Binary Relations from Tournament Solutions, and Back Again

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Abstract

We present a generalization of an abstract model of group choice in which the process of collective choice is modeled as a cooperative process of consideration and reconsideration of alternatives. We develop a general framework for studying choice from a finite set of alternatives, using the idea that one alternative may challenge – or displace from consideration – another in the course of a group choosing. From this binary relation – the "challenges" relation – new tournament solutions are obtained, the limit of which is the central object of the present study. The model presented generalizes that of contestation [Schwartz, 1990] and we characterize the set of alternatives that can be chosen in a collective choice setting when the process of collective choice is viewed as cooperatively considering and reconsidering alternatives. Basic properties of the family of tournament solutions studied is given as well.

1 Introduction

This paper examines the logical consequences of conceiving of collective choice as a process of consideration and reconsideration of alternatives. Specifically, this paper introduces a family of tournament solutions based on the binary relations obtained from criteria for choosing among alternatives (formally, from tournament solutions). In doing

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so, tournament solutions obtained as the top-set of binary relations is explored. We develop a general framework for studying choice from a finite set of alternatives, using the idea that one alternative may challenge – or displace from consideration – another in the course of considering and reconsidering alternatives as a collective choice. From this binary relation – the "challenges" relation – new tournament solutions are obtained, the limit of which is the central object of the present study. Identifying alternatives that are unchallenged or mutually challenged leads to a new family of tournament solutions, generalizing the Tournament Equilibrium Set (TEQ) of Schwartz [1990].

We conceive of group choice as a cooperative process of consideration and reconsideration of alternatives. For example, when choosing from from set of alternatives, a group might consider alternative x. If alternative x is proposed as collective choice, what could replace (i.e. challenge) x in a process of cooperative re-consideration? Schwartz [Schwartz, 1990] argues only alternatives that are majority preferred to x are even possible challenges, as a majority could prevent z from replacing x when x is majority-preferred to z. However, not all alternatives that are majority preferred to x could replace x as a final choice, Schwartz [1990] argues, because some of these alternatives are not real challenges as some would not be chosen by the group when considering possible alternatives to x. For example, if the group deems A to be a criteria arbitrating "better" and "worse" alternatives, some alternatives will fail according to A. Only alternatives deemed "better than x" will pose a real challenge to x. That is, not all possible challenges to x are actual challenges to x because some of these alternatives may fail the group's criteria for "betterness," A. If one interprets collective choice to be an eventual outcome of consideration and reconsideration in a cooperative game, the "challenges" relation may be used to define alternatives that may appear in final contracts. Alternatives that are challenged and do not challenge any other alternative are susceptible to being displaced from consideration, and hence one would not expect to see such an alternative as the outcome of collective choice.

Two factors are relevant for determining if one alternative may displace from consideration another: a *criteria of challenging* being a principal by which one alternative is deemed better than another (e.g. majority preference, covering, contestation, etc.) and the set of alternatives that are potential challengers, i.e. the set of alternatives a potential group choice is compared to. We operationalize the former via a binary relation capturing the idea that one alternative may be "better than" another. We model the latter via a *neighborhood map* – a method of selecting alternatives that are compared to a given alternative when a group considers it as a possible choice.

Perhaps surprisingly, we find many different notions of "challenging" lead to the same group choice. Indeed, when the set of possible challengers satisfies a minimal notion of majoritarianism (namely, equalling the Condorcet winner when there is one) we show that a large class of challenges-relations lead to the same collective choices. We show that the criteria of challenging – the criteria by which alternatives are evaluated – is far less important than the set of alternatives one is compared to. Indeed, in the limit the former plays virtually no role whatsoever in collective choice, while the later crucially distinguishes among sets of alternatives that may be considered possible collective choices. Examples of the new class of tournament solutions as well as their relation to well-known tournament solutions are given.

An extensive general treatment of tournaments can be found in Laslier [1997], who discusses tournaments as a tool for studying social choice under majority voting. Schwartz [1990] introduces one idea of "challenging" – the contestation relation and defines an important tournament solution, the tournament equilibrium set (TEQ) in terms of the top-set of the contestation relation, the properties of which have been studied in Dutta [1990], Laffond et al. [1993] and Houy [2009].[1] The present work departs from the above

[1] Some of the most basic properties of TEQ were unknown until the recent discovery of Brandt et al.

in a number of ways. Most importantly, we seek to identify the essential building-blocks of these models of collective choice. In doing so, we introduce a generalization of TEQ which yields a family of tournament solutions based only on *neighborhoods* – alternatives that are potential challengers to a collective choice. The utility of the exercise is not to suggest "superior" tournament solutions, but rather to shed light on the use and usefulness of binary relations obtained from tournament solutions for the study of collective choice.

After introducing the basics of tournaments (Section 2), we introduce a the challenges relation in Section 3, and a family of new tournament in solution 4. Section 5 introduces extensions and future work. Section 6 provides an illustration, relating the new family of tournament solutions to extant tournament solutions. Section 7 discusses this research, and concludes.

2 Preliminaries

2.1 Tournaments

Many of the definitions and basic results discussed in this section can be found in Laslier [1997]. A *tournament*, T, is nonempty finite set X equipped with a binary relation $\succ$ on X, such that for all $x, y, \in X$ exactly one of $x \succ y$ and $y \succ x$ holds. That is, a tournament $T := (X, \succ)$, is an irreflexive, complete and asymmetric binary relation on a finite set. We interpret $\succ$ as the "beats" relation[2] so that $x \succ y$, $x, y \in X$ means "alternative x beats alternative y." No ties are contemplated, and no transitivity assumptions are made on $\succ$. We refer to elements of X as *alternatives*.

[2013] who show that TEQ is not pair-wise intersecting and hence is not monotonic, and does not satisfy the strong-superset property [Laffond et al., 1993].

[2]Commonly taken to be a groups' majority preference relation

A tournament may alternatively be defined as a complete, asymmetric directed graph, with X being the set of vertices. Denote the set of all tournaments on X by $\mathcal{T}(X)$. A tournament is *irreducible* if and only if there exists a directed path (under the $\succ$ relation) between any two alternatives. Define the following sets, for any $x \in X$, $T^{-1}(x) = \{y|y \succ x\}$ and $T(x) = \{y|x \succ y\}$. For a binary relation R on X, let $R|_Y$, $Y \subseteq X$ be the binary relation on Y induced by R.[3] Trivially, if $T \in \mathcal{T}(X)$ and $Y \neq \emptyset$ is a subset of X, then $T|_Y := (Y, \succ|_Y)$ is a tournament. In an abuse of notion, if $T = (X, \succ)$ is a tournament, we denote $|T| := |X|$ and write $x \in T$ when $x \in X$.

A major concern of social choice (as well as computer scientists – see Brandt et al. [2009]) is to identify "winners" from a given tournament. To that end, define a *tournament solution* as follows.

Definition 2.1 A **tournament solution** on X is a function

$$S : \bigcup_{Y \subseteq X} \mathcal{T}(Y) \to 2^X$$

for $Y \neq \emptyset$ satisfying (1) $S(T|_Y) \subseteq Y$ for all $Y \subseteq X$; (2) non-emptiness: $S(T) \neq \emptyset$ for all $T \in \mathcal{T}(X)$; and (3) respect for isomorphism: relabeling the elements of X does not affect the solution.[4] In words, a tournament solution (or just *solution*) is a a function S assigning to each sub-tournament $(Y, \succ|_Y)$ of $(X, \succ)$, a nonempty subset of Y, called the *S-winners*, