates  $u_2, \dots, u_{12}$ . Our basic tool is the 1-parameter subgroup

$$(2.9) \quad \sigma_\lambda : (x_0, \dots, x_4) \mapsto (\lambda^{-2}x_0, \lambda^{-1}x_1, x_2, \lambda x_3, \lambda^2 x_4)$$

of  $G$ . For  $\lambda \neq 0$ ,  $(\gamma \circ \beta)(\lambda, u_2, \dots, u_{12})$  lies in  $\Delta$  if and only if its image under  $\sigma_\lambda^{-1}$  does. This image is

$$\begin{aligned} & (\gamma \circ \beta(\lambda, u_2, \dots, u_{12})) \cdot \sigma_\lambda^{-1} \\ &= (F + \lambda^6 x_4^3 + \lambda^6 u_2 x_4 x_3^2 + \dots + \lambda^6 u_{12} x_0^3) \cdot \sigma_\lambda^{-1} \\ &= F + x_4^3 + \lambda^2 u_2 x_4 x_3^2 + \dots + \lambda^{12} u_{12} x_0^3, \end{aligned}$$

which is the point of  $A$  with coordinates  $(\lambda^2 u_2, \dots, \lambda^{12} u_{12})$ . Therefore a defining equation for  $(\gamma \circ \beta)^{-1}(\widehat{\Delta})$  in  $\mathbb{C}^* \times A$  is

$$(2.10) \quad \delta'(\lambda^2 u_2, \dots, \lambda^{12} u_{12}) = 0.$$

The second idea is to determine the lowest-weight terms of  $\delta'$ , with respect to the weights  $\text{wt}(u_i) = i$ , by using our knowledge of  $\widehat{\Delta} \cap E$ . The closure in  $\mathbb{C} \times A$  of the variety (2.10) meets  $\{0\} \times A$  in the variety defined by  $\delta'_{\text{lowest}}(u_2, \dots, u_{12})$ , where  $\delta'_{\text{lowest}}$  consists of the lowest-weight terms of  $\delta'$ . By Lemma 2.10, we have  $\widehat{B} \cap \widehat{\Delta} = \widehat{B} \cap \widehat{\Delta}$  near  $(T, \tau_0)$ . Therefore  $\delta'_{\text{lowest}}$  defines  $(\gamma \circ \beta)^{-1}(E \cap \widehat{\Delta}) \subseteq \{0\} \times A$ . By Lemma 2.3(i), we know that  $E_T \cap \widehat{\Delta}$  consists of those  $(T, \tau)$  where  $\tau$  has a multiple point. This forces  $\delta'_{\text{lowest}}$  to be a power of the standard  $A_{11}$  discriminant  $\delta$ , which is defined by the property that  $\delta(u_2, \dots, u_{12}) = 0$  if and only if

$$s^{12} + u_2 s^{10} + u_3 s^9 + \dots + u_{12}$$

has a multiple root. Since  $\delta$  is quasihomogeneous of weight 132, we have

$$(2.11) \quad \delta'(u_2, \dots, u_{12}) = \delta(u_2, \dots, u_{12})^p + (\text{terms of weight} > 132p)$$

for some  $p \geq 1$ . Although it is not essential, we will soon see that  $p = 1$ .

The third idea is to use singularity theory to describe  $\Delta \cap A$  in a neighborhood of  $G_{0, \dots, 0}$ , in terms of a different set of local coordinates. We saw in Lemma 2.4 that  $G_{0, \dots, 0}$  has an  $A_{11}$  singularity at  $P$  (and no others, as one may check), and that the family  $A$  provides a versal deformation for it. Therefore there is a neighborhood  $W$  of  $G_{0, \dots, 0}$  in  $A$  such that  $\Delta \cap W$  is a copy of the standard  $A_{11}$  discriminant. That is, there are analytic coordinates  $v_2, \dots, v_{12}$  on  $W$ , centered at  $G_{0, \dots, 0}$ , such that  $\Delta \cap W$  is defined by  $\delta(v_2, \dots, v_{12}) = 0$ . This tells us that  $p = 1$ , because  $\delta'(u_2, \dots, u_{12}) = 0$  and  $\delta(v_2, \dots, v_{12}) = 0$  are defining equations for the same variety, so in particular they vanish to the same order at the origin. We suppose without loss of generality that  $W$  is

the unit polydisk with respect to the  $v_i$  coordinate system. It is on  $B^1 \times W \subseteq \mathbb{C} \times A$  that we will soon define the remaining map  $\alpha$ .

The fourth idea is to use a sort of rigidity of the  $A_{11}$  discriminant. Briefly: since the variety defined by  $\delta(u_2, \dots, u_{12}) = 0$  is close to that defined by  $\delta(v_2, \dots, v_{12}) = 0$ , the  $u_i$  must be close to the  $v_i$ . This relationship between coordinate systems will be crucial later in the proof. To make this idea precise, consider the biholomorphism-germ  $g$  of  $W$  at  $G_{0, \dots, 0}$  given by  $u_i \circ g = v_i$ . Since the image of the locus  $\delta(v_2, \dots, v_{12}) = 0$  is the locus  $\delta(u_2, \dots, u_{12}) = 0$ , and since the first of these is also the locus  $\delta'(u_2, \dots, u_{12}) = 0$ , we have  $\delta \circ g = \delta'$ . Now, a computer calculation using Maple [26] shows that the 11 terms of  $\delta$  specified by the hypothesis of Lemma 2.11 are nonzero, and (2.11) implies that  $\delta'$  has no terms of weight  $< 132$ . The lemma then implies

$$(2.12) \quad v_i = c_i u_i + p_i(u_2, \dots, u_{12}) + q_i(u_2, \dots, u_{12}) ,$$

where the  $c_i$ ,  $p_i$  and  $q_i$  have the properties stated there. We will need the inverse of this, giving the  $u_i$  in terms of the  $v_i$ . To compute this it suffices to work in the formal power series ring; one writes (2.12) as

$$u_i = \frac{1}{c_i} v_i - \frac{1}{c_i} p_i(u_2, \dots, u_{12}) - \frac{1}{c_i} q_i(u_2, \dots, u_{12}) ,$$

substitutes these expressions into themselves, and repeats this process infinitely many times. The result is

$$(2.13) \quad u_i = c'_i v_i + p'_i(v_2, \dots, v_{12}) + q'_i(v_2, \dots, v_{12}) ,$$

where the  $c'_i$ ,  $p'_i$  and  $q'_i$  satisfy the same conditions as the  $c_i$ ,  $p_i$  and  $q_i$ , with respect to the weights  $\text{wt}(v_i) = i$ .

The fifth idea is to combine the quasihomogeneous scalings of the  $u_i$  and  $v_i$  to define a map  $\alpha$  whose image will be the set  $V'$  whose existence is claimed by the theorem. For every  $0 < |\lambda| < 1$  we define  $\rho_\lambda : W \rightarrow W$  to be the quasihomogeneous scaling with respect to the  $v$ -coordinates:

$$\rho_\lambda(v_2, \dots, v_{12}) = (\lambda^2 v_2, \dots, \lambda^{12} v_{12}) .$$

For every  $\lambda \in \mathbb{C}^*$ , we define  $\eta_\lambda : A \rightarrow A$  to be the quasihomogeneous scaling with respect to the  $u$ -coordinates:

$$\eta_\lambda(u_2, \dots, u_{12}) = (\lambda^2 u_2, \dots, \lambda^{12} u_{12}) .$$

The  $\eta_\lambda$  are related to the 1-parameter group (2.9) by

$$(2.14) \quad \gamma \circ \beta(\lambda, \eta_{\lambda^{-1}}(a)) = \sigma_\lambda(a) ,$$

which can be verified by expressing both sides in terms of the  $u$ -coordinates and expanding. We define

$$\alpha : (B^1 - \{0\}) \times W \rightarrow (B^1 - \{0\}) \times A$$

by  $\alpha(\lambda, w) = (\lambda, \eta_{\lambda^{-1}}\rho_\lambda(w))$ . It is easy to see that  $\alpha$  is injective.

The first property of  $\alpha$  is that the preimage of the discriminant under it has a very simple form, namely

$$(2.15) \quad (\gamma \circ \beta \circ \alpha)^{-1}(\widehat{\Delta}) = (B^1 - \{0\}) \times (\Delta \cap W).$$

To see this, observe that

$$\gamma \circ \beta \circ \alpha(\lambda, w) = \gamma \circ \beta(\lambda, \eta_{\lambda^{-1}}\rho_\lambda(w)) = \sigma_\lambda(\rho_\lambda(w)),$$

using (2.14). The right hand side lies in  $\Delta$  if and only if  $\rho_\lambda(w)$  does, hence if and only if  $w$  does.

The second property of  $\alpha$  is that it extends to a holomorphic map  $B^1 \times W \rightarrow \mathbb{C} \times A$ . To prove this, it suffices by Riemann extension to show that it has a continuous extension. The key step is to compute  $\lim_{\lambda \rightarrow 0} \alpha(\lambda, w)$ . If  $w \in W$  then its  $v$ -coordinates are  $v_2(w), \dots, v_{12}(w)$ , and the  $v$ -coordinates of  $\rho_\lambda(w)$  are  $\lambda^2 v_2(w), \dots, \lambda^{12} v_{12}(w)$ . Using (2.13), the  $u$ -coordinates of  $\rho_\lambda(w)$  are

$$\begin{aligned} u_i(\rho_\lambda(w)) &= c'_i \lambda^i v_i(w) + p'_i(\lambda^2 v_2(w), \dots, \lambda^{12} v_{12}(w)) \\ &\quad + q'_i(\lambda^2 v_2(w), \dots, \lambda^{12} v_{12}(w)) \\ &= \lambda^i \cdot \left( c'_i v_i(w) + p'_i(v_2(w), \dots, v_{12}(w)) \right) \\ &\quad + (\text{terms of degree } > i \text{ in } \lambda). \end{aligned}$$

Therefore the  $u$ -coordinates of  $\eta_{\lambda^{-1}}\rho_\lambda(w)$  are

$$\begin{aligned} u_i(\eta_{\lambda^{-1}}\rho_\lambda(w)) &= \lambda^{-i} u_i(\rho_\lambda(w)) = c'_i v_i(w) + p'_i(v_2(w), \dots, v_{12}(w)) \\ &\quad + \text{terms involving } \lambda. \end{aligned}$$

The limit as  $\lambda \rightarrow 0$  obviously exists, and provides the desired extension.

The third property of  $\alpha$  is that after shrinking  $W$ , we may suppose that  $\alpha : B^1 \times W \rightarrow \mathbb{C} \times A$  is a biholomorphism onto its image. To see this we use the fact that

$$u_i(\alpha(0, w)) = c'_i v_i(w) + p'_i(v_2(w), \dots, v_{12}(w));$$

since the  $c'_i$  are nonzero and the  $p'_i$  have no linear terms, we may shrink  $W$  so that  $w \mapsto \alpha(0, w)$  is injective. Because  $\alpha : B^1 \times W \rightarrow \tilde{U}'$  is injective and  $\tilde{U}'$  is normal,  $\alpha$  is a biholomorphism from a neighborhood  $\tilde{V}'$  of  $(0, G_{0, \dots, 0}) \in B^1 \times W$  to a neighborhood of  $(0, G_{0, \dots, 0}) \in \tilde{U}'$ .



Now we are essentially done. Early on, we defined  $U' \cong B^1 \times B^{11}$  with the modest properties stated in the lemma, and then its 6-fold branched cover  $\tilde{U}'$ . We have just defined  $\alpha$  as a biholomorphism from  $B^1 \times W$  to its image in  $\tilde{U}'$ . The theorem promises a neighborhood  $\tilde{V}'$  of  $(T, \tau_0)^\sim$  in  $\tilde{U}'$  in which the discriminant and exceptional divisor have simple descriptions. In particular, we promised to have  $\tilde{V}' \cong B^1 \times W'$  for a neighborhood  $W'$  of  $(T, \tau_0)^\sim$  in  $\widehat{B} \cap E$ . We define  $W'$  as the (biholomorphic) image of  $W \subseteq A$  under  $w \mapsto \alpha(0, w)$ , think of  $W$  and  $W'$  as the same set, and regard  $\alpha$  as a map  $B^1 \times W' \rightarrow \tilde{U}'$ . We define  $\tilde{V}'$  as its image. All that remains is to check that  $\tilde{V}' \cap \beta^{-1}(E) = \{0\} \times W'$  and  $\tilde{V}' \cap \beta^{-1}(\widehat{\Delta}) = B^1 \times (W' \cap \widehat{\Delta})$ . The first of these is just unwinding definitions, and the second is also, together with (2.15).  $\square$

PROOF OF THEOREM 2.7. Theorem 2.8 gives us an analytic description of the 6-fold branched cover  $\tilde{V}'$  of a neighborhood of  $(T, \tau_0)$  in  $\widehat{B}$ . The idea is to take the quotient by the deck group  $\mathbb{Z}/6$  and see what we get. Recall that  $\tilde{U}' \subseteq \mathbb{C} \times A$ , where  $A \cong \mathbb{C}^{11}$ , and the deck group is generated by  $(\lambda, a) \mapsto (\lambda\zeta, a)$ , where  $\zeta = e^{\pi i/3}$ . We will write  $\xi$  for this map.  $(T, \tau_0)^\sim$  is the point  $(0, G_{0,\dots,0}) \in \tilde{U}'$ . Also,  $\alpha : B^1 \times W \rightarrow \tilde{U}'$  is an embedding onto a neighborhood  $\tilde{V}'$  of  $(T, \tau_0)^\sim$ , where  $W$  is a polydisk around  $G_{0,\dots,0}$  in  $A$ . We can't regard  $\xi$  as a self-map of  $B^1 \times W$ , because  $\tilde{V}'$  may not be a  $\xi$ -invariant subset of  $\tilde{U}'$ . However, we can take the intersection of the finitely many translates of  $\tilde{V}'$ , and let  $Z \subseteq B^1 \times W$  be the  $\alpha$ -preimage of this intersection. The action of  $\xi$  on  $Z$  can be worked out by using the definition of  $\alpha$ . The result is  $\xi(0, w) = (0, w)$ , and

$$\xi(\lambda, w) = (\lambda\zeta, \rho_\lambda^{-1} \circ \rho_\zeta^{-1} \circ \eta_\zeta \circ \rho_\lambda(w))$$

for  $\lambda \in B^1 - \{0\}$ . Now,  $\beta \circ \alpha$  carries  $Z/\langle \xi \rangle$  homeomorphically to a neighborhood of  $(T, \tau_0)$  in  $\widehat{B}$ .  $(T, \tau_0)$  corresponds to the image of  $(0, G_{0,\dots,0})$ , and  $\widehat{\Delta} \cap \widehat{B}$  to the image of  $Z \cap (B^1 \times (W \cap \widehat{\Delta}))$ . Therefore our goal is to describe  $Z/\langle \xi \rangle$  and the image therein of  $Z \cap (B^1 \times (W \cap \widehat{\Delta}))$ .

To take the quotient  $Z/\langle \xi \rangle$ , we first observe that the 'slice of pie'

$$\Sigma = \{ \lambda \in B^1 \mid \lambda = 0 \text{ or } \text{Arg } \lambda \in [0, \pi/3] \}$$

is a fundamental domain for  $\lambda \mapsto \lambda\zeta$  acting on  $B^1$ , and the quotient  $B^1/(\mathbb{Z}/6)$  is got by gluing one edge

$$E_0 = \{ \lambda \in B^1 \mid \lambda = 0 \text{ or } \text{Arg } \lambda = 0 \}$$

to the other

$$E_{\pi/3} = \{ \lambda \in B^1 \mid \lambda = 0 \text{ or } \text{Arg } \lambda = \pi/3 \}$$

in the obvious way. Similarly,  $Z/\langle\xi\rangle$  is homeomorphic to  $(Z \cap (\Sigma \times W))/\sim$ , where  $\sim$  is the equivalence relation that each  $(s, w) \in Z \cap (E_0 \times W)$  is identified with  $\xi(s, w) \in Z \cap (E_{\pi/3} \times W)$ . On the other hand, consider  $(\Sigma \times W)/\approx$ , where  $\approx$  is the equivalence relation that each  $(s, w) \in E_0 \times W$  is identified with  $(s\zeta, w)$ . The key step in the proof is the definition of the following map

$$\Phi : (Z_0 \cap (\Sigma \times W))/\sim \rightarrow (\Sigma \times W)/\approx ,$$

where  $Z_0$  is a suitable neighborhood of  $(0, G_{0,\dots,0})$  in  $\Sigma \times W$ , defined below. The map is  $\Phi(\lambda, w) = (\lambda, w)$  for  $\lambda \in E_0$ , and

$$\Phi(\lambda, w) = \left( \lambda, \rho_{(\text{Arg } \lambda)/(\pi/3)}^{-1} \circ (\rho_\lambda^{-1} \rho_\zeta^{-1} \eta_\zeta \rho_\lambda) \circ \rho_{(\text{Arg } \lambda)/(\pi/3)}(w) \right)$$

otherwise. Claim: there is a neighborhood  $Z_0$  of  $(0, G_{0,\dots,0})$  in  $\Sigma \times W$  on which  $\Phi$  is defined, i.e., such that if  $(\lambda, w) \in Z_0$  then either  $\lambda \in E_0$  or else

$$(2.16) \quad \eta_\zeta \rho_\lambda \rho_{(\text{Arg } \lambda)/(\pi/3)}(w) \in \rho_\zeta \rho_\lambda \rho_{(\text{Arg } \lambda)/(\pi/3)}(W) .$$

Assuming the claim for a moment, it is easy to complete the proof. The key point is that the restriction of  $\Phi$  to  $Z_0 \cap (E_0 \times W)$  is the obvious inclusion into  $E_0 \times W$ , while the  $\Phi$ -image of  $(\lambda, w) \in (E_{\pi/3} \times W)$  is  $\xi(\lambda\zeta^{-1}, w)$ . Furthermore, for any  $(\lambda, w) \in Z_0 \cap (\Sigma \times W)$ ,  $\Phi(\lambda, w)$  lies in  $\Sigma \times (W \cap \widehat{\Delta})$  if and only if  $(\lambda, w)$  itself does. (This uses the fact that the  $\rho_\lambda$  are quasihomogeneous scalings of  $W$  that preserve  $W \cap \widehat{\Delta}$ .) It follows that  $Z/\langle\xi\rangle$  is homeomorphic to a neighborhood of  $(0, G_{0,\dots,0})$  in  $(\Sigma \times W)/\approx$ , i.e., in  $B^1 \times W$ , with its intersection with  $\widehat{\Delta}$  being  $(\Sigma \times (\widehat{\Delta} \times W))/\approx$ , i.e.,  $B^1 \times (\widehat{\Delta} \times W)$ . This implies the theorem.

All that remains is to construct  $Z_0$ . First observe that  $Z$  contains  $B^1 \times \{G_{0,\dots,0}\} \subseteq B^1 \times W$ . Therefore  $Z$  contains  $B_r^1 \times W_0$  for some  $0 < r \leq 1$  and some neighborhood  $W_0$  of  $G_{0,\dots,0}$  in  $W$ . We claim that  $\Phi$  is defined for all  $(\lambda, w) \in \Sigma_r \times W_0$ , so that we may take  $Z_0 = \Sigma_r \times W_0$ . There is nothing to verify if  $\lambda \in E_0$ , so suppose  $0 < \text{Arg } \lambda \leq \pi/3$ . Then we define  $\lambda' = \lambda \cdot (\text{Arg } \lambda)/(\pi/3)$  and use the fact that  $|\lambda'| \leq |\lambda|$ , so  $(\lambda', w) \in Z$ . Therefore  $\xi$  is defined at  $(\lambda', w)$ , which is to say that have  $\eta_\zeta \rho_{\lambda'}(w) \in \rho_\zeta \rho_{\lambda'}(W)$ . Plugging in the definition of  $\lambda'$  yields (2.16).  $\square$

## CHAPTER 3

### Extension of the Period Map

In this chapter we extend the period map  $g : PC_0 \rightarrow P\Gamma \backslash \mathbb{C}H^{10}$  defined in chapter 1 to a larger domain. It turns out that  $g$  does extend to all of  $PC_s$  but not to  $PC_{ss}$ . Replacing  $P\Gamma \backslash \mathbb{C}H^{10}$  by its Baily-Borel compactification  $\overline{P\Gamma \backslash \mathbb{C}H^{10}}$  allows us to extend  $g$  to most of  $PC_{ss}$  but not all. The problem is that it does not extend to the chordal cubic locus. We will see a proof of this in chapter 4, but for now we just refer to this to motivate the blowing-up of the chordal cubic locus in  $PC$  to obtain  $\widehat{PC}$ , and extending  $g$  to a regular map  $(\widehat{PC})_{ss} \rightarrow \overline{P\Gamma \backslash \mathbb{C}H^{10}}$ . In this chapter we will construct the extension; we rely heavily on Lemma 1.5 and Theorems 2.1 and 2.2, which describe the local structure of the discriminant. Recall that we write  $\pi$  for the projection  $\widehat{PC} \rightarrow PC$ ,  $E$  for the exceptional divisor,  $E_T \cong P^{12}$  for the points of  $E$  lying over a chordal cubic  $T$ , and  $(T, \tau)$  for a point in  $E_T$ , where  $\tau$  is an unordered 12-tuple in the rational normal curve  $R_T$  of which  $T$  is the secant variety.

The first thing we need to do is describe  $(\widehat{PC})_s$  and  $(\widehat{PC})_{ss}$ , using [1] and [33]. To lighten the notation we will write just  $\widehat{PC}_s$  and  $\widehat{PC}_{ss}$ . In order to discuss GIT-stability we need to choose a line bundle on  $\widehat{PC}$ . The following lemma shows that this choice doesn't matter very much; it follows directly from the considerations of [33, sec. 2].

LEMMA 3.1. *For large enough  $d$ , the stable and semistable loci of  $\widehat{PC}$ , with respect to the standard  $SL(5, \mathbb{C})$ -action on*

$$(3.1) \quad \mathcal{O}(-E) \otimes \pi^*(\mathcal{O}(d)) ,$$

*are independent of  $d$ .* □

Our notation  $\widehat{PC}_s$  and  $\widehat{PC}_{ss}$  refers to the linearization (3.1) for large enough  $d$ . Reichstein's work allows us to describe these sets explicitly:

THEOREM 3.2. *Suppose  $T$  is a cubic threefold not in the closure of the chordal cubic locus, regarded as an element of  $\widehat{PC}$ . Then*

- (i)  $T$  is stable if and only if each singularity of  $T$  has type  $A_1$ ,  $A_2$ ,  $A_3$  or  $A_4$ ;

- (ii)  $T$  is semistable if and only if each singularity of  $T$  has type  $A_1, \dots, A_5$  or  $D_4$ ;
- (iii)  $T$  is strictly semistable with closed orbit in  $\widehat{PC}_{ss}$  if and only if  $T$  is projectively equivalent to one of the threefolds defined by

$$x_0x_1x_2 + x_3^3 + x_4^3$$

or

$$F_{A,B} = Ax_2^3 + x_0x_3^2 + x_1^2x_4 - x_0x_2x_4 + Bx_1x_2x_3,$$

with  $A, B \in \mathbb{C}$  and  $4A \neq B^2$ .

Now suppose instead that  $T$  is in the closure of the chordal cubic locus but is not a chordal cubic. Then every element of  $\pi^{-1}(T) \subseteq \widehat{PC}$  is unstable. Finally, suppose that  $T$  is a chordal cubic, and  $\tau$  is an unordered 12-tuple in the rational normal curve  $R_T$ , so that  $(T, \tau) \in E_T \subseteq \widehat{PC}$ . Then

- (iv)  $(T, \tau)$  is stable if and only if  $\tau$  has no points of multiplicity  $\geq 6$ ;
- (v)  $(T, \tau)$  is semistable if and only if  $\tau$  has no points of multiplicity greater than 6;
- (vi)  $(T, \tau)$  is strictly semistable with closed orbit in  $\widehat{PC}_{ss}$  if and only if  $\tau$  consists of two distinct points of multiplicity 6.

Finally, the points (vi) of  $\widehat{PC}$  lie in the closure of the union of the orbits of the  $T_{A,B}$  from (iii).

REMARKS. The first threefold described in (iii) has three  $D_4$  singularities, and is the unique such cubic threefold. The 2-parameter family  $T_{A,B}$  really describes only a 1-parameter set of orbits, because the projective equivalence class is determined by the ratio  $4A/B^2 \in \mathbb{C}P^1 - \{1\}$ . These threefolds have exactly two singularities, both of type  $A_5$ , except when  $A = 0$ , when there is also an  $A_1$  singularity. Every cubic threefold with two  $A_5$  singularities is projectively equivalent to one of these. If  $4A$  and  $B^2$  were allowed to be equal and nonzero, then  $F_{A,B}$  would define a chordal cubic. All of these assertions are proven in section 5 of [1].

PROOF. Throughout the proof, we will write  $L \subseteq PC$  for the closure of the chordal cubic locus. By Theorems 2.1 and 2.3 of [33], a point of  $\widehat{PC} - E$  is unstable as an element of  $\widehat{PC}$  if and only if either (a) it is unstable as an element of  $PC$ , or (b) it is GIT-equivalent in  $PC$  to an element of  $L$ . Referring to the stability of cubic threefolds, given by Theorems 1.3 and 1.4 of [1], this says that  $\widehat{PC}_{ss} - E$  is the set of  $T$ 's having no singularities of types other than  $A_1, \dots, A_5$  and  $D_4$ . This

justifies (ii). The same theorems of [33] say that a point of  $\widehat{PC} - E$  is stable as an element of  $\widehat{PC}$  if and only if it is stable as an element of  $PC$ . Referring again to [1], this says that  $\widehat{PC}_s - E$  is the set of  $T$  having no singularities of types other than  $A_1, \dots, A_4$ , justifying (i). Now we prove (iii). If  $T \in \widehat{PC}_{ss} - E$  is not in  $\widehat{PC}_s$ , then it has a singularity of type  $A_5$  or  $D_4$ . Then Theorem 1.3(i,ii) of [1] implies that  $T$  is GIT-equivalent in  $PC$ , hence in  $\widehat{PC}$ , to one of the threefolds given in (iii). Theorem 1.2 of [1] implies that the threefolds given explicitly in (iii) have closed orbits in  $PC_{ss}$ ; since the orbits miss  $L$ , they are also closed in  $\widehat{PC}_{ss}$ . It follows that these orbits are the only orbits in  $\widehat{PC}_{ss} - E$  that are strictly semistable and closed in  $\widehat{PC}_{ss}$ . This justifies (iii).

Now suppose  $T \in L$ . If  $T$  is not a chordal cubic then it is unstable by Theorem 1.4(i) of [1], so every point of  $\widehat{PC}$  lying over  $T$  is unstable by Theorem 2.1 of [33]. It remains only to discuss stability of pairs  $(T, \tau)$  with  $T$  a chordal cubic. Our key tool is Theorem 2.4 of [33]. This says that  $(T, \tau)$  is unstable if and only if it lies in the proper transform of the set of cubic threefolds that are GIT-equivalent to chordal cubics. So our job is to determine this proper transform. If  $\tau$  has a point of multiplicity  $> 6$ , then by Lemma 2.3(ii) it is a limit of threefolds having  $A_{n>5}$  singularities. Since  $T$  lies in  $PC_{ss}$  and  $PC_{ss}$  is open in  $PC$ ,  $(T, \tau)$  is a limit of semistable threefolds having  $A_{n>5}$  singularities. By Theorem 1.3 of [1], such threefolds are GIT-equivalent to chordal cubics. Then Reichstein's Theorem 2.4 shows that  $(T, \tau)$  is unstable. Reichstein's theorem also asserts that  $(T, \tau) \in \widehat{PC}_{ss}$  is non-stable if and only if it lies in the proper transform of  $PC_{ss} - PC_s$ . If  $\tau$  has a point of multiplicity 6, then Lemma 2.3(ii) shows that  $(T, \tau)$  is a limit of semistable threefolds having  $A_5$  singularities, so it is not stable. This justifies the 'if' parts of (iv) and (v).

Now, suppose  $\tau$  has no point of multiplicity  $> 6$ . Since  $PC_{ss}$  is open,  $T$  has a neighborhood  $U \subseteq PC$  with every member of  $U - L$  having only  $A_n$  and  $D_4$  singularities. (In fact,  $D_4$  singularities can be excluded, but this doesn't matter here.) By Lemma 1.5, every member of  $U - L$  admits in  $PC$  a simultaneous versal deformation of all its singularities. If some member of  $U - L$  had an  $A_{n \geq 6}$  singularity, then at some point of  $U - L$ ,  $\Delta$  would be locally modeled on the  $A_n$  discriminant (times a ball of the appropriate dimension). On the other hand, it follows from Theorem 2.2 that after shrinking  $U$  we may suppose that at every

point of  $U - L$ ,  $\Delta$  is locally modeled on

$$(3.2) \quad \bigcup_{i=1}^m K_1 \times \cdots \times K_{i-1} \times \Delta_i \times K_{i+1} \times \cdots \times K_m \times B^N,$$

where the notation is as in Lemma 1.5. In particular, the  $\Delta_i \subseteq K_i$  are copies of the  $A_k$  discriminants for various  $k$ 's that are at most 5. Since (3.2) is not a copy of an  $A_{n \geq 6}$  discriminant,  $U$  contains no points with an  $A_{n \geq 6}$  singularity, hence no points GIT-equivalent to chordal cubics. By Theorem 2.4 of [1],  $(T, \tau)$  is not unstable, which is to say that it is semistable. This proves the ‘only if’ part of (v). The same argument, using the fact that members of  $\widehat{PC}_{ss} - E$  have  $A_5$  or  $D_4$  singularities, proves the ‘only if’ part of (iv).

Finally, if  $\tau$  has a point of multiplicity 6, then  $(T, \tau)$ 's orbit closure in  $E_T \cap \widehat{PC}_{ss}$  contains  $(T, \tau')$ , where  $\tau'$  has two points of multiplicity 6. This is a classical fact about point-sets in  $P^1$ , and follows from Richardson's relative Hilbert-Mumford criterion [6, Thm. 4.2]. (The point is that once 6 points collide, one can use a 1-parameter group to drive the other 6 points together.) This proves (vi). To prove the last claim of the theorem, just observe that the restrictions of the  $F_{A,B}$  in (iii) to the singular locus of the standard chordal cubic (defined by  $F_{1,-2}$ ) consists of  $[1, 0, 0, 0, 0]$  and  $[0, 0, 0, 0, 1]$ , each with multiplicity 6. Let  $A \rightarrow 1$  and  $B \rightarrow -2$ .  $\square$

Now that we know how much to enlarge the domain of  $g$ , we will construct the extension. This relies on an analysis of the local monodromy group at a point of  $\widehat{PC}$ , by which we mean the following. In chapter 1 we considered the local system  $\Lambda(\mathcal{V}_0)$  over  $\mathcal{C}_0$  and its associated local system  $\mathcal{CH}(\mathcal{V}_0)$  of complex hyperbolic spaces. Now,  $\Lambda(\mathcal{V}_0)$  does not descend to a local system on  $PC_0$ , but  $\mathcal{CH}(\mathcal{V}_0)$  does, because the scalars  $\{I, \omega I, \bar{\omega} I\} \subseteq \text{GL}(5, \mathbb{C})$  act on each  $\Lambda(V)$  by scalar multiplication. After fixing a basepoint  $F \in \mathcal{C}_0$ , we defined

$$\rho : \pi_1(\mathcal{C}_0, F) \rightarrow \Gamma(V) := \text{Aut } \Lambda(V)$$

to be the monodromy of  $\Lambda(\mathcal{V}_0)$ . Analogously, we define, for  $T \in PC_0$ ,

$$(3.3) \quad P\rho : \pi_1(PC_0, T) \rightarrow P\Gamma(V) \subseteq \text{Isom}(\mathcal{CH}(V)).$$

Henceforth, all references to monodromy refer to  $P\rho$  unless otherwise stated. In the arguments below, we will compute the monodromy of various elements of  $\pi_1(PC_0)$ . For convenience we will perform various monodromy calculations with roots of  $\Lambda(V)$ , but these could all be rephrased in terms of elements of  $P\Gamma(V)$ .

Now suppose  $T_0$  or  $(T_0, \tau_0)$  is an element of  $\widehat{PC}$  and  $U$  is a suitable small neighborhood of it; for example,  $U$  could be as in Lemma 1.5 or Theorem 2.1. By the local fundamental group we mean  $\pi_1(U - (\widehat{\Delta} \cup E), T)$ , where  $T$  is a basepoint. By the local monodromy action we mean the restriction of  $P\rho$  to the local fundamental group, and by the local monodromy group we mean the image of this homomorphism in  $P\Gamma(V)$ . We will see that  $\widehat{PC}_s$  is exactly the subset of  $\widehat{PC}_{ss}$  where the local monodromy group is finite. In order to establish this, we will need to know the monodromy around a meridian of  $E$ :

LEMMA 3.3. *Suppose  $\gamma$  is a meridian around  $E$  in  $\widehat{PC}$ ,  $T$  is a point of  $\gamma$ , and  $P\rho(\gamma)$  is the monodromy action of  $\gamma$  on  $\mathbb{C}H(V)$ . Then there is a direct sum decomposition*

$$\Lambda(V) = \Lambda_1 \oplus \Lambda_{10} ,$$

where  $\Lambda_1$  is the span of a norm 3 vector  $s$ ,  $P\rho(\gamma)$  acts on  $\mathbb{C}H(V)$  as a hexaflection in  $s$ , and  $\Lambda_{10}$  is isometric to the sum of the last three summands in (1.8).

This lemma resembles Lemma 1.4; each shows that a certain monodromy action is a complex reflection in a norm 3 vector of  $\Lambda(V)$ . But there is an essential difference. We have already defined a root of  $\Lambda(V)$  to be any norm 3 vector  $r$ ; we refine the language by calling  $r$  a nodal or chordal root according to whether  $\langle r | \Lambda(V) \rangle = \theta\mathcal{E}$  or  $3\mathcal{E}$ . It is easy to see that every root is either nodal or chordal. Lemma 3.3 asserts that the monodromy of a meridian around  $E$  is a hexaflection in a chordal root. Lemma 1.4 asserts that the monodromy of a meridian around  $\widehat{\Delta}$  is a triflection in a root, and a simple argument shows that this root must be nodal. (Namely, by considering a threefold with an  $A_2$  singularity, one finds two meridians of  $\widehat{\Delta}$ , which by Lemma 1.6 act by the  $\omega$ -reflections in linearly independent roots  $r$  and  $r'$ , and satisfy the braid relation. This relation forces  $|\langle r | r' \rangle| = \sqrt{3}$ , so  $\langle r | r' \rangle$  is a unit times  $\theta$ .) The ‘nodal’ and ‘chordal’ language reflects the fact that these monodromy transformations arise by considering a degeneration to a nodal threefold or to a chordal cubic. We caution the reader that while it is true that every nodal (resp. chordal) root of  $\Lambda(V)$  comes from a nodal (resp. chordal) degeneration, we do not need this and have not yet proven it. In any case, in Lemma 6.2 we show that  $\Gamma$  is transitive on nodal and chordal roots of  $\Lambda$ . (The proof of Lemma 6.2 is independent of the rest of the monograph, so it could be read at this point.)

PROOF OF LEMMA 3.3: Let  $T_0$  be the standard chordal cubic, and  $\tau_0$  a 12-tuple in  $R_{T_0}$  concentrated at one point. By Theorem 2.1, the local fundamental group at  $(T_0, \tau_0)$  is  $\mathbb{Z} \times B_{12}$ , where  $\gamma$  is a generator of  $\mathbb{Z}$  and we write  $a_1, \dots, a_{11}$  for standard generators for the braid group. By Lemma 1.4, the  $a_i$  act on  $\mathbb{C}H(V)$  as triflections, and the 1-dimensional eigenspaces of (lifts of the  $a_i$  to)  $\Lambda(V)$  are spanned by vectors  $r_i$  of norm 3. We take  $\Lambda_{10}$  to be the span of the  $r_i$ . Following the proof of Theorem 1.7 shows that  $\Lambda_{10}$  is a copy of the direct sum of the last three summands of (1.8), that  $\Lambda_{10}$  is a summand of  $\Lambda(V)$ , and that  $\Lambda_{10}^\perp$  is spanned by a vector of norm 3. We write  $s$  for such a vector and  $\Lambda_1$  for its span. Since  $\gamma$  commutes with the  $a_i$ , any lift of  $P\rho(\gamma)$  to  $\Lambda(V)$  multiplies each  $r_i$  by a non-zero scalar. Since  $\langle r_i | r_{i+1} \rangle \neq 0$ , it multiplies all the  $r_i$  by the same scalar, so that it acts on  $\Lambda_{10}$  as that scalar. Since it acts by a scalar on the 1-dimensional lattice  $\Lambda_{10}^\perp$ , the order of  $P\rho$  is the order of the quotient of these scalars as elements of the unit group of  $\mathcal{E}$ . Therefore  $P\rho$  acts on  $\mathbb{C}H(V)$  as a complex reflection of order 2, 3 or 6, or acts trivially.

Now we show that  $P\rho(\gamma)$  has order 6. We may find a neighborhood of  $E - \widehat{\Delta}$  in  $\widehat{PC}$  which is a disk bundle over  $E - \widehat{\Delta}$ . (We do not need all the fibers to “have the same radius”—they can shrink as one approaches  $\widehat{\Delta}$ .) We choose a ball  $B$  around  $T_0$  in the chordal cubic locus, and write  $N$  for the restriction of this disk bundle to  $\pi^{-1}(B) \subseteq E$ .  $N$  is a neighborhood of  $E_{T_0} - \widehat{\Delta}$  in  $\widehat{PC} - \widehat{\Delta}$ , and we may suppose without loss of generality that  $\gamma$  and  $a_1, \dots, a_{11}$  lie in  $N - E$ . Now,  $N - E$  is a punctured-disk bundle over  $\pi^{-1}(B)$ , which in turn is a punctured  $P^{12}$ -bundle over  $B$ . Since  $B$  is a ball,  $\pi^{-1}(B) \rightarrow B$  trivializes topologically, so up to homotopy,  $N - E$  is a circle-bundle over  $E_{T_0} - \widehat{\Delta}$ . Now,  $\pi_1(E_{T_0} - \widehat{\Delta})$  is the 12-strand *spherical* braid group  $B_{12}(S^2)$ , so  $\pi_1(N - E)$  is a central extension of  $B_{12}(S^2)$  by  $\mathbb{Z} = \langle \gamma \rangle$ . Furthermore, the local description of the discriminant shows that the generators  $a_1, \dots, a_{11}$  map to the corresponding standard generators for  $B_{12}(S^2)$ . Now,  $w = a_1 \cdots a_{10} a_{11}^2 a_{10} \cdots a_1 \in B_{12}$  represents the braid in which the leftmost strand moves in a large circle around all the other strands. Since this is trivial in  $B_{12}(S^2)$ ,  $w$  is homotopic in  $N - E$  to a member of the central  $\mathbb{Z}$ , i.e., to a power of  $\gamma$ . One can write out the  $r_i \in \Lambda(V)$  explicitly, as in [2, sec. 5], and then matrix multiplication shows that  $w$  acts on  $\mathbb{C}H(V)$  with order 6. Since  $P\rho(\gamma)$  has order dividing 6, and some power of it has order 6,  $P\rho(\gamma)$  itself has order 6.  $\square$

Now we will extend the domain of  $g$ , in two steps. We will begin with the map  $g : P\mathcal{F}_0 \rightarrow \mathbb{C}H^{10}$  obtained from (1.10), where  $P\mathcal{F}_0$



is the quotient of  $\mathcal{F}_0$  by the action of  $\mathbb{C}^* \subseteq \mathrm{GL}(5, \mathbb{C})$  given in (1.9). We enlarge  $P\mathcal{F}_0$  to a space  $P\mathcal{F}_s$ , which is the branched cover of  $\widehat{PC}_s$  associated to the covering space  $P\mathcal{F}_0 \rightarrow PC_0$ . Formally, we define  $p : P\mathcal{F}_s \rightarrow \widehat{PC}_s$  to be the Fox completion of the composition  $P\mathcal{F}_0 \rightarrow PC_0 \rightarrow \widehat{PC}_s$ . That is, a point of  $P\mathcal{F}_s$  lying over a point  $T$  of  $\widehat{PC}_s$  is a function  $\alpha$  which assigns to each neighborhood  $W$  of  $T$  a connected component  $\alpha(W)$  of  $p^{-1}(W \cap PC_0)$ , in such a way that if  $W' \subseteq W$  then  $\alpha(W') \subseteq \alpha(W)$ .  $P\mathcal{F}_s$  has a natural topology; for details see [15]. By the naturality of the Fox completion, the actions of  $P\Gamma$  and  $PG$  extend to  $P\mathcal{F}_s$ .

Since  $P\mathcal{F}_s \rightarrow \widehat{PC}_s$  is branched over  $\widehat{\Delta} \cup E$ , it is clear that the local structure of  $\widehat{\Delta}$  and  $E$  plays a key role in the nature of  $P\mathcal{F}_s$ ; by studying it we will show that  $P\mathcal{F}_s$  is a complex manifold. The analysis follows (3.3)–(3.10) of [3], but is more complicated.

We first need to assemble some known results about certain complex reflection groups. Coxeter [12] noticed that for  $n = 1, \dots, 4$ , if one adjoins to the  $(n + 1)$ -strand braid group the relations that the  $n$  standard generators have order 3, then one obtains a finite complex reflection group. We call this group  $R_n$ . One can describe the group concretely by choosing vectors  $r_1, \dots, r_n$  that span an  $n$ -dimensional Euclidean complex vector space  $V_n$ , such that the  $i$ th generator acts as  $\omega$ -reflection in  $r_i$ . One may scale the roots in any convenient manner; we take  $r_i^2 = 3$  and refer to them as roots. Then the braid and commutation relations imply that  $|\langle r_i | r_{i\pm 1} \rangle| = \sqrt{3}$  and all other inner products vanish. By multiplying  $r_2, \dots, r_n$  in turn by scalars, we can take  $\langle r_i | r_{i+1} \rangle = \theta$  for all  $i$ . The group generated by the reflections in  $r_1, \dots, r_n$  is what we call  $R_n$ . In each case,  $r_1, \dots, r_n$  generate an  $\mathcal{E}$ -lattice, and it turns out that the reflections in  $R_n$  are exactly the reflections in the norm 3 vectors of this lattice. We write  $\mathcal{H}_n$  for the union of the orthogonal complements of all these vectors.

**THEOREM 3.4.** *For any  $n = 1, \dots, 4$ , the pair  $(V_n/R_n, \mathcal{H}_n/R_n)$  is biholomorphic to  $(\mathbb{C}^n, \Delta_{A_n})$ , where  $\Delta_{A_n}$  is the standard  $A_n$  discriminant.  $R_n$  acts freely on  $V_n - \mathcal{H}_n$ , so  $V_n - \mathcal{H}_n \rightarrow \mathbb{C}^n - \Delta_{A_n}$  is a covering map. The subgroup of  $B_{n+1} = \pi_1(\mathbb{C}^n - \Delta_{A_n})$  corresponding to this covering space is the kernel of the homomorphism  $B_{n+1} \rightarrow R_n$  described above. Finally,  $V_n \rightarrow \mathbb{C}^n$  is the Fox completion of the composition*

$$V_n - \mathcal{H}_n \rightarrow \mathbb{C}^n - \Delta_{A_n} \rightarrow \mathbb{C}^n .$$

**PROOF.** That  $V_n/R_n \cong \mathbb{C}^n$  is the same as the ring of  $R_n$ -invariants on  $V_n$  being a polynomial ring, which it is by work of Shephard and

Todd [36]. That  $\mathcal{H}_n/R_n$  corresponds to the  $A_n$  discriminant is part of the main result of Orlik and Solomon [30, cor. 2.26]. It is known that any finite complex reflection group acts freely on the complement of the mirrors of its reflections. This is a theorem of Steinberg [37]; there is also a shorter proof by Lehrer [23]. The subgroup  $H$  of  $B_{n+1}$  corresponding to the covering space contains the cubes of the meridians of  $\Delta_{A_n}$ , since  $R_n$  contains the triflections across the components of  $\mathcal{H}_n$ . Since modding out  $B_{n+1}$  by the cubes of the standard generators yields a copy of  $R_n$ , the cubes of meridians generate  $H$ , and  $B_{n+1}/H \cong R_n$  under the indicated homomorphism.

The claim about the Fox completion is a special case of the following: suppose  $G$  is a finite group acting linearly and faithfully on a finite-dimensional real vector space  $V$ , and contains no (real) reflections. Then, writing  $V_0$  for the open subset of  $V$  on which  $G$  acts freely,  $V \rightarrow V/G$  is the Fox completion of  $V_0 \rightarrow V_0/G \rightarrow V/G$ . (One just verifies that  $V \rightarrow V/G$  satisfies the definition of a completion of  $V_0 \rightarrow V/G$ . The absence of real reflections in  $G$  is required for  $V_0$  to be locally connected in  $V$ , in Fox's terminology.)  $\square$

Now we can describe the Fox completion  $P\mathcal{F}_s$ . First we describe it away from the chordal locus, and then at a point in the chordal locus.

**THEOREM 3.5.** *Suppose  $T \in \widehat{PC}_s - E$  has  $n_i$  singularities of type  $A_i$ , for each  $i = 1, \dots, 4$ . Suppose  $\check{T} \in P\mathcal{F}_s$  lies over  $T$ . Then near  $\check{T}$ ,  $P\mathcal{F}_s$  has a complex manifold structure, indeed a unique one for which  $P\mathcal{F}_s \rightarrow PC_s$  is holomorphic. With respect to this structure,  $\check{T}$  has a neighborhood in  $P\mathcal{F}_s$  biholomorphic to*

$$(B^1)^{n_1} \times (B^2)^{n_2} \times (B^3)^{n_3} \times (B^4)^{n_4} \times B^N,$$

where  $N = 34 - n_1 - 2n_2 - 3n_3 - 4n_4$ , such that  $P\mathcal{F}_0$  corresponds to

$$(B^1 - \mathcal{H}_1)^{n_1} \times (B^2 - \mathcal{H}_2)^{n_2} \times (B^3 - \mathcal{H}_3)^{n_3} \times (B^4 - \mathcal{H}_4)^{n_4} \times B^N.$$

The stabilizer of  $\check{T}$  in  $P\Gamma$  is isomorphic to  $R_1^{n_1} \times \dots \times R_4^{n_4}$ , acting in the obvious way, and the map to  $\widehat{PC}_s$  is the quotient by this group action.

**PROOF.** By Lemma 1.5,  $T$  has a neighborhood  $U \subseteq PC$  biholomorphic to

$$(3.4) \quad (B^1/R_1)^{n_1} \times \dots \times (B^4/R_4)^{n_4} \times B^N,$$

such that  $U \cap PC_0$  corresponds to (by Theorem 3.4)

$$(3.5) \quad ((B^1 - \mathcal{H}_1)/R_1)^{n_1} \times \dots \times ((B^4 - \mathcal{H}_4)/R_4)^{n_4} \times B^N,$$

and the local fundamental group is  $B_2^{n_1} \times \cdots \times B_5^{n_4}$ . We write  $T'$  for a basepoint in  $U - \widehat{\Delta}$ , so that we can refer to its associated fourfold  $V'$ . By Lemma 1.4, any standard generator of any of the braid group factors acts on  $\mathbb{C}H(V')$  as the  $\omega$ -reflection in a root  $r \in \Lambda(V')$ . We write  $H \subseteq \Gamma(V')$  for the group generated by all these reflections. The local monodromy group is by definition the projectivization of  $H$ .

By Lemma 1.6, distinct generators of the local fundamental group give linearly independent roots. Therefore the discussion before Theorem 3.4 shows that  $H$  is  $R_1^{n_1} \times \cdots \times R_4^{n_4}$ . Since the  $\mathcal{E}$ -sublattice spanned by the roots is positive-definite, it has lower dimension than  $\Lambda(V')$ , so  $H$  contains no scalars. Therefore  $P\rho(\pi_1(U - \widehat{\Delta}))$  is a copy of  $H$ . By Theorem 3.4, the covering space of  $U - \widehat{\Delta}$  associated to the kernel of this monodromy is

$$(B^1 - \mathcal{H}_1)^{n_1} \times \cdots \times (B^4 - \mathcal{H}_4)^{n_4} \times B^N,$$

with the deck group being  $H$ , acting in the obvious way. Furthermore, the Fox completion over  $U$  is then

$$(B^1)^{n_1} \times \cdots \times (B^4)^{n_4} \times B^N,$$

with  $\check{T}$  being the point at the center.

Since (3.4) is not just a homeomorphism but a biholomorphism,  $(B^1)^{n_1} \times \cdots \times (B^4)^{n_4} \times B^N \rightarrow U$  is complex analytic when the domain is equipped with the standard complex manifold structure. This gives the Fox completion a complex manifold structure such that  $P\mathcal{F}_s \rightarrow PC_s$  is holomorphic. A standard argument using Riemann extension shows that this structure is unique.  $\square$

**THEOREM 3.6.** *Suppose  $(T, \tau) \in E \cap \widehat{PC}_s$ , where  $\tau$  has  $n_i$  points of multiplicity  $i+1$ , for each  $i = 1, \dots, 4$ . Suppose  $(T, \tau)^\check{\vee} \in P\mathcal{F}_s$  lies over  $(T, \tau)$ . Then near  $(T, \tau)^\check{\vee}$ ,  $P\mathcal{F}_s$  has a complex manifold structure, indeed a unique one for which  $P\mathcal{F}_s \rightarrow \widehat{PC}_s$  is holomorphic. With respect to this structure,  $(T, \tau)^\check{\vee}$  has a neighborhood biholomorphic to*

$$B^1 \times (B^1)^{n_1} \times \cdots \times (B^4)^{n_4} \times B^{N-1},$$

where  $N$  is as in Theorem 3.5, such that the preimage of  $E$  corresponds to

$$(3.6) \quad \{0\} \times (B^1)^{n_1} \times \cdots \times (B^4)^{n_4} \times B^{N-1}$$

and the preimage of  $PC_0$  corresponds to

$$(B^1 - \{0\}) \times (B^1 - \mathcal{H}_1)^{n_1} \times \cdots \times (B^4 - \mathcal{H}_4)^{n_4} \times B^{N-1}.$$

The stabilizer of  $(T, \tau)^\sim$  in  $P\Gamma$  is isomorphic to  $\mathbb{Z}/6 \times R_1^{n_1} \times \cdots \times R_4^{n_4}$ , with the  $R_i$ 's acting in the obvious way. The  $\mathbb{Z}/6$  acts freely away from (3.6).

PROOF. This is much the same as the previous proof. The difference is that we don't have a local analytic description of  $\widehat{\Delta}$  near  $E$ , only weaker results, Theorems 2.1 and 2.2. We begin with the local monodromy analysis. Theorem 2.1 provides a neighborhood  $U$  of  $(T, \tau)$  with

$$\pi_1(U - (E \cup \widehat{\Delta})) \cong \mathbb{Z} \times (B_2)^{n_1} \times \cdots \times (B_5)^{n_4},$$

where a generator for the  $\mathbb{Z}$  factor is a meridian  $\gamma$  of  $E$ , and the standard generators for the braid group factors are meridians of  $\widehat{\Delta}$ . As in the previous proof, we write  $T'$  for a basepoint in  $U - (E \cup \widehat{\Delta})$ , so we can refer to the associated fourfold  $V'$ . We write  $H$  for the subgroup of  $\Gamma(V')$  generated by the reflections in the roots associated to the braid group factors. By Lemma 3.3,  $P\rho(\gamma)$  is a hexaflection of  $\mathbb{C}H(V')$ , which is the projectivization of a hexaflection  $S$  of  $\Lambda(V')$  in a chordal root  $s$  of  $\Lambda(V')$ . We write  $H'$  for  $\langle H, S \rangle$ . The local monodromy group  $P\rho(\pi_1(U - (E \cup \widehat{\Delta})))$  is the projectivization of  $H'$ . Following the previous proof shows that  $H \cong R_1^{n_1} \times \cdots \times R_4^{n_4}$ . We claim that  $s$  is orthogonal to all the roots of the braid group factors. To prove this, we use the fact that  $S$  commutes with  $H$ , so that for every nodal root  $r$  of a braid group factor, the triflection  $R$  in  $r$  carries  $s$  to a multiple of itself. Therefore, either  $s$  is orthogonal to all the  $r$ 's, or else it is proportional to one of them. The latter is impossible because then  $s$  would be both nodal and chordal, which is impossible. Since  $s$  is orthogonal to the  $r$ 's,  $H' = \mathbb{Z}/6 \times H$ . Arguing as in the previous proof,  $H'$  contains no scalars, so it maps isomorphically to its projectivization. Continuing as before proves the corollary, with "biholomorphic" replaced by "homeomorphic".

To prove the existence of the complex manifold structure, we proceed in two steps. First, we take  $\tilde{U}$  to be the 6-fold cover of  $U$ , branched over  $U \cap E$ . This clearly has a complex manifold structure such that the projection  $\tilde{U} \rightarrow U$  is holomorphic. Writing  $(T, \tau)^\sim$  for the preimage of  $(T, \tau)$ , Theorem 2.2 provides us with a neighborhood  $\tilde{V}$  of  $(T, \tau)^\sim$  with the properties stated there. The important property is that  $\tilde{V}$  is biholomorphic to  $B^1 \times (B^1/R_1)^{n_1} \times \cdots \times (B^4/R_4)^{n_4} \times B^{N-1}$ , such that  $\tilde{V} - \tilde{\Delta}$  corresponds to

$$B^1 \times ((B^1 - \mathcal{H}_1)/R_1) \times \cdots \times ((B^4 - \mathcal{H}_4)/R_4) \times B^{N-1}.$$

Taking the branched cover of  $\widehat{V}$  with deck group  $H$  gives the claimed complex manifold model of  $P\mathcal{F}_s$  near  $(T, \tau)^\vee$ . The uniqueness of the complex manifold structure again follows from Riemann extension.  $\square$

Extending the period map to  $P\mathcal{F}_s$  is now easy. If  $r$  is a root of  $\Lambda$ , then we will call  $r^\perp \subseteq \mathbb{C}H^{10}$  a discriminant hyperplane or chordal hyperplane according to whether  $r$  is a nodal or chordal root. By the chordal (resp. discriminant) locus of  $P\mathcal{F}_s$ , we mean the preimage of  $E \subseteq \widehat{PC}_s$  (resp.  $\widehat{\Delta}$ ).

**THEOREM 3.7.** *The period map  $P\mathcal{F}_0 \rightarrow \mathbb{C}H^{10}$  extends to a holomorphic map  $g : P\mathcal{F}_s \rightarrow \mathbb{C}H^{10}$ , which is invariant under  $PG$  and equivariant under  $P\Gamma$ . The chordal (resp. discriminant) locus of  $P\mathcal{F}_s$  maps into the chordal (resp. discriminant) hyperplanes of  $\mathbb{C}H^{10}$ .*

**PROOF.**  $P\mathcal{F}_s$  is a complex manifold by Theorems 3.5 and 3.6; since  $g$  is a map to a bounded domain, the extension exists by Riemann extension. The preimages of  $\widehat{\Delta}$  and  $E$  map into hyperplanes as claimed because of  $P\Gamma$ -equivariance. Namely, a generic point of  $P\mathcal{F}_s$  lying over  $\widehat{\Delta}$  has stabilizer  $\mathbb{Z}/3$  in  $P\Gamma$ , so it maps to the fixed-point set of  $\mathbb{Z}/3$  in  $\mathbb{C}H^{10}$ , which is a discriminant hyperplane. The same idea applies when  $\widehat{\Delta}$  is replaced by  $E$ .  $\square$

Since the period map of Theorem 3.7 is  $P\Gamma$ -equivariant, it induces a map

$$\widehat{PC}_s = P\Gamma \backslash P\mathcal{F}_s \rightarrow P\Gamma \backslash \mathbb{C}H^{10},$$

which we will now extend further. As before, we continue to use the notation  $g$ . The argument relies on a monodromy analysis near a threefold with an  $A_5$  or  $D_4$  singularity; we give the key point as a lemma:

**LEMMA 3.8.** *Suppose  $F \in \mathcal{C}_{ss}$  defines a threefold  $T$  with a singularity of type  $A_5$ , and let  $U$  be a neighborhood of  $F$  as in Lemma 1.5, with basepoint  $F'$ . Let  $a_1, \dots, a_5$  be standard generators for the corresponding factor  $B_6$  of  $\pi_1(U - \Delta, F')$ , and let  $r_1, \dots, r_5$  be roots of  $\Lambda(V')$ , by whose  $\omega$ -reflections the  $a_i$  act. Suppose  $\langle r_i | r_{i\pm 1} \rangle = \pm\theta$  and all other inner products are zero. Then  $\xi = r_1 - \theta r_2 - 2r_3 + \theta r_4 + r_5$  is a nonzero isotropic vector of  $\Lambda(V')$ , and  $(a_1 \dots a_5)^6$  acts on  $\Lambda(V')$  by the unitary transvection in  $\xi$ , namely*

$$x \mapsto x - \frac{\langle x | \xi \rangle}{\theta} \xi.$$

Now suppose the singularity has type  $D_4$  rather than  $A_5$ , and that  $a_1, a_2, a_3$  and  $b$  are standard generators for  $\mathcal{A}(D_4) \subseteq \pi_1(U - \Delta)$ , with  $b$  corresponding to the central node of the  $D_4$  diagram. Suppose  $r_1, r_2, r_3$

and  $r'$  are roots for the corresponding  $\omega$ -reflections of  $\Lambda(V')$ , scaled so that  $\langle r_i | r' \rangle = \theta$  for  $i = 1, 2, 3$ . Then  $\xi = r_1 + r_2 + r_3 - \theta r'$  is a nonzero isotropic vector of  $\Lambda(V')$ , and  $(a_1 a_2 a_3 b)^3$  acts on  $\Lambda(V')$  by the unitary transvection in  $\xi$ .

REMARK. The given words in the Artin generators generate the centers of the Artin groups, and in each case,  $\xi$  spans the kernel of the restriction of  $\langle | \rangle$  to the span of the roots. So it isn't surprising that the word acts by a transvection in a multiple of  $\xi$ . The point of the lemma is that this multiple is nonzero.

PROOF. We treat the  $A_5$  case first. We remark that the  $r_i$  are pairwise linearly independent by Lemma 1.6, and by scaling them we may assume that their inner products are as stated. We need the sharper result that  $r_1, \dots, r_5$  are linearly independent. Direct calculation using the given inner products shows that  $\xi$  is isotropic and orthogonal to  $r_1, \dots, r_5$ . One can show (Lemma 3.9 below) that the locus of cubic threefolds with an  $A_5$  singularity is irreducible, so to prove  $\xi \neq 0$ , it suffices to treat a single example. If  $T'$  has an  $A_6$  singularity, then one can write down the inner product matrix for its six roots and check that it is nondegenerate. Therefore its six roots are linearly independent. In particular, the first five are, so  $\xi \neq 0$ . The fact that  $(a_1 \cdots a_5)^6$  acts as the transvection in  $\xi$  is a matrix calculation—one writes down any linearly independent set of roots in  $\mathbb{C}^{10,1}$  with these inner products, and just multiplies the reflections together suitably.

The same idea works for the  $D_4$  case. To show  $\xi \neq 0$ , one must check (i) the irreducibility of the set of threefolds with a  $D_4$  singularity, (ii) that there is a threefold with a  $D_5$  singularity admitting a versal deformation in  $PC$ , and (iii) the inner product matrix for 5 roots corresponding to the generators of  $\mathcal{A}(D_5)$  is nondegenerate. Then one does a matrix calculation to check the action of  $(a_1 a_2 a_3 b)^3$ .  $\square$

We used the following result in the proof of the lemma.

LEMMA 3.9. *The locus of cubic threefolds having an  $A_5$  singularity is irreducible.*

PROOF SKETCH: The proof of Theorem 4.2 in [1] shows that, given a cubic threefold  $T$  with an  $A_5$  singularity, one can choose homogeneous coordinates so that many coefficients of  $F$  vanish. A more precise way to say this is that  $T$  is projectively equivalent to a member of a certain  $P^{19} \subseteq P^{34} = PC$ , and furthermore that a generic member of this  $P^{19}$  has exactly one  $A_5$  singularity. Since  $\mathrm{PGL}_5\mathbb{C}$  and this open subset of  $P^{19}$  are irreducible, so is the  $A_5$  locus.  $\square$

REMARK. A similar argument applies to the  $A_2$  locus, whose irreducibility we used in the proof of Lemma 1.6. In that case there is a more conceptual argument, which with effort can also be made to work for the  $A_5$  case. The Jacobian scheme  $J$  of an  $A_2$  singularity is an “infinitesimal curve”, i.e., a 0-dimensional irreducible scheme with 1-dimensional tangent space. The length of  $J$  is 2. One way to phrase the recognition principle for simple singularities [9, Lemma 1 and Corollary] is that the generic hypersurface whose Jacobian scheme contains  $J$  has an  $A_2$  singularity there, rather than some more-complicated singularity. Now, the family of copies of  $J$  in  $P^4$  is irreducible, since  $\mathrm{PGL}_5\mathbb{C}$  acts transitively, and the irreducibility of the  $A_2$  locus follows easily. The same ideas apply in the  $A_5$  case, but then  $J$  has length 5 and  $\mathrm{PGL}_5\mathbb{C}$  is no longer transitive. It turns out that if  $J$  lies in a hyperplane then the cubic threefolds whose Jacobian scheme contains  $J$  have singularities worse than  $A_5$ . And only one  $\mathrm{PGL}_5\mathbb{C}$ -orbit of copies of  $J$  fails to lie in a hyperplane (namely,  $J$  lies in a rational normal quartic curve). So the argument still applies.

THEOREM 3.10. *The period map  $\widehat{PC}_s \rightarrow P\Gamma \backslash CH^{10}$  extends to a holomorphic map  $g : \widehat{PC}_{ss} \rightarrow \overline{P\Gamma \backslash CH^{10}}$ . This map sends  $\widehat{PC}_{ss} - \widehat{PC}_s$  to the boundary points of the Baily-Borel compactification.*

PROOF. The extension of  $g$  to  $\widehat{PC}_{ss} - E$  follows section 8 of [3], but is a little more complicated. After explaining this, we will extend  $g$  to  $\widehat{PC}_{ss} \cap E$  by modifying the argument, in the same way that we modified the proof of Theorem 3.5 to prove Theorem 3.6.

We begin by supposing  $T \in \widehat{PC}_{ss} - (\widehat{PC}_s \cup E)$ , with defining function  $F$ . We adopt the notation of Lemma 1.5, so that  $T$  has  $m$  singularities  $s_1, \dots, s_m$ , and  $U$  is a neighborhood of  $F$  in  $\mathcal{C}_{ss}$  with the properties stated there. Let  $\check{U}$  be the cover of  $U$  universal among those with two-fold branching over  $U \cap \Delta$ . By Brieskorn’s description [7] of versal deformations of simple singularities,  $\check{U}$  is biholomorphic to a neighborhood of the origin in  $\mathbb{C}^{34}$ , such that the preimage  $\check{\Delta}$  of  $\Delta$  is the union of the reflection hyperplanes for the Coxeter group  $W_1 \times \dots \times W_m$ , where  $W_i$  is the Coxeter group of the same type as the singularity  $s_i$ . Let  $\beta$  be a loop lying in a generic line through the origin in  $\mathbb{C}^{35}$ , and encircling the origin once positively.

We claim that the sixth power of  $\rho(\beta)$  is nontrivial and unipotent. (Discussing  $\rho(\beta)$  requires choosing a basepoint  $T' \in U - \Delta$ , a framing of this threefold  $T'$ , and then a basepoint in  $\check{U} - \check{\Delta}$  lying over this framed threefold. These choices are immaterial.)

If  $T$  has only one singularity, of type  $A_n$ , then in terms of the standard generators  $a_1, \dots, a_n$  for  $\pi_1(U - \Delta) \cong B_{n+1}$ ,  $\beta = (a_1 \dots a_n)^{n+1}$ . The  $a_i$  act by  $\omega$ -reflections in linearly independent roots  $r_1, \dots, r_n$  of  $\Lambda(V)$ , with  $r_i \perp r_j$  except for  $\langle r_i | r_{i\pm 1} \rangle = \pm\theta$ . This lets one work out the action of  $\beta$ , by choosing such roots and multiplying matrices together. (Two choices of such roots are equivalent under  $U(9, 1)$ , so the conjugacy class of  $\rho(\beta^6)$  in  $U(9, 1)$  is independent of the choice.) Direct calculation shows that in the cases  $n = 1, 2, 3$  and  $4$ , the order of  $\rho(\beta)$  has order  $3, 2, 3$  and  $6$ . If  $n = 5$  then  $\rho(\beta^6)$  is a nontrivial unipotent by Lemma 3.8, hence of infinite order. (No calculation is required to see that  $\rho(\beta)$  has finite order if  $n < 5$ , because  $\rho(B_{n+1})$  is the finite group  $R_n$ .) If  $T$  has only one singularity, of type  $D_4$ , with standard generators  $a_1, a_2, a_3$  and  $b, b$  corresponding to the central node, then  $\beta = (a_1 a_2 a_3 b)^6$ , and again Lemma 3.8 shows that  $\rho(\beta)$  is a nontrivial unipotent.

If  $T$  has  $m$  singularities, then  $\beta = \beta_1 \dots \beta_m$ , where each  $\beta_i$  is as  $\beta$  above, one for each singularity. Since the  $\beta_i$ 's commute,  $\rho(\beta^6)$  is a product of nontrivial commuting unipotent isometries, one for each  $A_5$  or  $D_4$  singularity. Since  $T$  is not stable, there is at least one  $A_5$  or  $D_4$  singularity. If there is only one, then this proves that  $\rho(\beta^6)$  is nontrivial and unipotent.

If there are more than one, then the product is unipotent since it is a product of commuting unipotents. A little extra work is required to show that it is nontrivial, i.e., that no cancellation occurs. Suppose  $s$  and  $s'$  are two singularities of  $T$ , each of type  $A_5$  or  $D_4$ . ( $T$  cannot have both an  $A_5$  and a  $D_4$  singularity, but this isn't needed here.) Let  $\xi$  and  $\xi'$  be the isotropic vectors from Lemma 3.8. They are orthogonal because they correspond to distinct singularities. Since  $\Lambda(V')$  has signature  $(9, 1)$ ,  $\xi$  and  $\xi'$  are proportional. By using the formula defining a unitary transvection, one can check that the product of the transvections in  $\xi$  and  $\xi'$  is a transvection in a nonzero multiple of  $\xi$ . The same argument applies when there are more than two singularities of type  $A_5$  or  $D_4$ . There is only one  $PG$ -orbit of such threefolds, which has three  $D_4$  singularities.

Because a power of  $\rho(\beta)$  is nontrivial and unipotent, it fixes a unique point of  $\partial CH^{10}$ . By unipotence, this fixed point is represented by a null vector of  $\Lambda$ , so there is an associated boundary point  $\eta$  of  $\overline{PG \setminus CH^{10}}$ . We claim that for every neighborhood  $Z$  of  $\eta$ , there is a neighborhood  $\check{Y}$  of the origin in  $\check{U}$ , such that the composition

$$\check{U} - \check{\Delta} \rightarrow U - \Delta \rightarrow PG \setminus CH^{10} \rightarrow \overline{PG \setminus CH^{10}}$$



carries  $\check{Y} - \check{\Delta}$  into  $Z$ . This uses a hyperbolic analysis argument (see p. 708 of [3]). It follows that for every neighborhood  $Z$  of  $\eta$ , there is a neighborhood  $Y$  of  $F$  in  $\mathcal{C}$ , such that  $g(Y - \Delta) \subseteq Z$ . Then, by Riemann extension,  $g : Y - \Delta \rightarrow \overline{P\Gamma \backslash \mathbb{C}H^{10}}$  extends holomorphically to  $Y$ , carrying  $F$  to  $\eta$ .

Now suppose  $(T, \tau) \in \widehat{PC}_{ss} \cap E$ , with  $\tau$  having a point of multiplicity six, and let  $\check{U}$  and  $\check{V}$  be as in Theorem 2.2. Then let  $\check{V}$  be the universal cover of  $\check{V}$  with 2-fold branching over  $\check{\Delta} \subseteq \check{V}$ . We write  $\check{\Delta}$  and  $\check{E}$  for the preimages of  $\Delta$  and  $E$  in  $\check{V}$ . From here, the treatment is exactly the same as above, except that  $\check{E}$  is present. This affects nothing, because the monodromy around  $\check{E}$  is trivial. (The monodromy around  $E$  has order 6, and  $\check{U}$  is the 6-fold cover of  $U$  branched over  $U \cap E$ .)  $\square$

**THEOREM 3.11.**  $\overline{P\Gamma \backslash \mathbb{C}H^{10}}$  has exactly two boundary points. The threefolds having a  $D_4$  singularity map to one, and those having an  $A_5$  singularity map to the other. The points  $(T, \tau)$  of  $E$ , where  $\tau$  has a point of multiplicity 6, map to the latter boundary point.

We will call the boundary points the  $A_5$  and  $D_4$  cusps of  $\overline{P\Gamma \backslash \mathbb{C}H^{10}}$ .

**PROOF.** We know that the preimage in  $\widehat{PC}_{ss}$  of the boundary consists of  $\widehat{PC}_{ss} - \widehat{PC}_s$ . The GIT equivalence classes in  $\widehat{PC}_{ss} - \widehat{PC}_s$  are represented by the points (iii) and (vi) of Theorem 3.2. The threefolds  $T_{A,B}$  form a 1-parameter family, limiting to (vi). Therefore they all map to a single boundary point. We call the union of these GIT equivalence classes the  $A_5$  component of  $\widehat{PC}_{ss} - \widehat{PC}_s$ . Only one GIT equivalence class remains, which we call the  $D_4$  component of  $\widehat{PC}_{ss} - \widehat{PC}_s$ . Therefore there are at most two boundary points.

It remains to show that the  $A_5$  and  $D_4$  threefolds map to distinct boundary points. We will use the fact that the  $D_4$  point of  $\widehat{PC}_{ss}/\mathrm{SL}(5, \mathbb{C})$  is an isolated point of the strictly semistable locus (by Theorem 3.2). Theorem 3.10 implies that  $\widehat{PC}_s/G \rightarrow P\Gamma \backslash \mathbb{C}H^{10}$  is proper, so any  $G$ -invariant neighborhood of the  $D_4$  locus, minus the  $D_4$  locus itself, surjects to a deleted neighborhood of the  $D_4$  cusp in  $\overline{P\Gamma \backslash \mathbb{C}H^{10}}$ . Now, the images of the  $A_5$  and  $D_4$  components in  $\widehat{PC}_{ss}/\mathrm{SL}(5, \mathbb{C})$  are disjoint closed sets, so the  $A_5$  and  $D_4$  components have disjoint  $PG$ -invariant neighborhoods in  $\widehat{PC}_{ss}$ . Using this together with the injectivity of the period map  $PC_0/G \rightarrow P\Gamma \backslash \mathbb{C}H^{10}$  (Theorem 1.9) shows that no  $A_5$  threefold can map to the  $D_4$  cusp. So the two cusps are distinct.  $\square$

## CHAPTER 4

### Degeneration to a Chordal Cubic

The aim of this chapter is to identify the limit Hodge structure for the degeneration of cyclic cubic fourfolds associated to a generic degeneration to cubic threefolds to a chordal cubic. The following theorem is this chapter's contribution to the proof of the main theorem of the monograph, Theorem 6.1.

**THEOREM 4.1.** *The period map  $g : \widehat{PC}_{ss} \rightarrow P\Gamma \backslash \mathbb{C}H^{10}$  carries the chordal locus onto a divisor.*

This immediately implies that the chordal cubics are points of indeterminacy for the rational map  $\mathcal{C}_{ss} \dashrightarrow \overline{P\Gamma \backslash \mathbb{C}H^{10}}$  obtained from the period map  $\mathcal{C}_s \rightarrow P\Gamma \backslash \mathbb{C}H^{10}$ .

The theorem is a consequence of Theorem 4.1.1 below, which most of the chapter is devoted to proving. This theorem describes the limit Hodge structure in terms of a Hodge structure studied by Deligne and Mostow [28],[13], associated to a 12-tuple of points in  $P^1$ .

#### 4.1. Statement of Results

Let us establish notation and definitions. The chordal cubic  $T$  is the secant variety of a rational normal curve  $R$  in  $P^4$ .  $R$  is the whole singular locus of  $T$ . Consider a pencil  $\{T_t\}$  of cubic threefolds with  $T_0 = T$ . In homogeneous coordinates, the family near  $T_0$  is

$$\left\{ ([x_0 : \dots : x_4], t) \in P^4 \times \Delta \mid F(x_0, \dots, x_4) + tG(x_0, \dots, x_4) = 0 \right\},$$

where  $F$  defines  $T = T_0$ ,  $G$  is some other member of the pencil, and  $\Delta$  is the unit disk in  $\mathbb{C}$ . We assume that the pencil is generic in the sense that  $T_t$  is smooth for all sufficiently small nonzero  $t$ . By scaling  $G$  we may suppose that  $T_t$  is smooth for all  $t \in \Delta - \{0\}$ . We also make the genericity assumption that  $G = 0$  cuts out on  $R$  a set  $B$  of 12 distinct points, the “infinitesimal base locus”.

Taking the threefold covers of  $P^4$  over the  $T_t$  gives a family  $\mathcal{V}$ , which is the restriction to  $\Delta \subseteq \mathcal{C}$  of the family called  $\mathcal{V}$  in the rest of

the monograph. Explicitly,

$$\mathcal{V} = \{ ([x_0 : \dots : x_5], t) \in P^5 \times \Delta \mid F(x_0, \dots, x_4) + tG(x_0, \dots, x_4) + x_5^3 = 0 \} .$$

All fibers of  $\mathcal{V}$  are smooth except for  $V_0$ .

By Lemma 3.3, the monodromy of  $\mathcal{V}|_{\Delta - \{0\}}$  on  $H^4$  has finite order, so there is a well-defined limit Hodge structure. To discuss this limiting Hodge structure, we define the notion of an ‘‘Eisenstein Hodge structure.’’ This is an Eisenstein module  $H$  with a complex Hodge structure on the vector space  $H \otimes_{\mathcal{E}} \mathbb{C}$ , where multiplication by  $\omega$  preserves the Hodge decomposition. A morphism of such objects is a homomorphism of Eisenstein modules which is compatible with the Hodge decomposition.

The Eisenstein Hodge structure of interest to us throughout the monograph is  $\Lambda(V)$ , for  $V$  a smooth cyclic cubic 4-fold. Recall that its underlying abelian group is  $H_0^4(V; \mathbb{Z})$ , and that the  $\mathcal{E}$ -module structure is defined by taking  $\omega$  to act as  $\sigma^*$ , where  $\sigma$  is from (1.1). Then  $H_0^4(V; \mathbb{R})$  is identified with  $\Lambda(V) \otimes_{\mathcal{E}} \mathbb{C}$  and is hence a complex vector space. The projection to  $\sigma$ 's  $\omega$ -eigenspace identifies it with  $H_{\sigma=\omega}^4(V; \mathbb{C})$ . In light of this, the Hodge decomposition

$$(4.1.1) \quad H_{\sigma=\omega}^4(V; \mathbb{C}) = H_{\sigma=\omega}^{3,1}(V) \oplus H_{\sigma=\omega}^{2,2}(V)$$

gives an Eisenstein Hodge structure on  $\Lambda(V)$ .

The limit mixed Hodge structure of a degeneration can be computed from the mixed Hodge structure of the central fiber provided that the degeneration is *semistable*. One computes the limit using the Clemens-Schmid sequence [11]. A semistable model  $\hat{\mathcal{V}}$  for a degeneration  $\mathcal{V}$  is a family over  $\Delta$  disk with the following properties:

(a) There is a surjective morphism  $\hat{\mathcal{V}} \rightarrow \mathcal{V}$  which makes the diagram

$$\begin{array}{ccc} \hat{\mathcal{V}} & \longrightarrow & \mathcal{V} \\ \downarrow & & \downarrow \\ \Delta & \longrightarrow & \Delta \end{array}$$

commute, where the bottom arrow is  $t = s^n$  for some  $n$ , with  $s$  a parameter on the left  $\Delta$  and  $t$  a parameter on the right, and the restriction of  $\hat{\mathcal{V}}$  to  $\Delta - \{0\}$  is the pullback of  $\mathcal{V}|_{\Delta - \{0\}}$ . We write  $V$  and  $\hat{V}$  for the central fibers of  $\mathcal{V}$  and  $\hat{\mathcal{V}}$ , and  $V_t$  and  $\hat{V}_s$  for other fibers.

(b) The total space of  $\hat{\mathcal{V}}$  is smooth and the central fiber  $s = 0$  is a normal crossing divisor (NCD) with smooth components, each of multiplicity one.

It is known that every one-parameter degeneration has a semistable model, obtainable by artful combination of three moves: base extension, blowing up, and normalization. In our case we will apply one base extension, replacing  $t$  by  $s^6$ , followed by three blowups along various ideal sheaves.

In our case the Clemens-Schmid sequence reads as follows:

$$\cdots \rightarrow H^4(\hat{\mathcal{V}}, \hat{\mathcal{V}}^*) \rightarrow H^4(\hat{V}) \rightarrow \lim_{s \rightarrow 0} H^4(\hat{V}_s) \xrightarrow{N} H^4(\hat{V}_s) \rightarrow \cdots$$

where  $N$  is the logarithm of the monodromy transformation. The asterisk indicates the restriction of  $\hat{\mathcal{V}}$  to the punctured disk. The terms of the sequence are abelian groups equipped with Hodge structures, but in our calculations we will only need to work with rational coefficients. Since the monodromy of  $\mathcal{V}^*$  on  $H^4$  of the fiber has order 6 (by Lemma 3.3), the base extension ensures that the monodromy of the semistable model is trivial. Therefore  $N = 0$  and the sequence reduces to

$$(4.1.2) \quad \cdots \rightarrow H^4(\hat{\mathcal{V}}, \hat{\mathcal{V}}^*) \rightarrow H^4(\hat{V}) \rightarrow \lim_{s \rightarrow 0} H^4(\hat{V}_s) \rightarrow 0.$$

Since the monodromy is trivial, the limit is a (pure) Hodge structure and is presented as a quotient of the mixed Hodge structure on  $H^4(\hat{V})$ .

Because we are interested in the ‘‘Eisenstein part’’ of the sequence, we make the following definition. Let  $H$  be a  $\mathbb{Z}$ -module on which an automorphism  $\zeta$  of order  $n$  acts. Let  $p_k$  be the cyclotomic polynomial of degree  $k$ , where  $k$  is a divisor of  $n$ . Let  $H_{(k)}$  be the kernel of  $p_k(\zeta)$  acting on  $H$ . Then there is a decomposition of  $H \otimes \mathbb{Q}$  into a direct sum of subspaces  $H_{(k)} \otimes \mathbb{Q}$ . Moreover,  $H$  is commensurable with the direct sum of the  $H_{(k)}$ . Since the characteristic polynomial for the action of  $\zeta$  is  $p_k$  on  $H_{(k)}$ , we call  $H_{(k)}$  the  $k$ th characteristic submodule (or subspace) of  $H$ , or alternatively, the  $k$ -part of  $H$ .

Passing to the 3-parts of the terms of (4.1.2), we have the sequence

$$(4.1.3) \quad \cdots \rightarrow H_{(3)}^4(\hat{\mathcal{V}}, \hat{\mathcal{V}}^*) \rightarrow H_{(3)}^4(\hat{V}) \rightarrow \lim_{s \rightarrow 0} H_{(3)}^4(\hat{V}_s) \rightarrow 0,$$

which need not be exact. However, it is exact when one takes  $\mathbb{Q}$  as the coefficient group.

The Eisenstein Hodge structure of  $\hat{V}$  is the focus of the rest of this chapter; we describe it in Theorem 4.1.1 below. The description is in terms of a curve  $C$  determined by the inclusion  $B \subseteq R$ . Identifying  $R$  with  $P^1$ ,  $B$  is the zero locus of a homogeneous polynomial  $f$  of degree 12. The curve  $C$  is a certain 6-fold cover of  $P^1$  branched over

$B$ , namely

$$(4.1.4) \quad C = \{ [x:y:z] \in P(1, 1, 2) \mid f(x, y) + z^6 = 0 \} ,$$

and it has an automorphism

$$(4.1.5) \quad \zeta([x:y:z]) = [x:y:-\omega z]$$

of order 6. The Griffiths residue calculus, Proposition 1.2, shows that  $H^{1,0}(C)$  is the vector space consisting of the residues of the rational differentials

$$(4.1.6) \quad \frac{a(x, y, z)\Omega}{f(x, y) + z^6}$$

where  $\Omega = xdydz - ydx dz + zdx dy$  and the total weight is zero, so  $a$  is a polynomial of weight 8. Now,  $\zeta$  scales  $z$  and  $\Omega$  by  $-\omega$ , and it follows that  $H_{\zeta=-\bar{\omega}}^{1,0}(C)$  is 1-dimensional, spanned by (4.1.6) with  $a = z^4$ . One can also check that  $H_{\zeta=-\bar{\omega}}^{0,1}(C)$  is 9-dimensional. We define an Eisenstein module structure  $\Lambda_{10}(C)$  by taking the underlying group to be  $H_{(6)}^1(C; \mathbb{Z})$  and taking  $-\bar{\omega}$  to act as  $\zeta^*$ . By the same considerations as above,  $\Lambda_{10}(C) \otimes_{\mathcal{E}} \mathbb{C}$  is identified with  $H_{\zeta=-\bar{\omega}}^1(C; \mathbb{C})$ , so the Hodge structure of  $C$  gives an Eisenstein Hodge structure on  $\Lambda_{10}(C)$ . The notation  $\Lambda_{10}(C)$  is intended to emphasize the parallel between the isomorphism in Theorem 4.1.1 and the description of  $\Lambda(V)$  in Theorem 1.7.

**THEOREM 4.1.1.** *There is an isogeny of Eisenstein Hodge structures*

$$\Lambda(\hat{V}) \rightarrow \mathcal{E} \oplus \Lambda_{10}(C)$$

*of weight  $(-2, -1)$ , where  $\mathcal{E}$  indicates a 1-dimensional Eisenstein lattice with Hodge type  $(1, 0)$ .*

**REMARK.** The total weight of the map is  $-3$ . It is odd for a map of Hodge structures to have odd total weight. Such odd weight morphisms were first considered by van Geemen in his paper on half twists [16].

**PROOF OF THEOREM 4.1, GIVEN THEOREM 4.1.1.** By Theorem 4.1.1 we see that  $H_{(3)}^4(\hat{V})$  has rank 22 as a  $\mathbb{Z}$ -module, which is the same as the rank of the limit Hodge structure. Therefore (4.1.3) shows that the Eisenstein Hodge structures of  $\Lambda(\hat{V})$  and  $\lim_{s \rightarrow 0} \Lambda(V_s)$  are isogenous. The period map for the family of Hodge structures  $\Lambda_{10}(C)$  has rank 9, by a standard calculation with the Griffiths residue calculus; see Proposition 1.2 and the calculation in the proof of Lemma 1.8. Therefore, as the pencils in  $PC$  through  $T$  vary, their limiting Hodge

structures sweep out a 9-dimensional family. It follows from Theorem 3.10 that the period map is proper, so the image of the chordal locus is closed, hence a divisor.

We remark that Mostow [28] (see also [13]) proved that the Eisenstein Hodge structure on  $H_{(6)}^1(C)$  determines the point set  $B$  in  $P^1$  up to projective transformation. This is a global Torelli theorem, rather than just the local one provided by the residue calculation referred to above.  $\square$

## 4.2. Overview of the Calculations

This section outlines the calculations required for proving Theorem 4.1.1. In order to discuss  $\hat{V}$  we need to describe briefly our particular semistable model  $\hat{\mathcal{V}}$ . In the next section, we define  $\mathcal{V}_0$  as the degree 6 base extension of  $\mathcal{V}$ ,  $\mathcal{V}_1$  as a blowup of  $\mathcal{V}_0$ ,  $\mathcal{V}_2$  as a blowup of  $\mathcal{V}_1$  and  $\mathcal{V}_3$  as a blowup of  $\mathcal{V}_2$ .  $\hat{\mathcal{V}}$  is  $\mathcal{V}_3$ . We write  $E_1 \subseteq \mathcal{V}_1$ ,  $E_2 \subseteq \mathcal{V}_2$  and  $E_3 \subseteq \mathcal{V}_3$  for the exceptional divisors of the blowups, and indicate proper transforms of  $V$  and the  $E_i$  in subsequent blowups by adding primes. For example,  $V'$  is the proper transform of  $V$  in  $\mathcal{V}_1$ , and the central fiber of  $\mathcal{V}_3$  is

$$\hat{V} = V''' \cup E_1'' \cup E_2' \cup E_3.$$

Using the Mayer-Vietoris and Leray spectral sequences, we show in Lemma 4.4.2 that only  $V'''$  and  $E_1''$  contribute to  $H_{(3)}^4(\hat{V}; \mathbb{Q})$ , and these contributions depend only on  $V$  and  $E_1$ . That is,

$$(4.2.1) \quad H_{(3)}^4(\hat{V}; \mathbb{Q}) \cong H_{(3)}^4(V; \mathbb{Q}) \oplus H_{(3)}^4(E_1; \mathbb{Q}).$$

A key point in the argument is that  $H_{(3)}^*(A) = 0$  when  $A$  is any intersection of two or more components. In Lemma 4.4.3 we show that the first term on the right side of (4.2.1) is the Eisenstein Hodge structure  $\mathcal{E} = \mathbb{Z}[\omega]$ , where all elements are of type  $(2, 2)$ , and in Lemma 4.4.4 we show that the second term is isomorphic to  $H_{(6)}^1(C)$  as an Eisenstein Hodge structure. Theorem 4.1.1 follows.

The argument is rather involved, for which reason we present a roadmap in Figures 1 and 2. The configuration of components of the central fibers  $\mathcal{V}_1(0)$ ,  $\mathcal{V}_2(0)$ , and  $\mathcal{V}_3(0)$  of the successive blowups are represented by a graph, with some indication of the the spaces attached to vertices, edges, and faces.

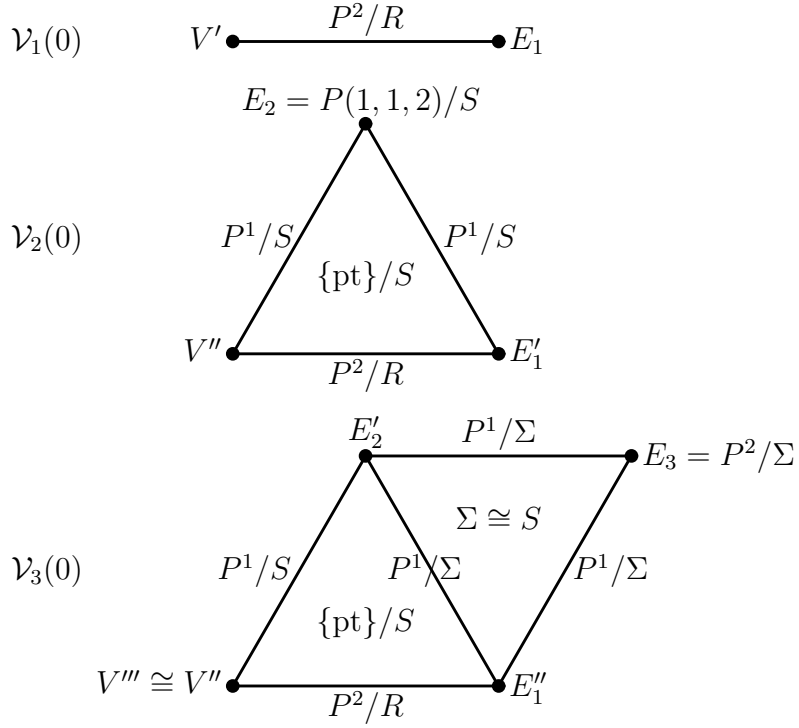


FIGURE 1. Components of the central fibers and their intersections. Each node represents a component. Edges and faces represent intersections of components. The notation  $X/Y$  means an  $X$ -bundle over  $Y$ .  $R$  is a copy of  $P^1$ , and  $S \cong \Sigma$  is a  $P^1$ -bundle over  $R$ .  $E_1$  is described by (4.3.4) and surrounding text, and the other spaces are described in paragraphs marked ‘Geometry of the central fiber’.

### 4.3. Semistable reduction

We now construct a semistable model  $\hat{\mathcal{V}}$  for  $\mathcal{V}$ . Once constructed, it will be called  $\mathcal{V}_3$ . As noted above, our approach is to apply one base extension followed by three blowups, the first of which is centered in  $R$ .

Let  $\mathcal{V}_0$  denote the result of base extension of  $\mathcal{V}$ , where  $t = s^6$  is the parameter substitution. Thus

$$\mathcal{V}_0 = \{([x_0 : \dots : x_4 : z], s) \in P^5 \times \Delta \mid F + s^6 G + z^3 = 0\},$$

with  $s = 0$  defining the central fiber  $\mathcal{V}_0(0)$ . This central fiber is just the fourfold  $V$  defined by  $F + z^3 = 0$ , which is singular along  $R$ .

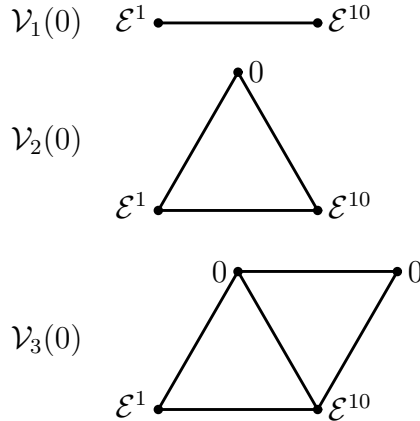


FIGURE 2.  $H_{(3)}^*$  of spaces in figure 1. It vanishes for the spaces corresponding to edges and faces of the figure, by Lemma 4.4.1.  $H_{(3)}^4$  is shown for the vertex spaces. The ones marked 0 vanish by Lemma 4.4.1. The others are free Eisenstein modules of the indicated dimensions. The three marked  $\mathcal{E}^1$  (resp.  $\mathcal{E}^{10}$ ) are all isomorphic via the blowdown maps, by Lemma 4.4.3 (resp. 4.4.4).

The next step is to blow up  $\mathcal{V}_0$  by blowing up  $P^5 \times \Delta$  along an ideal sheaf  $\mathcal{J}$  supported in  $R$ . Let  $\mathcal{I}$  denote the ideal sheaf of  $R$  in  $P^4$ . Extend it, keeping the same name, to the ideal sheaf on  $P^5 \times \Delta$  that defines

$$(\text{the cone on } R \text{ in } P^5, \text{ with vertex } [0:0:0:0:0:1]) \times \Delta .$$

Let  $\mathcal{J}$  be the ideal sheaf

$$(4.3.1) \quad \mathcal{J} = \langle \mathcal{I}^2, \mathcal{I}zs, \mathcal{I}s^3, z^3, z^2s^2, zs^4, s^6 \rangle .$$

Finally, let  $\widehat{P^5 \times \Delta}$  be the blowup of  $P^5 \times \Delta$  along  $\mathcal{J}$ , and let  $\mathcal{V}_1$  be the proper transform of  $\mathcal{V}_0$ . The central fiber  $\mathcal{V}_1(0)$  is the pullback of  $s = 0$ , and consists of the proper transform  $V'$  of  $V$  and the exceptional divisor  $E_1$  of  $\mathcal{V}_1 \rightarrow \mathcal{V}_0$ .

The choice of ideal sheaf to blow up came from the local analytic model (4.3.3) of  $\mathcal{V}_0 \subseteq P^5 \times \Delta$ . There, it is obvious that one “should” use a weighted blowup along  $R$  with weights  $(3, 3, 3, 2, 1)$ . For a well-defined global construction, one needs to choose an ideal sheaf whose description in a coordinate patch gives this weighted blowup. We worked backwards from (4.3.3) and arrived at  $\mathcal{J}$ .

The family  $\mathcal{V}_1$  is nearly semistable; one could call it an “orbifold semistable model”. The precise meaning of this is that the inclusion of



the central fiber into  $\mathcal{V}_1$  is locally modeled on the inclusion of a normal crossing divisor into a smooth manifold, modulo a finite group. We will now proceed to resolve the quotient singularities.

Let  $S$  be the singular locus of  $\mathcal{V}_1$ . This turns out to lie in  $V' \cap E_1$  and be a  $P^1$  bundle over  $R$ . Let  $\mathcal{K}$  be the ideal sheaf generated by (i) the regular functions vanishing to order 2 along  $S$ , and (ii) the regular functions vanishing to order 3 at a generic point of  $V'$ . Let  $\mathcal{V}_2$  be the blowup of  $\mathcal{V}_1$  along  $\mathcal{K}$ . The central fiber  $\mathcal{V}_2(0)$  consists of the proper transforms  $V''$  and  $E'_1$  of  $V'$  and  $E_1$ , together with the new exceptional divisor  $E_2$ . The choice of ideal sheaf  $\mathcal{K}$  is explained after Lemma 4.3.4.

The family  $\mathcal{V}_2$  is also an “orbifold semistable model.” It has quotient singularities along a surface  $\Sigma$  which projects isomorphically to  $S$ , and is smooth away from  $\Sigma$ . Define  $\mathcal{V}_3$  to be the ordinary blowup of  $\mathcal{V}_2$  along  $\Sigma$ . This is the desired semistable model  $\hat{\mathcal{V}}$ . Its central fiber  $\mathcal{V}_3(0)$  is as always the pullback of  $s = 0$ , and consists of the proper transforms  $V'''$ ,  $E''_1$  and  $E'_2$ , and the exceptional divisor  $E_3$ . Of course,  $\sigma$  acts on  $\mathcal{V}_0$  by  $z \rightarrow \omega z$ . Since it preserves  $\mathcal{J}$ , it acts on  $\mathcal{V}_1$ , and similarly for  $\mathcal{V}_2$  and  $\mathcal{V}_3$ .

**THEOREM 4.3.1.**  $\mathcal{V}_3 \rightarrow \Delta$  is a semistable model for the family  $\mathcal{V} \rightarrow \Delta$ .

The rest of this section is devoted to proving this theorem and describing the various varieties in enough detail for the cohomology calculations in section 4.4. We will discuss weighted blowups and give two lemmas, and then construct and study the three blowups. We provide more computational detail than we would if our blowups were just ordinary blowups.

We describe weighted blowups as follows. Let  $P(a) = P(a_1, \dots, a_n)$  be the weighted projective space with weights  $a = (a_1, \dots, a_n)$ . The *weighted blowup* of  $\mathbb{C}^n$  with weights  $a$  is the closure of the graph of the rational map  $f : \mathbb{C}^n \dashrightarrow P(a)$ , where  $f(z_1, \dots, z_n) = [z_1 : \dots : z_n]$  is the natural projection.

One can write down orbifold coordinate charts for the weighted blowup as in [20, pp. 166–167]. However, note that the definition of weighted blowup on p. 166 of that reference contains a misprint. The map  $f$  used to define the graph should be the one we are using, not  $f(z_1, \dots, z_n) = [z_1^{a_1} : \dots : z_n^{a_n}]$ .

Weighted blowups are a special case of blowups of  $\mathbb{C}^n$  at an ideal supported at the origin. Suppose given weights  $(a_1, \dots, a_n)$ . Let  $d$  be the least common multiple of the weights. Let  $h(z_1, \dots, z_n)$  be the

vector of monomials of weight  $d$  in some order, considered as a rational map from  $\mathbb{C}^n$  to  $P^{N-1}$ , where  $N$  is the number of monomials. Let  $\Gamma_h$  be the closure of the graph of  $h$ . By the naturality of blowups,  $\Gamma_h$  is the blowup of  $\mathbb{C}^n$  along the ideal generated by the monomials of weight  $d$ . Let  $v(z_1, \dots, z_n)$  be the same vector of monomials regarded as a map from  $P(a)$  to  $P^{N-1}$ . This ‘‘Veronese map’’ is well-defined, and it embeds  $P(a)$  in  $P^{N-1}$ . Finally, let  $f$  be the natural quotient (rational) map from  $\mathbb{C}^n$  to  $P(a)$ , and let  $\Gamma_f$  be the closure of its graph. Then  $v \circ f = h$ . Since  $v$  is an embedding, it induces an isomorphism  $\Gamma_f \cong \Gamma_g$ . To summarize:

LEMMA 4.3.2. *Suppose given weights  $(a_1, \dots, a_n)$ . Let  $d$  be their least common multiple, and let  $\mathcal{J}$  be the ideal generated by the monomials of weight  $d$ . Then the blowup of  $\mathbb{C}^n$  at the origin with weights  $(a_1, \dots, a_n)$  is isomorphic to the blowup of  $\mathbb{C}^n$  along  $\mathcal{J}$ .  $\square$*

Let  $X \subset \mathbb{C}^n$  be an analytic hypersurface with equation  $f_d(z) + f_{d+1}(z) + \dots$ , where  $f_k$  has weight  $k$  and  $d > 1$ . Then the proper transform  $\hat{X}$  of  $X$  under the weighted blowup is obtained as follows: delete the origin and replace it by the hypersurface in  $P(a)$  defined by  $f_d(z) = 0$ . That is, replace the origin by the weighted projectivization of the weighted tangent cone of  $X$ . We call  $\hat{X}$  the weighted blowup of  $X$  at the point corresponding to the origin.

Next we give a local equation for the chordal cubic at a point of its singular locus.

LEMMA 4.3.3. *Let  $T$  be the chordal cubic,  $R$  its rational normal curve, and  $P$  a point of  $R$ . Then there are local analytic coordinates  $x, u, v, w$  on a neighborhood of  $P$  in  $P^4$ , in which  $R$  is defined by  $u = v = w = 0$  and  $T$  by*

$$(4.3.2) \quad u^2 + v^2 + w^2 = 0.$$

PROOF. A hyperplane  $H$  transverse to  $R$  at  $P$  meets  $T$  in a cubic surface with a node at  $P$ . Consequently there are analytic coordinates  $u, v, w$  on  $H$  near  $P$  such that the equation for  $T \cap H$  is  $u^2 + v^2 + w^2 = 0$ . Now, there exists a 1-parameter subgroup  $X$  of  $\text{Aut } T = \text{PGL}(2, \mathbb{C})$  that moves  $P$  along  $R$ , and we take  $x$  as a coordinate on  $X$ . The natural map  $T \times X \rightarrow P^4$  gives the claimed local analytic coordinates.  $\square$

REMARK. If  $H$  is a hyperplane in  $P^4$  with a point of contact of order 4 with  $R$ , then there exist global algebraic coordinates on  $P^4 - H$  such that  $T - H$  is given by (4.3.2). The proof is an unenlightening sequence of coordinate transformations.

**First blowup.** One checks that  $\mathcal{V}_0 \subseteq P^5 \times \Delta$  is smooth away from  $R$  and transverse to  $\{s = 0\}$  away from  $R$ . To study  $\mathcal{V}_0$  near a point  $P$  of  $R$ , introduce local analytic coordinates  $x, u, v, w, z, s$  around  $P$  as follows. Begin with affine coordinates  $x, u, v, w$  around  $P$  in  $P^4$ , as in Lemma 4.3.2. Let  $z$  and  $s$  be the same  $z$  and  $s$  used above. Then a neighborhood of  $P$  in  $\mathcal{V}_0$  is

$$\{(x, u, v, w, z, s) \in (\text{some open set in } \mathbb{C}^6) \mid \\ u^2 + v^2 + w^2 + s^6 G(x, u, v, w) + z^3 = 0\} .$$

If  $P \in B$  (recall that  $B$  is the infinitesimal base locus defined in the first paragraph of section 4.1), then we may use the transversality of  $R$  and  $\{G = 0\}$  to change coordinates by  $x \rightarrow x + (\text{a function of } u, v, w)$ , so that  $\{G = 0\}$  is the same set as  $\{x = 0\}$ . That is,  $G$  is  $x$  times a nonvanishing function. Absorb a sixth root of this nonvanishing function into  $s$ . This yields a neighborhood of  $P$  of the form

$$(4.3.3) \quad \{(x, u, v, w, z, s) \in (\text{some open set in } \mathbb{C}^6) \mid \\ u^2 + v^2 + w^2 + s^6 x + z^3 = 0\} ,$$

where  $\mathcal{I} = \langle u, v, w \rangle$  and  $\sigma$  acts by  $z \rightarrow \omega z$ . If  $P \notin B$  then the same analysis leads to (4.3.3), but with the  $s^6 x$  term replaced by  $s^6$ . We will not discuss this case, because it is implicitly treated in the  $P \in B$  case (by looking at points of  $R$  near points of  $B$ ).

We defined the ideal sheaf  $\mathcal{J}$  in (4.3.1), defined  $\widehat{P^5 \times \Delta}$  as the blowup of  $P^5 \times \Delta$  along  $\mathcal{J}$ , and  $\mathcal{V}_1$  as the proper transform of  $\mathcal{V}_0$ . To understand its geometry, we write down generators for  $\mathcal{J}$  in our local coordinates and recognize  $\widehat{P^5 \times \Delta}$  as a weighted blowup along  $R$ . Since  $\mathcal{I} = \langle u, v, w \rangle$ ,  $\mathcal{J}$  is generated by the monomials of weight six, where we give the variables  $u, v, w, z, s$  the weights 3, 3, 3, 2, 1. By Lemma 4.3.2,  $\mathcal{V}_1$  is the weighted blowup of  $\mathcal{V}_0$  along the  $x$ -axis. By the discussion following that lemma, above a neighborhood of  $P$ , the exceptional divisor  $E_1$  is

$$(4.3.4) \quad \{(x, [u : v : w : z : s]) \in (\text{an open set in } \mathbb{C}) \times P(3, 3, 3, 2, 1) \mid \\ u^2 + v^2 + w^2 + z^3 + xs^6 = 0\} .$$

That is, over  $R - B$ ,  $E_1$  is a smooth fibration with fiber isomorphic to the hypersurface in  $P(3, 3, 3, 2, 1)$  defined by

$$(4.3.5) \quad u^2 + v^2 + w^2 + z^3 + s^6 = 0 .$$

The special fibers lie above the points of  $B \subset R$  and are copies of the weighted homogeneous hypersurface with equation  $u^2 + v^2 + w^2 + z^3 = 0$ .

The method we use for our detailed coordinate computations follows [20, pp. 166–167]. The blowup is covered by open sets  $B_u, B_v, B_w, B_z$  and  $B_s$ , each the quotient of an open set  $A_u, \dots, A_s$  of  $\widehat{\mathbb{C}^6}$  by a group of order 3, 3, 3, 2 or 1. We will treat  $A_u \rightarrow B_u \subseteq \widehat{P^5 \times \Delta}$  in some detail; the treatment of  $v$  and  $w$  is the same, and we will only briefly comment on  $B_z$  and  $B_s$ . One may take coordinates  $x, \dot{u}, \dot{v}, \dot{w}, \dot{z}, \dot{s}$  on  $A_u$ , with  $B_u = A_u/\langle \eta \rangle$ , where  $\eta$  is the transformation  $\dot{u} \rightarrow \omega \dot{u}, \dot{z} \rightarrow \omega \dot{z}, \dot{s} \rightarrow \bar{\omega} \dot{s}$ . The map  $A_u \rightarrow P^5 \times \Delta$  is given by  $u = \dot{u}^3, v = \dot{v}\dot{u}^3, w = \dot{w}\dot{u}^3, z = \dot{z}\dot{u}^2$  and  $s = \dot{s}\dot{u}$ ; these functions of  $\dot{u}, \dots, \dot{s}$  are  $\eta$ -invariant, so they define a map  $B_u \rightarrow P^5 \times \Delta$ . The preimage in  $A_u$  of the exceptional divisor of  $\widehat{P^5 \times \Delta} \rightarrow P^5 \times \Delta$  is  $\{\dot{u} = 0\}$ . The pullback to  $A_u$  of the defining equation of  $\mathcal{V}_0$  is  $\dot{u}^6(1 + \dot{v}^2 + \dot{w}^2 + \dot{s}^6x + \dot{z}^3) = 0$ , so the preimage in  $A_u$  of the proper transform  $\mathcal{V}_1$  is the hypersurface  $H$  given by  $1 + \dot{v}^2 + \dot{w}^2 + \dot{s}^6x + \dot{z}^3 = 0$ . One checks that this is a smooth hypersurface. Also, the pullback to  $A_u$  of the central fiber  $s = 0$  is  $\dot{s}\dot{u} = 0$ , and one checks that  $H$  meets  $\{\dot{u} = 0\}, \{\dot{s} = 0\}$  and  $\{\dot{u} = \dot{s} = 0\}$  transversely. Therefore  $s = 0$  defines a NCD in  $H$  with smooth components of multiplicity one. The one complication is that  $\mathcal{V}_1 \cap B_u$  is  $H/\langle \eta \rangle$  rather than  $H$  itself. Since  $\eta$  has fixed points in  $H$ ,  $\mathcal{V}_1$  turns out to be singular. We will resolve these singularities after discussing  $B_z$  and  $B_s$ .

The analysis of  $\mathcal{V}_1 \cap B_s$  is easy, since  $B_s = A_s$ . One writes down the equation for  $\mathcal{V}_1 \cap A_s$ , and checks smoothness and transversality to  $\{s = 0\}$ . In fact,  $A_s$  meets only one component of the central fiber,  $E_1$ . The analysis of  $\mathcal{V}_1 \cap B_z$  is only slightly more complicated. We have  $B_z = A_z/(\mathbb{Z}/2)$ . One writes down the equation for the preimage of  $\mathcal{V}_1$  in  $A_z$ , and checks its smoothness and that  $s = 0$  defines a NCD with smooth components. Then one observes that this hypersurface misses the fixed points of  $\mathbb{Z}/2$  in  $A_z$ , so that the same conclusions apply to  $\mathcal{V}_1 \cap B_z$ .

Now we return to  $\mathcal{V}_1 \cap B_u$ . It is convenient to make the coordinate change  $\dot{v} \rightarrow (\dot{v} + \dot{w})/2, \dot{w} \rightarrow (\dot{v} - \dot{w})/2i$  on  $A_u$ , so that  $\dot{v}^2 + \dot{w}^2$  is replaced by  $\dot{v}\dot{w}$ . Then the fixed-point set of  $\eta$  in  $H \subseteq A_u$  is  $\{(x, 0, \dot{v}, \dot{w}, 0, 0) \mid 1 + \dot{v}\dot{w} = 0\}$ . Therefore every fixed point has  $\dot{v} \neq 0$ , and we can use  $1 + \dot{v}\dot{w} + \dot{s}^6x + \dot{z}^3 = 0$  to solve for  $\dot{w}$  in terms of the other variables. The result is that there is an open set  $W$  in  $\mathbb{C}^5$ , with coordinates  $x, \dot{u}, \dot{v}, \dot{z}, \dot{s}$ , mapping isomorphically onto its image in  $H$ , with its image containing all the fixed points of  $\eta$  in  $H$ . The induced action of  $\eta$  on  $W$  is  $(x, \dot{u}, \dot{v}, \dot{z}, \dot{s}) \rightarrow (x, \omega \dot{u}, \dot{v}, \omega \dot{z}, \bar{\omega} \dot{s})$ . Therefore, a point on the singular locus of  $\mathcal{V}_1$  has a neighborhood in  $\mathcal{V}_1$  locally modeled on  $(W \subseteq \mathbb{C}^5)/\text{diag}[1, \omega, 1, \omega, \bar{\omega}]$ . Therefore the singular locus of  $\mathcal{V}_1$  is a

smooth surface  $S$ . The preimage in  $W$  of the central fiber  $s = 0$  is  $\dot{s}\dot{u} = 0$ . Note that this is  $\eta$ -invariant, hence well-defined on  $W/\langle\eta\rangle$ . The preimage in  $A_u$  of the exceptional divisor  $E_1$  is  $\dot{u} = 0$  and that of  $V'$  is  $\dot{s} = 0$ . Note that neither  $\dot{u}$  nor  $\dot{s}$  is  $\eta$ -invariant, so these equations don't make sense on  $W/\langle\eta\rangle$ . This reflects the fact that neither  $E_1$  nor  $V'$  is a Cartier divisor, but their sum is. (It should not alarm the reader that  $E_1$  is not Cartier, even though blowing up an ideal always gives a Cartier divisor; it is  $3E_1$  rather than  $E_1$  which is Cartier.) Finally, a lift to  $W$  of  $\sigma : \mathcal{V}_1 \rightarrow \mathcal{V}_1$  is  $\dot{z} \rightarrow \omega\dot{z}$ . This shows that  $\sigma$  acts trivially on  $S$ .

*Geometry of the central fiber.* The central fiber of  $\mathcal{V}_1$  is the variety

$$\mathcal{V}_1(0) = V' \cup E_1,$$

where the exceptional divisor has already been described as a fibration over  $R$ . The components have multiplicity one. The first component is a blowup of  $V$  along  $S$ . The exceptional divisor of  $V' \rightarrow V$  is  $V' \cap E_1$ , which in coordinates is given by (4.3.4) with the extra condition  $s = 0$ . Therefore,  $V' \cap E_1$  is a fiber bundle over  $R$  with fiber isomorphic to the hypersurface  $u^2 + v^2 + w^2 + z^3 = 0$  in  $P(3, 3, 3, 2)$ . One can check that this hypersurface is a copy of  $P^2$  (by projecting away from  $[0:0:0:1]$ ).

**Second blowup.** Our aim is now to resolve the singularities of  $\mathcal{V}_1$ . Henceforth, we will work with the local description of  $\mathcal{V}_1$  in terms of  $W/\langle\eta\rangle$ , rather than regarding  $\mathcal{V}_1$  as a hypersurface in a 6-dimensional space. We will see below (Lemma 4.3.4) that blowing up the ideal sheaf  $\mathcal{K}$  to get  $\mathcal{V}_2$  can be described in terms of  $W/\langle\eta\rangle$  as follows. Recall that  $W$  has coordinates  $x, \dot{u}, \dot{v}, \dot{z}, \dot{s}$  with  $\eta = \text{diag}[1, \omega, 1, \omega, \bar{\omega}]$ . Define  $\widehat{W}$  to be the weighted blowup of  $W$  along the  $x\text{-}\dot{v}$  plane, where  $\dot{u}$  and  $\dot{z}$  have weight 1 and  $\dot{s}$  has weight 2. It is natural to use these weights, because with respect to them,  $\eta$  is weighted-homogeneous, hence fixes the exceptional divisor pointwise. Then the map

$$\left( \text{the preimage of } W/\langle\eta\rangle \subseteq \mathcal{V}_1 \text{ in } \mathcal{V}_2 \right) \rightarrow W/\langle\eta\rangle$$

turns out to be equivalent to  $\widehat{W}/\langle\eta\rangle \rightarrow W/\langle\eta\rangle$ . This is the content of Lemma 4.3.4 below. As before, one covers  $\widehat{W}$  by open sets  $B_{\dot{u}}, B_{\dot{z}}$  and  $B_{\dot{s}}$ , which are quotients of open sets  $A_{\dot{u}}, A_{\dot{z}}$  and  $A_{\dot{s}}$  of  $\mathbb{C}^5$  by cyclic groups of orders 1, 1 and 2. By working in local coordinates, one can check that  $\eta$  acts on  $B_{\dot{u}} = A_{\dot{u}}$  by multiplying a single coordinate by  $\omega$ . Therefore  $B_{\dot{u}}/\langle\eta\rangle \subseteq \mathcal{V}_2$  is smooth. One also checks that  $s = 0$  defines a NCD with smooth components of multiplicity one. (The multiplicity is three along the preimage of  $E_2$  in  $B_{\dot{u}}$ , but in  $B_{\dot{u}}/\langle\eta\rangle$  the multiplicity is only one.) Exactly the same considerations apply to  $B_{\dot{z}}/\langle\eta\rangle \subseteq \mathcal{V}_2$ .

A similar analysis leads to the conclusion that  $B_{\mathfrak{s}}/\langle\eta\rangle$  is isomorphic to an open set in  $\mathbb{C}^5$ , modulo  $\mathbb{Z}/2$ , acting by negating three coordinates. Therefore  $\mathcal{V}_2$  is singular along a surface  $\Sigma$  which maps isomorphically to  $S$ . (It follows that  $\sigma$  acts trivially on  $\Sigma$ .) Furthermore, at a singular point,  $\mathcal{V}_2$  is locally modeled on

$$\mathbb{C}^2 \times (\text{the cone in } \mathbb{C}^6 \text{ on the Veronese surface } P^2 \subseteq P^5).$$

One checks that away from  $\Sigma$ , the central fiber  $\mathcal{V}_2(0)$  in  $B_{\mathfrak{s}}/\langle\eta\rangle$  is a NCD with smooth components of multiplicity one.

*Geometry of the central fiber.* The central fiber of  $\mathcal{V}_2$  is the variety

$$\mathcal{V}_2(0) = V'' \cup E'_1 \cup E_2.$$

One can check the following:

(a) The new exceptional divisor  $E_2$  is a  $P(1, 1, 2)$ -bundle over  $S$ . Note that  $P(1, 1, 2)$  is isomorphic to a cone in  $P^3$  over a smooth plane conic. The vertices of these cones comprise  $\Sigma$ .

(b) The map  $V'' \rightarrow V'$  has exceptional divisor  $V'' \cap E_2$ , which is a  $P^1$ -bundle over  $S$ . (Each fiber  $P^1$  is a smooth section of the quadric cone introduced in (a). In particular,  $V'' \cap \Sigma = \emptyset$ .)

(c) The map  $E'_1 \rightarrow E_1$  has exceptional divisor  $E'_1 \cap E_2$ , which is a  $P^1$ -bundle over  $S$ . (Each fiber  $P^1$  is a line through the vertex of the cone in (a).)

(d)  $V'' \cap E'_1$  projects isomorphically to  $V' \cap E_1$ . This is obvious away from  $S$ , and is true over  $S$  because the fiber of  $V'' \cap E'_1$  over a point of  $S$  is the intersection of the  $P^1$ 's in (b) and (c), which is a point.

(e) The triple intersection  $V'' \cap E'_1 \cap E_2$  is a point-bundle over  $S$ , hence isomorphic to  $S$ . This follows from (b) and (c).

**Third blowup.** The family  $\mathcal{V}_2$  is singular along the surface  $\Sigma$ , where it is locally modeled on  $\mathbb{C}^2 \times (\mathbb{C}^3/\{\pm I\})$ . A single ordinary blowup along  $\Sigma$ , i.e., blowing up along the ideal sheaf of  $\Sigma$ , resolves the singularity, and it is obvious that  $E_3$  is a  $P^2$ -bundle over  $\Sigma$ . One checks that the central fiber is a NCD with smooth components of multiplicity one. A convenient way to do the computations is to take the ordinary blowup of  $\mathbb{C}^5$  along  $\mathbb{C}^2$ , and then quotient by  $\mathbb{Z}/2$ . (This gives the same blowup.)

*Geometry of the central fiber.* The central fiber of  $\mathcal{V}_3$  is the variety

$$\mathcal{V}_3(0) = V''' \cup E''_1 \cup E'_2 \cup E_3.$$

The components are of multiplicity one and smooth and transverse. We have already observed that  $E_3$  is a  $P^2$ -bundle over  $\Sigma$ . Its intersections with  $E''_1$  and  $E'_2$  are  $P^1$ -bundles over  $\Sigma$ ; in fact the  $P^1$ 's are lines in

the  $P^2$ 's. The triple intersection  $E_3 \cap E_2' \cap E_1''$  projects isomorphically to  $\Sigma$ .  $E_3$  does not meet  $V'''$ , so  $V''' = V''$  and the intersections of  $V'''$  with  $E_1''$  and  $E_2'$  are the same as  $V''$ 's intersections with  $E_1'$  and  $E_2$ , described earlier. Since  $E_1' \cap E_2$  is a  $P^1$ -bundle over  $S$  (part (b) of the ‘Geometry of the central fiber’ of the second blowup), blowing up  $S$  makes  $E_1'' \cap E_2'$  into a  $P^1$ -bundle over  $\Sigma$ .

All that remains for the proof of Theorem 4.3.1 is to explain why blowing up the ideal sheaf  $\mathcal{K}$  coincides with our description  $\widehat{W}/\langle\eta\rangle \rightarrow W/\langle\eta\rangle$  of the second blowup. We formalize this as a lemma:

LEMMA 4.3.4. *Suppose  $W$  is an open set in  $\mathbb{C}^5$ , with coordinates  $x, \dot{u}, \dot{v}, \dot{z}$  and  $\dot{s}$ , and is invariant under  $\eta = \text{diag}[1, \omega, 1, \omega, \bar{\omega}]$ . Suppose  $V' \subseteq W/\langle\eta\rangle$  is the image of  $\{\dot{s} = 0\} \subseteq W$ , and let  $\mathcal{K}$  be the ideal sheaf on  $W/\langle\eta\rangle$  generated by the regular functions vanishing to order 3 at a generic point of  $V'$  and the regular functions vanishing to order 2 along the singular locus  $S$  of  $W/\langle\eta\rangle$ . Let  $\widehat{W}$  be the weighted blowup of  $W$  along  $\dot{u} = \dot{z} = \dot{s} = 0$ , with  $\dot{u}$  and  $\dot{z}$  having weight 1 and  $\dot{s}$  having weight 2. Then  $W \rightarrow W/\langle\eta\rangle$  induces an isomorphism from  $\widehat{W}/\langle\eta\rangle$  to the blowup of  $W/\langle\eta\rangle$  along  $\mathcal{K}$ .*

PROOF. We begin by observing that the nine monomials  $x, \dot{v}, \dot{u}^3, \dot{u}^2\dot{z}, \dot{u}\dot{z}^2, \dot{z}^3, \dot{u}\dot{s}, \dot{z}\dot{s}$  and  $\dot{s}^3$  generate the invariant ring of  $\eta$ , so that evaluating them embeds  $W/\langle\eta\rangle$  in  $\mathbb{C}^9$ . It is easy to see that  $\mathcal{K}$  is the ideal generated by (i) the quadratic monomials in  $\dot{u}^3, \dot{u}^2\dot{z}, \dots, \dot{s}^3$ , and (ii) the linear function  $\dot{s}^3$ . (We remark that the function  $\dot{s}^3$  vanishes to order 3 along  $V'$ , at a generic point of  $V'$ , and generates the ideal of such functions. However, even though  $S$  lies in  $V'$ ,  $\dot{s}^3$  only vanishes to order one at a point of  $S$ , because  $\dot{s}^3$  is one of the coordinate functions on  $\mathbb{C}^9$ . This is related to the fact that  $V'$  is not a Cartier divisor, but  $3V'$  is.) Therefore, blowing up  $\mathcal{K}$  amounts to defining  $\widehat{\mathbb{C}^9}$  as the weighted blowup of  $\mathbb{C}^9$  along the  $x$ - $\dot{v}$  plane, with weights  $1, \dots, 1, 2$ , and taking the proper transform therein of  $W/\langle\eta\rangle$ . One can cover  $\widehat{W}$  and  $\widehat{\mathbb{C}^9}$  by open sets, write down the rational map from  $\widehat{W}$  to  $\widehat{\mathbb{C}^9}$  explicitly, and check that it is regular and induces an embedding  $\widehat{W}/\langle\eta\rangle \rightarrow \widehat{\mathbb{C}^9}$ .  $\square$

REMARK. The proof conceals the origin of the choice of  $\mathcal{K}$ . We found it as follows. We knew we wanted to take the  $(1, 1, 2)$  weighted blowup of  $W$ , which is to say, blow up the ideal  $\langle\dot{u}^2, \dot{u}\dot{z}, \dot{z}^2, \dot{s}\rangle$ . The problem is that its generators are not  $\eta$ -invariant, so they do not define functions on  $W/\langle\eta\rangle$ . The solution was to blow up the cube of this ideal rather than the ideal itself. We wrote down the generators of the cube, and expressed them in terms of the generators for the invariant

ring of  $\eta$ . Every generator of the ideal was a quadratic monomial in the generating invariants, except for  $s^3$ , which was itself one of the generating invariants. This suggested that our weighted blowup of  $\mathbb{C}^9$  would give the desired blowup  $\widehat{W}/\langle\eta\rangle \rightarrow W/\langle\eta\rangle$ . Then we just checked the construction.

#### 4.4. Cohomology computations

In this section we describe the 3-part of the middle cohomology of the central fiber  $\widehat{V}$  of the semistable model  $\mathcal{V}_3$ , in a sequence of lemmas.

LEMMA 4.4.1. *Let  $M'$  and  $M$  be algebraic varieties on which an automorphism  $\sigma$  of finite order acts. Let  $f : M' \rightarrow M$  be a morphism with connected fibers which is equivariant with respect to this action. Suppose  $S \subset M$  is a subspace, let  $D = f^{-1}(S)$ , and assume that (a)  $f : M' - D \rightarrow M - S$  is an isomorphism, (b)  $f : D \rightarrow S$  is a fiber bundle, and (c)  $\sigma$  acts trivially on  $S$  and trivially on the rational cohomology of the fiber. Then the map of characteristic subspaces*

$$H_{(k)}^*(M; \mathbb{Q}) \rightarrow H_{(k)}^*(M'; \mathbb{Q})$$

*is an isomorphism for each  $k \neq 1$ .*

PROOF. Consider the Leray spectral sequence for the map  $f : M' \rightarrow M$ . Its abutment is the cohomology of  $M'$ . Its  $E_1$  terms are of the form  $E_1^{p,q} = H^p(M, R^q f_* \mathbb{Q})$ . The automorphism  $\sigma$  acts on  $M, M'$ , and the spectral sequence. Thus we can speak of the spectral sequence for the  $k$ -part of the cohomology of  $M'$ . Because  $f$  is an isomorphism over  $M - S$  and has connected fibers over  $S$ , there are two kinds of terms of the spectral sequence. For the first, the support of the coefficient sheaf is all of  $M$ . These are the terms  $H^p(M, (f_* \mathbb{Q})_{(k)})$ , which are isomorphic to  $H_{(k)}^p(M, \mathbb{Q})$ . For the other terms the support of the coefficient sheaf is  $S$ . These are the terms  $H^p(S, (R^q f_* \mathbb{Q})_{(k)})$  with  $q > 1$ . However, since the action on the cohomology of the fibers is trivial, these spaces are zero. Therefore the spectral sequence degenerates and we have the isomorphism

$$H_{(k)}^*(M, \mathbb{Q}) \cong H_{(k)}^*(M, f_* \mathbb{Q}) \cong H_{(k)}^*(M', \mathbb{Q}).$$

This completes the proof.  $\square$

LEMMA 4.4.2. *The 3-part of the middle cohomology of the central fiber of  $\mathcal{V}_3$ , with  $\mathbb{Q}$  coefficients, is*

$$H_{(3)}^4(\widehat{V}; \mathbb{Q}) \cong H_{(3)}^4(V; \mathbb{Q}) \oplus H_{(3)}^4(E_1; \mathbb{Q}).$$



PROOF. First, if  $A$  is an intersection of two or more components of  $\hat{V}$ , then  $H_{(3)}^4(A; \mathbb{Q}) = 0$ . One sees this by using Lemma 4.4.1 and the explicit descriptions of the intersections, given in section 4.3. That is, each of  $E_3 \cap E'_2$  and  $E_3 \cap E''_1$  is a  $P^1$ -bundle over  $\Sigma$ , and  $\sigma$  acts trivially on  $\Sigma$ . Also,  $E_3 \cap E'_2 \cap E''_1$  projects isomorphically to  $\Sigma$ . Finally, the nonempty intersections involving  $V'''$  are  $V''' \cap E''_1$ ,  $V''' \cap E'_2$  and  $V''' \cap E''_1 \cap E'_2$ , to which essentially the same argument applies. Then the Mayer-Vietoris spectral sequence implies that with  $\mathbb{Q}$  coefficients we have

$$H_{(3)}^4(\hat{V}) = H_{(3)}^4(V''') \oplus H_{(3)}^4(E''_1) \oplus H_{(3)}^4(E'_2) \oplus H_{(3)}^4(E_3).$$

We continue to use Lemma 4.4.1.  $E_3$  is a  $P^2$ -bundle over  $\Sigma$ , so the last term vanishes. Also,  $E'_2 \rightarrow E_2$  induces an isomorphism on  $H_{(3)}^4(\cdot; \mathbb{Q})$ , and similarly for  $E''_1 \rightarrow E'_1 \rightarrow E_1$  and  $V''' = V'' \rightarrow V' \rightarrow V$ . Finally,  $E_2$  is a  $P(1, 1, 2)$ -bundle over  $S$ , and  $\sigma$  acts trivially on  $S$ , so  $H_{(3)}^4(E_2; \mathbb{Q}) = 0$  and the lemma follows.  $\square$

LEMMA 4.4.3. *There is an isomorphism*

$$H_{(3)}^4(V) \cong \mathcal{E},$$

where  $\mathcal{E}$  is the rank one free Eisenstein module of type  $(2, 2)$ .

PROOF. By Morse theory,  $P^5$  is obtained from a neighborhood of  $V$  by attaching cells of dimension five and larger. Thus the cohomology of  $V$  is the same as that of  $P^5$  in dimensions less than four. It is known that  $V$  is a topological manifold (it follows from Lemma 4.3.2 that near  $R$ ,  $V$  is modeled on a sum of three squares and a cube; apply the first example of §9 of [27]). Therefore Poincaré duality implies that  $H^4(V)$  is torsion-free and that  $V$ 's cohomology in all dimensions except four is that of  $P^4$ .

A chordal cubic  $T$  has three natural strata: (i)  $R$  itself, (ii) the union of the tangent lines to  $R$ , minus their points of tangency, and (iii) the union of the secant lines, minus the points in which they cut  $R$ . These strata are a  $P^1$ , a  $\mathbb{C}$  bundle over  $P^1$ , and a  $\mathbb{C}^*$ -bundle over  $P^1 \times P^1$ . Adding the Euler characteristics of the strata, we find that  $\chi(T) = 4$ . Therefore the Euler characteristic of  $V$ , a three-sheeted cover of  $P^4$  branched along  $T$ , is seven. It follows that  $H^4(V)$  is free of rank three. Since any  $\sigma$ -invariant cohomology pulls back from  $P^4$ , we conclude that  $H_{(3)}^4(V)$  has rank two. It is therefore a one-dimensional Eisenstein module, necessarily of type  $(2, 2)$ .  $\square$

**Note.** Suppose that  $\omega_t$  is a family of classes in  $H^{3,1}(V_t)_{\bar{\omega}}$ . Let  $\delta_t$  be a family of homology classes whose limit as  $t$  approaches zero is a

generator of the Eisenstein factor in the above decomposition. Then

$$\lim_{t \rightarrow 0} \int_{\delta_t} \omega_t = 0,$$

since the  $\mathcal{E}$ -component of the limit of  $\omega_t$  has type  $(2, 2)$ . Thus the linear equation

$$\int_{\delta} \omega = 0$$

is a necessary condition for a period vector to represent a chordal cubic. In other words, the limiting periods for degenerations to chordal cubics near  $V$  lie in the hyperplanes  $\delta^\perp$ .

Our final lemma treats the “interesting” part of the Hodge structure of the central fiber, describing it in terms of that of the curve  $C$  given by the equation (4.1.4).

LEMMA 4.4.4. *There is an isogeny*

$$H_{(3)}^4(E_1) \rightarrow H_{(6)}^1(C)$$

*of Eisenstein Hodge structures, of weight  $(-2, -1)$ .*

PROOF. It suffices to exhibit an isomorphism

$$(4.4.1) \quad H_{(3)}^4(E_1, \mathbb{Q}) \cong H_{(6)}^1(C; \mathbb{Q})$$

that carries each component  $H_{(3)}^{p,q}(E_1)$  of the Hodge decomposition,  $p + q = 4$ , to  $H_{(6)}^{p-2, q-1}(C)$ . The proof uses the existence of maps  $p : E_1 \rightarrow R$  and  $p' : C \rightarrow R$  via which we can compute the 3- and 6-parts of the cohomology using the Leray spectral sequence. The first order of business is to exhibit isomorphisms  $(R^3 p_* \mathbb{Q})_{(3)} \cong (p'_* \mathbb{Q})_{(6)}$ , and then to show that

$$H_{(3)}^4(E_1, \mathbb{Q}) \cong H^1(R^3 p_* \mathbb{Q})_{(3)} \cong H^1(R, (p'_* \mathbb{Q})_{(6)}) \cong H_{(6)}^1(C; \mathbb{Q})$$

The smooth fibers  $F$  of the map  $p$  can be identified with the hypersurface  $u^2 + v^2 + w^2 + z^3 + s^6 = 0$  in  $P(3, 3, 3, 2, 1)$ . By the Griffiths residue calculus, Proposition 1.2, we see that  $h^{3,0}(F) = 0$  and  $h^{2,1}(F) = 1$ , where a generator of  $H^{2,1}(F)$  is given by the residue of the form

$$(4.4.2) \quad \Phi = \frac{\Omega}{(u^2 + v^2 + w^2 + z^3 + s^6)^2}.$$

The  $(2, 1)$  cohomology belongs to the 3-part since  $\sigma\Omega = \omega\Omega$  and the denominator is invariant. By the Lefschetz hyperplane theorem, the cohomology of  $F$  is like that of a weighted projective space in other degrees, see the Corollary to Theorem 4.2.2 of [14]. It follows that for

points  $x \in R - B$ , (a)  $(R^3 p_* \mathbb{Q}_x)_{(3)}$  is a rank one Eisenstein module, (b)  $(R^i p_* \mathbb{Q}_x)_{(3)} = 0$  if  $i \neq 3$ .

Via the projection  $(u, v, w, z, s) \mapsto (w, z, s)$ , the fiber  $F$  can be presented as a conic bundle over  $P(3, 2, 1)$  with discriminant the Eisenstein elliptic curve  $\mathcal{E}$ ,  $w^2 + z^3 + s^6 = 0$ . This description gives an alternative computation of the cohomology of  $F$ .

The points of the base locus  $B$  are critical points of  $p$ . Near a critical point, the map is given by projection onto the  $x$ -axis from the variety  $u^2 + v^2 + w^2 + z^3 + xs^6 = 0$ . Group the terms of the equation as  $(u^2 + v^2) + (w^2 + z^3 + xs^6) = 0$ . The second term describes the degeneration of an Eisenstein elliptic curve to a cusp. Its monodromy can be written as  $(-1) \otimes \omega$  via the Sebastiani-Thom theorem [35]. The monodromy of  $p$  is the double suspension of this, given by  $(-1) \otimes (-1) \otimes (-1) \otimes \omega$ , as in the proof of Lemma 1.3. The transformation  $\sigma$ , which acts on weighted projective space by  $[u : v : w : z : s] \mapsto [u : v : w : \omega z : s]$ , acts on cohomology by  $1 \otimes 1 \otimes 1 \otimes \omega$ . Thus the local monodromy at a critical point corresponding to a positively oriented loop  $\gamma$  of winding number one is  $\rho(\gamma) = -\sigma$ . It follows that the sheaf  $(R^3 p_* \mathbb{Q})_{(3)}$  is supported on  $R - B$ , and on that space it is a local system associated to the representation  $\rho$ . This representation sends the “standard generators” of  $\pi_1(R - B)$  to  $-\sigma$ .

The potentially nonzero initial terms in the Leray spectral sequence of  $(p_* \mathbb{Q})_{(3)}$  are  $H^i(R, (R^3 p_* \mathbb{Q})_{(3)})$  for  $i = 0, 1, 2$ . A nonzero class for the case  $i = 0$  would represent an invariant cycle for the local system  $(R^3 p_* \mathbb{Q})_{(3)}$ . By the discussion above, there are no nonzero invariants of the monodromy representation. Therefore  $H^0(R, (R^3 p_* \mathbb{Q})_{(3)}) = 0$ .

The sheaf cohomology group  $H^2(R, (R^3 p_* \mathbb{Q})_{(3)})$  is the same as the cohomology  $H^2(R - B, (R^3 p_* \mathbb{Q})_{(3)})$  of the indicated local system. But  $R - B$  has the homotopy type of a 1-complex, so this group vanishes. Because the Leray spectral sequence for the 3-part has just a single initial term, one has the isomorphism

$$(4.4.3) \quad H^4(E_1, \mathbb{Q})_{(3)} \cong H^1(R, (R^3 p_* \mathbb{Q})_{(3)}).$$

The isomorphism

$$(4.4.4) \quad H^1(C, \mathbb{Q})_{(6)} \cong H^1(R, (p'_* \mathbb{Q})_{(6)})$$

is elementary: it expresses the fact that action of the sixth roots of unity on  $C$  is local.

To show that the coefficient systems are isomorphic, consider the monodromy representation of  $(p'_* \mathbb{Q})_{(6)}$ . It is the restriction to the 6-part of the monodromy representation on  $p'_* \mathbb{Q}$ . Let  $t$  be a local coordinate

on  $R$  such that  $t = 0$  is a point of  $B$ . Then the cover  $C \rightarrow R$  is given locally by  $(z, t) \mapsto t$  on the analytic set  $z^6 = t$ . The fiber over  $t$  is the set  $\{ \zeta^i t^{1/6} \mid i = 0, \dots, 5 \}$ , where  $t^{1/6}$  is some fixed sixth root of  $t$ . The fiber of  $p'_*\mathbb{Q}$  is the rational vector space generated by this set. The monodromy for a positively oriented loop  $\gamma$  of winding number one is given by multiplication by  $\zeta = \exp(2\pi i/6)$ . The action of  $\zeta$  also gives the decomposition into characteristic sheaves. One can check that  $(p'_*\mathbb{Q})_{(6)}$  is the rational vector space generated by  $1 - \zeta^2 - \zeta^3 + \zeta^5$  and  $\zeta + \zeta^2 - \zeta^4 - \zeta^5$ , and that multiplication by  $\zeta$  restricted to it is a transformation of order six. There is a choice of Eisenstein structure on  $(p'_*\mathbb{Q})_{(6)}$ : the one generated by  $\omega$  and the one generated by  $\omega^2$ . We choose the latter so that  $\rho(\gamma) = -\sigma$ . Thus the monodromy of the two local systems in question can be expressed in the same way in terms of the Eisenstein structure. Consequently there is an isomorphism of local systems,

$$(Rp_*^3\mathbb{Q})_{(3)} \cong (p'_*\mathbb{Q})_{(6)}.$$

Combining the preceding with the isomorphisms (4.4.3) and (4.4.4), we obtain the isomorphism (4.4.1).

We claim next that the isomorphism (4.4.1) underlies an isomorphism of complex Hodge structures. A Hodge structure is defined via a theory of differential forms and a harmonic theory. These exist both for cohomology with coefficients in the complex numbers  $\mathbb{C}$  and with coefficients in a local system, provided that the local system is unitary. This is the case for the eigensystems of the local systems considered above. Consider the local system  $R^3p_*\mathbb{Z}$ . There is a decomposition

$$(4.4.5) \quad R^3p_*\mathbb{Z} \otimes \mathbb{C} = (R^3p_*\mathbb{C})_\omega \oplus (R^3p_*\mathbb{C})_{\bar{\omega}}$$

into eigenspaces of the deck transformation  $\sigma$ . Each of the local systems on the right is unitary. Moreover, each has a Hodge type inherited from the Hodge structure on the cohomology of the fiber. To determine the relation between the types and the eigenvalues, recall that the form  $\Phi$  in equation (4.4.2) is of type  $(2, 1)$  and transforms as  $\sigma\Phi = \omega\Phi$ . Since  $H^3(F)$  is two-dimensional, it follows that

$$(R^3p_*\mathbb{C})_\omega = (R^3p_*\mathbb{C})_{\bar{\omega}}^{2,1}.$$

Therefore the complex Hodge structure

$$H^4(E_1, C)_\omega \cong H^1(R - B, (R^3p_*\mathbb{C})_\omega) = H^1(R - B, (R^3p_*\mathbb{C})_{\bar{\omega}}^{2,1})$$

has type  $(3, 1) + (2, 2)$ . Similar considerations yield an isomorphism

$$H_{(6),\omega}^1(C) \cong H_{(6),\omega}^1(R - B, (p'_*\mathbb{C})),$$

where the decomposition is relative to the action of  $\zeta$  and the local system can be considered as a local system of Hodge structures of type  $(0, 0)$ . Finally, the isomorphism

$$(R^3 p_* \mathbb{C})_\omega \cong (p'_* \mathbb{C})_{(6), \omega} \Leftrightarrow (R^3 p_* \mathbb{C})_{(3)}^{2,1} \cong (p'_* \mathbb{C})_{(6), \omega}^{0,0}$$

of unitary systems of complex Hodge structures gives an isomorphism of complex Hodge structures

$$H_\omega^4(E_1, \mathbb{C}) \cong H_{(6), \omega}^1(C).$$

of type  $(-2, -1)$ . The proof is now complete. □

REMARK. (1) As motivation for the choice of Eisenstein structure on  $C$ , we compute the action of  $\sigma$  on a nonzero element — in fact, a generator — of the space  $H^{1,0}(C)_{(6)}$ . Consider the expression

$$\Psi = \frac{\Omega(x, y)}{z}.$$

It defines a holomorphic one-form on  $C$ . The sixth root of unity acts by  $\Psi \mapsto \zeta^{-1}\Psi$ , so that  $\Psi = \Psi_{(6)}$ . Moreover,  $\sigma\Psi = \omega\Psi$ . Thus a holomorphic form in the 6-part of  $H^1(C)$  has eigenvalue  $\omega$  with respect to  $\sigma$ .

(2) Another approach to the Hodge structure of  $E_1$  is the following. One can show that  $E_1$  is birational to the hypersurface  $Z$  in  $P(1, 1, 6, 6, 6, 4)$  given by the equation

$$f(x, y) + u^2 + v^2 + w^2 + z^3 = 0.$$

(This is plausible because the projection  $(x, y, u, v, w, z) \mapsto (x, y)$  blows up  $Z$  along  $\{x = y = 0\}$ , yielding  $\hat{Z}$ . And  $\hat{Z}$  fibers over  $P^1$  with the same generic fiber and same special fibers as in the fibration of  $E_1$  over  $R$ .) Presumably,  $E_1 \dashrightarrow Z$  induces an isogeny of Hodge structures. Standard calculations with the Griffiths Jacobian calculus yields the Hodge numbers  $(0, 1, 9, 0, 0)$  for  $H^4(Z)_\omega$ , with the period map having 9-dimensional image.

## CHAPTER 5

### Degeneration to a Nodal Cubic

The goal of this chapter is to identify the limit Hodge structure for the degeneration of cyclic cubic fourfolds associated to a generic nodal degeneration of cubic threefolds. It is very similar to the previous chapter, so we will be more brief. The result of this chapter that is used in the proof of the main theorem of the monograph (Theorem 6.1) is the following:

**THEOREM 5.1.** *The period map  $g : \widehat{PC}_{ss} \rightarrow P\Gamma \backslash \mathbb{C}H^{10}$  carries the discriminant locus onto a divisor.*

One should expect such a result, because at a node of a cubic threefold  $T$ , the tangent cone is a cone over  $P^1 \times P^1$ , and the lines on  $T$  through the node sweep out a  $(3, 3)$  curve in the  $P^1 \times P^1$ . The generic genus 4 curve arises this way, providing 9 moduli. Our approach is to show that the interesting part of the limiting Hodge structure is that of the K3 surface  $K$  which is the 3-fold cover of  $P^1 \times P^1$  branched over this curve. This Hodge structure is the same as the one studied by Kondō in his work on moduli of genus four curves [22].

Suppose  $T$  is a generic nodal cubic threefold; we choose homogeneous coordinates  $x_0, \dots, x_4$  such that the node is at  $[1:0:\dots:0] \in P^4$  and  $T$  has defining equation

$$F(x_0, \dots, x_4) = x_0(x_1x_2 + x_3x_4) + f(x_1, \dots, x_4) = 0,$$

where  $f$  is a homogeneous cubic. Then  $V \subseteq P^5$  is defined by  $F + x_5^3 = 0$ , and we will project  $V$  away from  $P = [1:0:0:0:0:0]$  to the  $P^4$  with homogeneous coordinates  $x_1, \dots, x_5$ . Every reference to  $P^4$  will be to this  $P^4$ . We may suppose by genericity that the intersection  $K$  of  $x_1x_2 + x_3x_4 = 0$  and  $f(x_1, \dots, x_4) + x_5^3 = 0$  in  $P^4$  is smooth. The importance of this surface is that projection away from the cusp of  $V$  at  $P = (1, 0, 0, 0, 0, 0)$  is an isomorphism from  $V$ , minus  $P$  and the preimage of  $K$ , to its image. Over  $K$ , the projection is a  $\mathbb{C}$ -bundle. The notation reflects the fact that  $K$  is a K3 surface, since it is a smooth  $(2, 3)$ -intersection in  $P^4$ . We will write  $Q$  for the quadric  $x_1x_2 + x_3x_4 = 0$  in  $P^4$ . The vertex of  $Q$  is not in  $K$ , because  $K$  is smooth.

By scaling the variables, we may suppose that  $F + tx_0^3$  defines a smooth threefold for all  $t \in \Delta - \{0\}$ . This pencil of threefolds is well-suited to our coordinates, but any pencil gives the same limiting Hodge structure, provided that its generic member is smooth. As in chapter 4 we will write  $\mathcal{V}$  for the associated family of cubic fourfolds:

$$\mathcal{V} = \left\{ ([x_0 : \dots : x_5], t) \in P^5 \times \Delta \mid \right. \\ \left. x_0(x_1x_2 + x_3x_4) + f(x_1, \dots, x_4) + x_5^3 + tx_0^3 = 0 \right\}.$$

One checks that  $\mathcal{V}$  is smooth away from the cusp  $P$  of the central fiber, and of course the central fiber itself is smooth away from  $P$ . We let  $\mathcal{V}_0$  be the degree six base extension, got by setting  $t = s^6$ . By Lemma 1.4, the monodromy of  $\mathcal{V}$  on  $H^4$  over  $\Delta - \{0\}$  has order 3; therefore of monodromy of  $\mathcal{V}_0$  is trivial. Below, we will define  $\mathcal{V}_1$ ,  $\mathcal{V}_2$  and  $\mathcal{V}_3$  by repeated blowups. The last blowup  $\mathcal{V}_3$  turns out to be a semistable model for the degeneration. The reason we take six rather than three as the degree of the base extension is that this choice makes the multiplicities of the components of the central fiber of  $\mathcal{V}_3$  all be 1. As in chapter 4 we write  $E_i$  for the exceptional divisor of  $\mathcal{V}_i \rightarrow \mathcal{V}_{i-1}$ , and we use primes to indicate proper transforms of  $V$  and the  $E_i$ .

Regarding  $x_1, \dots, x_5$  as affine coordinates on  $\mathbb{C}^5 = \{x_0 \neq 0\} \subseteq P^5$ , we have

$$\mathcal{V}_0 \cap (\mathbb{C}^5 \times \Delta) = \left\{ (x_1, \dots, x_5, s) \in \mathbb{C}^5 \times \Delta \mid \right. \\ \left. x_1x_2 + x_3x_4 + f(x_1, \dots, x_4) + x_5^3 + s^6 = 0 \right\}.$$

By the Morse lemma, there is a neighborhood  $W$  of the origin in  $\mathbb{C}^4$  with analytic coordinates  $y_1, \dots, y_4$ , such that the  $x_i$  and  $y_i$  agree to first order, and

$$x_1x_2 + x_3x_4 + f(x_1, \dots, x_4) = y_1y_2 + y_3y_4.$$

To emphasize the analogy with the chordal case, we will write  $z$  for  $x_5$ . Then

$$\mathcal{V}_0 \cap (W \times \mathbb{C} \times \Delta) = \left\{ (y_1, \dots, y_4, z, s) \in W \times \mathbb{C} \times \Delta \mid \right. \\ \left. y_1y_2 + y_3y_4 + z^3 + s^6 = 0 \right\}.$$

The obvious thing to do is blow up the origin with weights (3,3,3, 3,2,1). To describe this intrinsically, we define  $\mathcal{I}$  to be the ideal sheaf on  $P^5 \times \Delta$  defining the subvariety

$$(\text{the line in } P^5 \text{ joining } [1:0:0:0:0] \text{ and } [0:0:0:0:1]) \times \Delta,$$

and we define  $\mathcal{J}$  as in (4.3.1). We define  $\widehat{P^5 \times \Delta}$  as the blowup of  $P^5 \times \Delta$  along  $\mathcal{J}$ , and  $\mathcal{V}_1$  as the proper transform of  $\mathcal{V}_0$ . The exceptional divisor in  $\widehat{P^5 \times \Delta}$  is a copy of  $P(3, 3, 3, 3, 2, 1)$ , and the exceptional divisor  $E_1$  of  $\mathcal{V}_1 \rightarrow \mathcal{V}_0$  is the hypersurface in  $P(3, 3, 3, 3, 2, 1)$  defined by

$$(5.1) \quad y_1 y_2 + y_3 y_4 + z^3 + s^6 = 0 .$$

As in chapter 4, we will use primes to indicate proper transforms of  $V$  and the various exceptional divisors. The exceptional divisor of  $V' \rightarrow V$  is the hypersurface in  $P(3, 3, 3, 3, 2)$  defined by

$$y_1 y_2 + y_3 y_4 + z^3 = 0 \text{ and } s = 0 .$$

This lets one see that  $\sigma$  acts trivially on  $V' \cap E_1$ , because  $\sigma = \text{diag}[1, 1, 1, 1, \omega]$  acts on  $P(3, 3, 3, 3, 2)$  in the same way as the quasihomogeneous scaling by  $\bar{\omega}$ , which of course acts trivially. Finally, calculations strictly analogous to those of chapter 4 show that  $\mathcal{V}_1$  is smooth away from the surface  $S$  in  $P(3, 3, 3, 3) \cong P^3$  defined by (5.1) and  $s = z = 0$ .  $S$  is a smooth quadric surface. Furthermore, any point of  $S$  admits a neighborhood in  $\mathcal{V}_1$  isomorphic to a neighborhood of the origin in  $\mathbb{C}^5$ , modulo  $\eta = \text{diag}[\omega, 1, 1, \omega, \bar{\omega}]$ . This is exactly the same as the local model near the singular set of chapter 4's  $\mathcal{V}_1$ .

The second blowup  $\mathcal{V}_2$  is the blowup of  $\mathcal{V}_1$  along the ideal sheaf  $\mathcal{K}$ , where  $\mathcal{K}$  is defined exactly as in chapter 4: it is generated by the regular functions which either vanish to order 2 along  $S$ , or vanish to order 3 at a generic point of  $V'$ . Then the exceptional divisor  $E_2$  is a  $P(1, 1, 2)$ -bundle over  $S$ . Now,  $P(1, 1, 2)$  is isomorphic to a cone in  $P^3$  over a smooth plane conic, so each fiber has a singular point. These turn out to be singular in  $\mathcal{V}_2$  as well, and constitute the entire singular locus  $\Sigma$  of  $\mathcal{V}_2$ , so  $\Sigma$  is a copy of  $S$ . Furthermore, near  $\Sigma$ ,  $\mathcal{V}_2$  is locally modeled on  $\mathbb{C}^2 \times (\mathbb{C}^3 / \{\pm I\})$ , just as in chapter 4. It turns out, also as in chapter 4, that  $V'' \cap \Sigma = \emptyset$ . Finally, we define  $\mathcal{V}_3$  as the ordinary blowup of  $\mathcal{V}_2$  along  $\Sigma$ . Then  $E_3$  is a  $P^2$ -bundle over  $\Sigma$ . One can check that  $\mathcal{V}_3$  is smooth, and that the central fiber is a normal crossing divisor with smooth components  $V'''$ ,  $E_1'$ ,  $E_2'$  and  $E_3$  of multiplicity one.

Now we will study the central fiber of  $\mathcal{V}_3$ , in order to determine the limiting Hodge structure. The first step is to rid ourselves of most of the complication introduced by our blowups. Then we will study what remains, the Hodge structures of  $V'''$  and  $E_1$ .

$$\text{LEMMA 5.2. } H_{(3)}^4(V''' \cup E_1' \cup E_2' \cup E_3; \mathbb{Q}) = H_{(3)}^4(V''; \mathbb{Q}) \oplus H_{(3)}^4(E_1; \mathbb{Q}).$$

PROOF. This is analogous to Lemma 4.4.2. We use cohomology with rational coefficients throughout the proof. For the first step, the



essential facts are the following. (i) The restriction of the projection  $\mathcal{V}_3 \rightarrow \mathcal{V}_2$  to the central fiber is an isomorphism to  $V'' \cup E'_1 \cup E_2$ , except over  $\Sigma$ . (ii) Over  $\Sigma$ , it is a  $P^2$ -bundle. (iii)  $\sigma$  fixes  $\Sigma$  pointwise. Then Lemma 4.4.1 implies

$$H_{(3)}^4(V''' \cup E'' \cup E'_1 \cup E_2 \cup E_3) = H_{(3)}^4(V'' \cup E'_1 \cup E_2) .$$

For the second step, the essential facts are (iv)  $E_2$  is a  $P(1, 1, 2)$ -bundle over  $S$ , (v)  $E_2 \cap E'_1$  and  $E_2 \cap V''$  are  $P^1$ -bundles over  $S$ , (vi)  $E_2 \cap E'_1 \cap V''$  projects isomorphically to  $S$ , and (vii)  $\sigma$  fixes  $S$  pointwise. Then Mayer-Vietoris implies

$$H_{(3)}^4(V'' \cup E'_1 \cup E_2) = H_{(3)}^4(V'' \cup E'_1) .$$

For the third step, the essential fact is (viii)  $\sigma$  acts trivially on  $V' \cap E_1$ . One can check that  $V'' \cap E'_1$  is the proper transform of  $V' \cap E_1$ , so  $\sigma$  acts trivially on  $V'' \cap E'_1$ . Then Mayer-Vietoris implies

$$H_{(3)}^4(V'' \cup E'_1) = H_{(3)}^4(V'') \oplus H_{(3)}^4(E'_1) .$$

For the final step, the essential facts are (ix)  $E'_1 \cap E_2$  is a  $P^1$ -bundle over  $S$ , so  $E'_1 \rightarrow E_1$  is an isomorphism except over  $S$ , over which it is a  $P^1$ -bundle, and (x)  $\sigma$  acts trivially on  $S$ . Then Lemma 4.4.1 implies  $H_{(3)}^4(E'_1) = H_{(3)}^4(E_1)$ . This completes the proof.  $\square$

LEMMA 5.3.  $H_{(3)}^4(E_1; \mathbb{Q})$  is 2-dimensional, of type  $(2, 2)$ .

PROOF. This is an exercise in the Griffiths residue calculus, Proposition 1.2. Regard  $E_1$  as a hypersurface in  $P(3, 3, 3, 3, 2, 1)$  with defining equation  $F = y_1y_2 + y_3y_4 + z^3 + s^6$ , as in (5.1). Because  $\Omega$  has weight fifteen, the nonzero rational differentials  $A\Omega/F^k$  with numerator polynomial of lowest degree are those with  $k = 3$  and  $A = c_1zs + c_2s^3 + (\text{terms in the } y_i)$ . Let  $\text{res}(A\Omega/F^k)$  be the Poincaré residue. The primitive part of  $H^4$  is spanned by  $\text{res}(sz\Omega/F^3)$  and  $\text{res}(s^3\Omega/F^3)$ . The automorphism  $\sigma$  acts on these by multiplication by  $\omega^2$  and  $\omega$ , respectively. Therefore  $H_{(3)}^4(E_1; \mathbb{Q})$  is 2-dimensional. It clearly has type  $(2, 2)$ .  $\square$

LEMMA 5.4.  $V''$  admits a morphism to  $P^4$  which is an isomorphism except over the K3 surface  $K$  and the vertex  $v$  of the quadric cone  $Q$ . Over  $K$ ,  $V''$  is a smooth  $P^1$ -bundle, and the preimage of  $v$  is  $E'_1 \cap V''$ .

PROOF. Let  $\phi$  be the rational map  $V \dashrightarrow P^4$  given by projection away from the cusp  $P$ . Then  $\phi$  induces rational maps  $\phi' : V' \dashrightarrow P^4$  and  $\phi'' : V'' \dashrightarrow P^4$ . One can check by lengthy local coordinate calculations that  $\phi''$  is a morphism, not just a rational map, and has the properties claimed in the lemma. We just give a summary.

Consider a line  $\ell$  in  $P^5$  through  $P$ . If  $\ell$  lies in  $V$  then  $\phi(\ell - \{P\})$  lies in  $K$ . We write  $D$  for the union of the lines on  $V$  through  $P$ . If  $\ell$  makes contact of order only 2 with  $V$  at  $P$ , then it meets  $V$  at exactly one further point, and does so transversely. If it makes contact of order exactly 3, then  $\ell$  meets  $V$  only at  $P$ , and the corresponding point of  $P^4$  is in  $Q$  but not in  $\phi(V - \{P\})$ . We conclude that  $\phi$  identifies  $V - D$  with  $P^4 - Q$ , and realizes  $D - \{P\}$  as a  $\mathbb{C}$ -bundle over  $K \subseteq Q$ . We write  $D''$  for the proper transform of  $D$  in  $V''$ .

In this proof we only care about divisors in  $V''$ , not in  $\mathcal{V}_2$ , so we will write  $e_1$  for  $V' \cap E_1$ ,  $e'_1$  for  $V'' \cap E'_1$  and  $e_2$  for  $V'' \cap E_2$ . One can check that  $e'_1$  is the proper transform of  $e_1$ .

Local calculations show that  $\phi'$  is regular except along  $S$ , and carries  $e_1 - S$  to  $v$ . Further calculations show that  $\phi''$  is regular on all of  $V''$ , and therefore  $\phi''$  carries  $e'_1$  to  $v$ . The behavior of  $\phi''$  on  $e_2$  is easy to understand. We know that  $e_2$  is a  $P^1$ -bundle over the smooth quadric surface  $S$ , and that  $Q$  is a cone over a smooth quadric surface. Therefore it is not surprising that (i)  $\phi''$  carries each fiber  $P^1$  of  $e_2$  isomorphically onto a line in  $Q$  through  $v$ , and (ii) the only points of  $e_2$  that  $\phi''$  carries to  $v$  are in  $e_2 \cap e'_1$ . It follows from (ii) that the fiber of  $\phi''$  over  $v$  is exactly  $e'_1$ .

It remains to show that  $\phi''$  is an isomorphism over  $P^4 - (K \cup \{v\})$  and a smooth  $P^1$ -bundle over  $K$ . Because  $\phi''$  maps  $V''$  onto  $P^4$ , the image of  $e_2$  must contain  $Q - (K \cup \{v\})$  and hence be equal to  $Q$ . Since  $P^4$  is smooth and  $D''$  and  $e'_1$  are the only divisors in  $V''$  that  $\phi''$  can crush to lower-dimensional varieties, Zariski's main theorem implies that  $\phi''$  is an isomorphism except over the image  $K \cup \{v\}$  of  $D'' \cup e'_1$ . So all that remains is to show that  $\phi''^{-1}(K)$  is a smooth  $P^1$ -bundle over  $K$ . First, by (i) above, each point of  $K$  has only one preimage in  $e_2$ , so each point of  $k$  has preimage  $\mathbb{C} \cup \{\text{point}\}$  in  $V''$ . It follows that  $D''$  is the full preimage of  $K$ . Also,  $\phi''$  restricts to an isomorphism  $e_2 - e'_1 \rightarrow Q - \{v\}$ . This implies that  $\phi''^{-1}(K) \cap e_2$  is a copy of  $K$ .

$D''$  is obviously smooth away from  $e_2$ . To show it is smooth at a point  $d$  of  $D'' \cap e_2$ , it suffices to exhibit a smooth curve in  $V''$  through  $d$  that is transverse to  $D''$ . Since  $\phi''^{-1}(K)$  is a copy of  $K$ , we can just take a curve in  $e_2$  transverse to this copy of  $K$ . Since  $e_2 - e'_1$  maps isomorphically to its image, the image curve is transverse to  $K$ , so the original curve must be transverse to  $D''$ . To show that  $D''$  is a fibration over  $K$ , we will show that the rank of  $\phi''|_{D''}$  is 2 everywhere. This clearly holds at every point of  $D'' - e_2$ . If the rank were  $< 2$  at a point  $d$  of  $D'' \cap e_2$ , then the rank of  $\phi'' : V'' \rightarrow P^4$  at  $d$  would be  $< 3$ , which is impossible since  $\phi''|_{e_2}$  is a local isomorphism at  $d$ .  $\square$

LEMMA 5.5. *The map  $\phi'' : V'' \rightarrow P^4$  of Lemma 5.4 induces an isomorphism*

$$H_{(3)}^4(V''; \mathbb{Q}) \cong H_{(3)}^2(K; \mathbb{Q}).$$

PROOF. From Lemma 5.4, one knows that

- (i) if  $x \in P^4 - (K \cup \{v\})$ , then  $\phi''^{-1}(x)$  is a point;
- (ii) The part of  $V''$  over  $K$  is a  $P^1$ -bundle;
- (iii)  $\phi''^{-1}(v) = E'_1 \cap V''$  is a copy of the hypersurface  $y_1y_2 + y_3y_4 + z^3$  in  $P(3, 3, 3, 3, 2)$ ; by projecting away from  $[0: \dots : 0: 1]$ , one checks that this is a copy of  $P^3$ . We saw above that  $\sigma$  acts trivially on it.

It follows that the only terms of the Leray spectral sequence which can contribute to  $H_{(3)}^4(V'')$  are

- (i')  $H_{(3)}^4(P^4, \phi''_*\mathbb{Q})$ ,
- (ii')  $H_{(3)}^2(K, R^2\phi''_*\mathbb{Q})$ ,
- (iii')  $H_{(3)}^0(v, R^4\phi''_*\mathbb{Q})$ .

Only the middle term (ii') is nonzero, and it is  $H_{(3)}^2(K; \mathbb{Q})$ . The lemma follows.  $\square$

LEMMA 5.6. *There is an isogeny of Eisenstein Hodge structures*

$$H_{(3)}^4(\hat{V}) \rightarrow H_{(3)}^2(K) \oplus \mathcal{E}$$

*of weight  $(-1, -1)$ , where  $\mathcal{E}$  indicates the 1-dimensional Eisenstein lattice with Hodge structure pure of type  $(1, 1)$ .*

PROOF. This follows from the isomorphisms

$$\begin{aligned} H_{(3)}^4(\hat{V}; \mathbb{Q}) &\cong H_{(3)}^4(V''; \mathbb{Q}) \oplus H_{(3)}^4(E_1; \mathbb{Q}) \\ &\cong H_{(3)}^2(K; \mathbb{Q}) \oplus \mathbb{Q}^2 \end{aligned}$$

of Lemmas 5.2, 5.3 and 5.5, together with the fact that the isomorphism  $H_{(3)}^4(V'') \rightarrow H_{(3)}^2(K; \mathbb{Q})$  is a map of Hodge structures of weight  $(-1, -1)$ . The latter fact comes from the fact that the restriction of  $V'' \rightarrow P^4$  to the preimage of  $K$  is a  $P^1$ -bundle over  $K$ .  $\square$

Now,  $K$  lies in the quadric cone  $Q$ , and projection away from the vertex realizes  $K$  as a 3-fold cover of  $P^1 \times P^1$  branched over a  $(3, 3)$  curve  $C$ . This curve has genus 4, and the generic genus 4 curve arises this way. Kondō [22] showed that  $H_{(3)}^2(K; \mathbb{Q})$  has dimension 20, and that the Hodge structure of  $K$  contains enough information to recover  $C$ , up to an automorphism of  $P^1 \times P^1$ . Therefore, as  $C$  varies over the  $(3, 3)$  curves in  $P^1 \times P^1$ , the associated K3 surfaces provide a 9-dimensional family of Hodge structures.

PROOF OF THEOREM 5.1: This is essentially the same as Theorem 4.1. The previous lemma and the exactness of the 3-part of the Clemens-Schmid sequence

$$\cdots \rightarrow H_{(3)}^4(\mathcal{V}_3, \mathcal{V}_3^*; \mathbb{Q}) \rightarrow H_{(3)}^4(\hat{V}; \mathbb{Q}) \rightarrow \lim_{s \rightarrow 0} H_{(3)}^4(\hat{V}_s; \mathbb{Q}) \rightarrow 0$$

imply that the limit Eisenstein Hodge structure is isogenous to that of  $K$  (plus a 1-dimensional summand). Then Kondō's work shows that the Eisenstein Hodge structures form a 9-dimensional family.  $\square$

## CHAPTER 6

### The Main Theorem

Recall that  $\mathcal{M}_0^f$  and  $\mathcal{M}_0$  are the moduli spaces of framed and unframed smooth cubic threefolds. We also define  $\mathcal{M}_{ss}$  as the GIT moduli quotient  $PC_{ss}/\mathrm{SL}(5, \mathbb{C})$ , and  $\mathcal{M}_s$  as the corresponding stable locus. Since we needed to blow up  $PC$  before extending the period map, we also define

$$\begin{aligned}\widehat{\mathcal{M}}_s^f &= P\mathcal{F}_s/PG \\ \widehat{\mathcal{M}}_s &= \widehat{PC}_s/PG \\ \widehat{\mathcal{M}}_{ss} &= \widehat{PC}_{ss}/\mathrm{SL}(5, \mathbb{C}).\end{aligned}$$

These moduli spaces fit into the commutative diagram:

$$\begin{array}{ccc} \mathcal{M}_0^f & \longrightarrow & \widehat{\mathcal{M}}_s^f \\ \downarrow & & \downarrow \\ \mathcal{M}_0 & \longrightarrow & \widehat{\mathcal{M}}_s \longrightarrow \widehat{\mathcal{M}}_{ss} \end{array}$$

We saw in Lemma 1.8 that  $\mathcal{M}_0^f$  is a complex manifold. Since  $PG$  acts properly on  $\widehat{PC}_s$ , we see that  $\mathcal{M}_0$ ,  $\widehat{\mathcal{M}}_s^f$  and  $\widehat{\mathcal{M}}_s$  are analytic spaces. We will see below that  $\mathcal{M}_s^f$  is smooth. Since it is a GIT quotient,  $\widehat{\mathcal{M}}_{ss}$  is a compact algebraic variety.

The period maps  $P\mathcal{F}_s \rightarrow \mathbb{C}H^{10}$  and  $\widehat{PC}_{ss} \rightarrow \overline{P\Gamma \backslash \mathbb{C}H^{10}}$  are  $PG$ -invariant, and hence induce maps on  $\widehat{\mathcal{M}}_s^f$  and  $\widehat{\mathcal{M}}_{ss}$ . The main theorem of the monograph, Theorem 6.1, says that the first of these maps is an isomorphism and that the second is almost an isomorphism. For the statement of the theorem, it is convenient to refer to the discriminant locus of  $\widehat{\mathcal{M}}_s^f$ , by which we mean the image of the discriminant locus of  $P\mathcal{F}_s$ , and similarly for the chordal locus. Let  $\mathcal{H}_\Delta$  (resp.  $\mathcal{H}_c$ ) be the union of the discriminant (resp. chordal) hyperplanes in  $\mathbb{C}H^{10}$  (defined just before Theorem 3.7), and let  $\mathcal{H} = \mathcal{H}_\Delta \cup \mathcal{H}_c$ .

**THEOREM 6.1.** *The period map  $g : \widehat{\mathcal{M}}_s^f \rightarrow \mathbb{C}H^{10}$  is an isomorphism. It identifies the discriminant (resp. chordal) locus of  $\widehat{\mathcal{M}}_s^f$  with  $\mathcal{H}_\Delta$  (resp.  $\mathcal{H}_c$ ), and  $\mathcal{M}_0^f$  with  $\mathbb{C}H^{10} - \mathcal{H}$ . The induced map*

$g : \widehat{\mathcal{M}}_{ss} \rightarrow \overline{P\Gamma \backslash \mathbb{C}H^{10}}$  is an isomorphism except over the  $A_5$  cusp, whose preimage is a rational curve. Finally, this map induces an isomorphism of  $\mathcal{M}_s$  with  $P\Gamma \backslash (\mathbb{C}H^{10} - \mathcal{H}_c)$ .

PROOF OF THEOREM 6.1: As a blowup of a GIT quotient,  $\widehat{\mathcal{M}}_{ss}$  is compact. And Theorem 3.10 asserts that the period map carries  $\widehat{\mathcal{M}}_{ss} - \widehat{\mathcal{M}}_s$  to the boundary of  $\overline{P\Gamma \backslash \mathbb{C}H^{10}}$ . Therefore  $\widehat{\mathcal{M}}_s \rightarrow P\Gamma \backslash \mathbb{C}H^{10}$  is proper. It follows that  $\widehat{\mathcal{M}}_s^f \rightarrow \mathbb{C}H^{10}$  is proper. The discriminant locus is carried into  $\mathcal{H}_\Delta$  by Theorem 3.7, and has image a divisor by Theorem 5.1. Therefore its image is a union of discriminant hyperplanes. By Lemma 6.2 below, all discriminant hyperplanes are  $P\Gamma$ -equivalent, so the discriminant locus has image exactly  $\mathcal{H}_\Delta$ . The same argument, using Theorem 4.1, shows that the chordal locus has image  $\mathcal{H}_c$ .

Now we claim that  $\widehat{\mathcal{M}}_s^f \rightarrow \mathbb{C}H^{10}$  is a local isomorphism. First observe that its restriction to the preimage of  $\mathbb{C}H^{10} - \mathcal{H}$  is an isomorphism. The reason for this is that, by Theorem 3.7,  $g^{-1}(\mathcal{H})$  is contained in the union of the discriminant and chordal divisors, thus the preimage of  $\mathbb{C}H^{10} - \mathcal{H}$  is contained in  $\mathcal{M}_0^f$ . But it follows from Theorem 1.9 that the restriction of  $g$  to  $\mathcal{M}_0^f$  is an isomorphism onto its image. Therefore  $g : \widehat{\mathcal{M}}_s^f \rightarrow \mathbb{C}H^{10}$  is a proper modification of  $\mathbb{C}H^{10}$ . (See [17, pp. 214–215].) Since  $\mathbb{C}H^{10}$  is smooth,  $g$  is a local isomorphism at a point  $x$  of  $\widehat{\mathcal{M}}_s^f$  unless it crushes some divisor passing through  $x$  to a variety of lower dimension. Since  $\mathcal{M}_0^f$  maps isomorphically to its image, and, by Theorems 4.1 and 5.1, the discriminant and chordal loci are mapped onto divisors, no divisors are crushed. Therefore the period map is everywhere a local isomorphism. Since it is a generic isomorphism, it is an isomorphism. It follows that it identifies  $\mathcal{M}_0^f$  with  $\mathbb{C}H^{10} - \mathcal{H}$ .

(One can avoid the machinery of proper modifications by applying Zariski's main theorem to

$$(6.1) \quad P\Gamma' \backslash \widehat{\mathcal{M}}_s^f \rightarrow P\Gamma' \backslash \mathbb{C}H^{10},$$

where  $P\Gamma'$  is a torsion-free finite index subgroup of  $P\Gamma$ . Because this is a birational isomorphism of algebraic varieties, and the target space is smooth, one can apply a version of Zariski's main theorem, Theorem 3.20 of [29], to deduce that (6.1) is a local isomorphism, hence an isomorphism. That  $\widehat{\mathcal{M}}_s^f \rightarrow \mathbb{C}H^{10}$  is an isomorphism follows.)

To prove the claim about  $\widehat{\mathcal{M}}_{ss}$ , it suffices to examine the cusps of  $\overline{P\Gamma \backslash \mathbb{C}H^{10}}$ . The  $D_4$  cusp is the image of just one point of  $\widehat{\mathcal{M}}_{ss}$ , so the period map is finite there. Since the Baily-Borel compactification is a normal analytic space, it is an isomorphism there. The preimage

of the  $A_5$  cusp consists of the (classes of the) threefolds  $T_{A,B}$  from Theorem 3.2(iii), together with their limiting point, the (class of the) points from Theorem 3.2(vi). These form a rational curve because they are parameterized by  $4A/B^2$ .

For the last claim, we observe that  $\mathcal{M}_s$  is  $\widehat{\mathcal{M}}_s$  minus the chordal locus. This follows from a comparison of the GIT analyses in Theorem 3.2 and [1]. This makes the last claim obvious.  $\square$

We used the following lemma in the proof of the theorem.

LEMMA 6.2. *Any two discriminant (resp. chordal) hyperplanes in  $\mathbb{C}H^{10}$  are  $P\Gamma$ -equivalent.*

PROOF. Suppose  $r \in \Lambda$  is a chordal root. Since  $\langle r|r \rangle = 3$  and  $\langle r|\Lambda \rangle = 3\mathcal{E}$ ,  $\langle r \rangle$  is a summand of  $\Lambda$ . Its orthogonal complement therefore has the same determinant as  $\Lambda_{10} = E_8^{\mathcal{E}} \oplus E_8^{\mathcal{E}} \oplus \begin{pmatrix} 0 & \theta \\ \theta & 0 \end{pmatrix}$ , namely  $-3^5$ . Also,  $\theta(r^\perp)^* = r^\perp$ , since  $r^\perp \subseteq \Lambda$ . There is a unique  $\mathcal{E}$ -lattice  $L$  of determinant  $-3^5$  and signature  $(9, 1)$  satisfying  $L \subseteq \theta L^*$ , by Lemma 2.6 of [5]. Therefore  $r^\perp \cong \Lambda_{10}$ . If  $s$  is another chordal root then the same argument shows that  $\Lambda = \langle s \rangle \oplus \Lambda_{10}$ . So there is an isometry of  $\Lambda$  carrying  $r$  to  $s$ . This proves transitivity on chordal roots.

We will prove three claims below. (i) Any two index 3 sublattices of  $\Lambda_{10}$  are isometric; we write  $\Lambda'_{10}$  for such a lattice. (ii) For any nodal root  $r \in \Lambda$ ,  $r^\perp \cong \Lambda'_{10}$ . (iii) There are exactly two enlargements of  $r^\perp \oplus \langle r \rangle$  to a copy of  $\Lambda$  in which  $r$  is a nodal root, and these are exchanged by negating  $r^\perp$  while leaving  $r$  fixed. Given the claims, a standard argument shows that  $r$  can be carried to any other nodal root  $s$  by an element of  $\Gamma$ . Namely, by (ii), there is an isometry  $r^\perp \oplus \langle r \rangle \rightarrow s^\perp \oplus \langle s \rangle$  carrying  $r$  to  $s$ . This isometry carries the enlargement  $\Lambda$  of  $r^\perp \oplus \langle r \rangle$  to an enlargement  $M$  of  $s^\perp \oplus \langle s \rangle$ . By (iii),  $M$  is either  $\Lambda$  itself or else is carried to  $\Lambda$  by negating  $s^\perp$ . The result is an isometry of  $\Lambda$  carrying  $r$  to  $s$ .

Now we prove (i). We will even show that  $\text{Aut } \Lambda_{10}$  acts transitively on the index 3 sublattices of  $\Lambda_{10}$ . The key ingredient is a symplectic form on  $\Lambda_{10}/\theta\Lambda_{10}$ . For any  $\mathcal{E}$ -lattice  $L$  satisfying  $L \subseteq \theta L^*$ , the  $\mathbb{F}_3$ -vector space  $L/\theta L$  admits an antisymmetric pairing, defined as follows: if  $v, w \in L$  have images  $\bar{v}, \bar{w} \in L/\theta L$ , then  $(\bar{v}, \bar{w}) \in \mathbb{F}_3$  is the reduction of  $\frac{1}{\theta} \langle v|w \rangle$  modulo  $\theta$ . It is easy to check that if  $v \in L$  has norm 3 and has inner product  $\theta$  with some element of  $L$ , then  $\bar{v}$  does not lie in the kernel of  $(\cdot, \cdot)$ , and the reflections in  $v$  act on  $L/\theta L$  as the transvections in  $\bar{v}$ . Since  $\Lambda_{10} = \theta\Lambda_{10}^*$ , the pairing on  $\Lambda_{10}/\theta\Lambda_{10}$  is nondegenerate. Index 3 sublattices of  $\Lambda_{10}$  correspond to hyperplanes in the  $\mathbb{F}_3$ -vector space, so

to prove transitivity of  $\text{Aut } \Lambda_{10}$  on such subspaces, it suffices to show that it acts as  $\text{Sp}(10, \mathbb{F}_3)$ . This is easy, since every root of  $\Lambda_{10}$  gives a transvection.

Before proving (ii), we prove the weaker claim that for any nodal root  $r \in \Lambda$ ,

$$r^\perp \oplus \langle r \rangle \cong \Lambda'_{10} \oplus (3) \cong (3) \oplus E_8^\mathcal{E} \oplus E_8^\mathcal{E} \oplus (-3) \oplus (3).$$

Because  $\langle r | \Lambda \rangle = \theta\mathcal{E}$ ,  $r^\perp \oplus \langle r \rangle$  has index 3 in  $\Lambda$  and contains  $\theta\Lambda$ . Therefore it is completely determined by its image  $S$  in  $W = \Lambda/\theta\Lambda$ , which is  $\bar{r}^\perp$ . The kernel of the pairing is 1-dimensional, coming from the first summand of (1.8), and  $\bar{r}$  is not in this kernel. Therefore  $\bar{r}^\perp$  is the preimage under  $W \rightarrow W/\ker W$  of a hyperplane in  $W/\ker W$ . As we saw above,  $\text{Aut } \Lambda_{10} \subseteq \Gamma$  acts on  $W/\ker W = \Lambda_{10}/\theta\Lambda_{10}$  as  $\text{Sp}(10, \mathbb{F}_3)$ , so it acts transitively on hyperplanes in  $W/\ker W$ . It follows that the isomorphism class of  $r^\perp \oplus \langle r \rangle$  is independent of  $r$ , so it can be described by working a single example. We take  $r = (0, \dots, 0, 1, \omega)$  in the coordinates of (1.8), and write  $a$  for  $(1, 0, \dots, 0)$  and  $b$  for  $(0, \dots, 0, 1, \bar{\omega})$ . Then  $r^\perp$  is  $(3) \oplus E_8^\mathcal{E} \oplus E_8^\mathcal{E} \oplus (-3)$ , with  $a$  spanning the first summand and  $b$  spanning the last.

Now we prove (ii). It suffices to prove that if  $s$  is a norm 3 vector of  $N = \Lambda'_{10} \oplus (3)$ , such that  $\langle s \rangle$  is a summand, then  $s^\perp \cong \Lambda'_{10}$ . In order to do this we need to refer to the  $\mathbb{F}_3$ -valued symmetric bilinear form on  $\theta N^*/N$ , got by reducing inner products modulo  $\theta$ . Because  $\theta(E_8^\mathcal{E})^* = E_8^\mathcal{E}$ ,  $\theta N^*/N \cong \mathbb{F}_3^3$ , with a basis consisting of  $\bar{a}$ ,  $\bar{b}$  and  $\bar{r}$ , which are the reductions modulo  $N$  of  $a/\theta$ ,  $b/\theta$  and  $r/\theta$ . The norms of  $\bar{a}$ ,  $\bar{b}$  and  $\bar{r}$  are 1,  $-1$  and 1. Since  $\langle s \rangle$  is a summand of  $N$ ,  $\langle s | N \rangle = 3\mathcal{E}$ , so  $s/\theta \in \theta N^*$ . We write  $\bar{s}$  for the image of  $s/\theta$  in  $\theta N^*/N$ , and observe that  $\bar{s}^2 = \langle s/\theta | s/\theta \rangle = 1$ . Enumerating the elements of  $\theta N^*/N$  of norm 1, we find that  $\bar{s}$  is one of  $\pm\bar{a}$ ,  $\pm\bar{r}$ , or  $\pm\bar{a} \pm \bar{b} \pm \bar{r}$ . In every case there is a 1-dimensional isotropic subspace  $S$  of  $\theta N^*/N$  orthogonal to  $\bar{s}$ . Two examples: if  $\bar{s} = \bar{a}$  then we can take  $S$  to be the span of  $\bar{b} + \bar{r}$ , and if  $\bar{s} = \bar{a} + \bar{b} + \bar{r}$  then we can take  $S$  to be the span of  $\bar{a} + \bar{b}$ . We define  $N^+$  to be the preimage of  $S$  in  $\theta N^*$ . This is spanned by  $N$  and a vector of norm divisible by 3, whose inner products with elements of  $N$  are divisible by  $\theta$ . Therefore  $\theta(N^+)^* \subseteq N^+$ . Now, the determinant of  $\Lambda$  is  $-3^6$ , so the index 3 lattice  $N$  has determinant  $-3^7$ , so  $N$ 's index 3 superlattice  $N^+$  has determinant  $-3^6$ . Also,  $\langle s | N^+ \rangle = 3$ , because  $S \perp \bar{s}$ . Therefore  $\langle s \rangle$  is a summand of  $N^+$ , so its complement in  $N^+$  has determinant  $-3^5$  and is therefore a copy of  $\Lambda_{10}$ . Since the complement of  $s$  in  $N$  has index three in the complement in  $N^+$ , it is  $\Lambda'_{10}$  by (i), proving (ii).



Finally we prove (iii). Using  $N$ ,  $\bar{a}$ ,  $\bar{b}$  and  $\bar{r}$  as above, we must determine the enlargements  $M$  of  $\Lambda'_{10} \oplus \langle r \rangle$  that are copies of  $\Lambda$  in which  $r$  is a nodal root. Such an  $M$  must satisfy  $M \subseteq \theta M^*$  and contain a vector having inner product  $\theta$  with  $r$ . Such an  $M$  is the preimage of an isotropic line in  $\theta N^*/N$  which is not orthogonal to  $\bar{r}$ . There are just two such subspaces, the spans of  $\bar{r} - \bar{b}$  and  $\bar{r} + \bar{b}$ . The first enlargement is  $\Lambda$  itself. The two enlargements are exchanged by negating  $r^\perp$ . This proves (iii).  $\square$

## CHAPTER 7

# The Monodromy Group and Hyperplane Arrangement

We have already discussed several facets of the action of  $\Gamma$  on various objects associated to  $\Lambda$ . Theorem 3.11 shows that  $\Gamma$  acts with two orbits on primitive null vectors, corresponding to the two boundary points of  $\overline{P\Gamma \backslash \mathbb{C}H^{10}}$ , and Lemma 6.2 shows that  $\Gamma$  acts with two orbits on roots of  $\Lambda$ , corresponding to discriminant and chordal hyperplanes. In this chapter, we gather a few more results of this flavor, and determine the image of the monodromy representation.

**THEOREM 7.1.** *For  $F \in \mathcal{C}_0$ , the monodromy representation*

$$\pi_1(\mathcal{C}_0, F) \rightarrow \Gamma(V)$$

*has image consisting of all isometries of  $\Lambda(V)$  with determinant a cube root of unity.*

**PROOF.** We first show that the image of  $P\rho(\pi_1(\mathcal{C}_0, F)) \subseteq P\Gamma(V)$  is all of  $P\Gamma(V)$ . This amounts to the connectedness of  $\mathcal{F}_0$ , which is the same as the connectedness of  $P\mathcal{F}_0$ , which is the same as the connectedness of  $P\mathcal{F}_s$ , which is the same as the connectedness of  $\widehat{\mathcal{M}}_s^f$ , which by Theorem 6.1 is the same as the connectedness of  $\mathbb{C}H^{10}$ , which is obvious.

Now it suffices to show that  $\rho(\pi_1(\mathcal{C}_0, F))$  contains  $\bar{\omega}I$  and does not contain  $-I$ . To see the latter, observe that  $\pi_1(\mathcal{C}_0)$  is generated by meridians, which  $\rho$  maps to  $\omega$ -reflections, so every element of  $\pi_1(\mathcal{C}_0)$  acts on  $\Lambda(V)$  by an isometry with determinant a cube root of 1. (Another way to see that  $-I$  is not in the monodromy group is to use the ideas of the proof of Theorem 1.9: the negation map of  $H_0^4(V; \mathbb{Z})$  does not extend to an isometry of  $H^4(V; \mathbb{Z})$  fixing  $\eta(V)$ .)

Now we prove that  $\bar{\omega}I$  lies in the monodromy group. Consider the cubic threefold  $T$  defined by

$$F = x_0x_3^2 - x_0x_2x_4 + x_1^2x_4 + x_2^2x_3 .$$

By the method of [1, §2], it has an  $A_7$  singularity at  $[1:0:0:0:0]$ , an  $A_4$  singularity at  $[0:0:0:0:1]$ , and no other singularities. Using Lemmas 1.4 and 1.6 shows that  $\rho(\pi_1(\mathcal{C}_0, F))$  contains an image of the product  $B_8 \times B_5$  of the 8-strand and 5-strand braid groups, with standard generators  $a_1, \dots, a_7$  and  $b_1, \dots, b_4$  mapping to the  $\omega$ -reflections in roots  $r_1, \dots, r_7$  and  $s_1, \dots, s_4$  of  $\Lambda(V)$ , no two of which are proportional. The method described before Theorem 3.4 shows that the  $r_i$  (resp.  $s_i$ ) span a nondegenerate  $\mathcal{E}$ -lattice  $R$  (resp.  $S$ ) of dimension 7 (resp. 4). Since  $R$  and  $S$  are orthogonal and nondegenerate, they meet only at 0, so  $\Lambda(V) = R \oplus S$  up to finite index. A generator  $c_8$  (resp.  $c_5$ ) for the center of  $B_8$  (resp.  $B_5$ ) is  $(a_1 \dots a_7)^8$  (resp.  $(b_1 \dots b_4)^5$ ), so it acts on  $R$  (resp.  $S$ ) as a scalar of determinant  $\omega^{56}$  (resp.  $\omega^{20}$ ), i.e., as the scalar  $\bar{\omega}$  (resp.  $\pm\bar{\omega}$ ). Therefore one of  $c_5 c_8^{\pm 1}$  acts on  $\Lambda(V)$  by  $\bar{\omega}I$ .  $\square$

**THEOREM 7.2.** (i) *No two chordal hyperplanes meet in  $\mathbb{C}H^{10}$ .* (ii) *If a discriminant hyperplane and a chordal hyperplane meet in  $\mathbb{C}H^{10}$ , then they are orthogonal.*

**PROOF.** If  $r$  and  $s$  are a nodal and a chordal root, then  $s^2 = r^2 = 3$  and  $\langle r|s \rangle$  is divisible by 3. If  $\langle r|s \rangle = 0$  then they are orthogonal, and otherwise the span of  $r$  and  $s$  has inner product matrix which is not positive-definite. In this case,  $r^\perp$  and  $s^\perp$  do not meet in  $\mathbb{C}H^{10}$ . This proves (ii).

Now we prove (i). Suppose  $r$  and  $s$  are non-proportional chordal roots. By the argument for (ii), if  $r^\perp$  and  $s^\perp$  meet, then  $r \perp s$ . So we must show that  $r \perp s$  is impossible. By Lemma 6.2, we may take  $s$  to be  $(1, 0, \dots, 0)$  in the coordinates of (1.8). If  $r \perp s$  then  $r$  lies in  $\Lambda_{10}$ . Now,  $r$  is primitive because its norm is 3, and by  $\theta\Lambda_{10}^* = \Lambda_{10}$ ,  $r$  makes inner product  $\theta$  with some element of  $\Lambda_{10}$ . Therefore  $r$  cannot be a chordal root.  $\square$

**THEOREM 7.3.** *The discriminant and chordal loci of  $\widehat{\mathcal{M}}_s$  have finite branched covers which are isomorphic.*

**PROOF.** Consider the nodal root  $r = (0, \dots, 0, 1, \omega)$  and chordal root  $s = (1, 0, \dots, 0)$ , in the notation of (1.8). The discriminant (resp. chordal) locus of  $\widehat{\mathcal{M}}_s$  is the image in  $P\Gamma \backslash \mathbb{C}H^{10}$  of  $r^\perp$  (resp.  $s^\perp$ ). We have

$$r^\perp \oplus \langle r \rangle = \langle s \rangle \oplus E_8^\mathcal{E} \oplus E_8^\mathcal{E} \oplus (-3) \oplus \langle r \rangle .$$

Let  $\Gamma'$  be the subgroup of  $\Gamma$  preserving the sublattice  $r^\perp \oplus \langle r \rangle$ , which has index  $(3^{10} - 1)/2 = 29524$  in  $\Gamma$ . The images of  $r^\perp$  and  $s^\perp$  in  $P\Gamma' \backslash \mathbb{C}H^{10}$  are isomorphic because  $r^\perp \oplus \langle r \rangle$  admits an isometry exchanging  $r$  and  $s$ .  $\square$

REMARK. The same argument proves Theorem 3 in Kondō's work [22] on genus 4 curves, and a similar result relating nodal and hyperelliptic hyperplanes in his uniformization of the moduli space of genus 3 curves by the 6-ball.

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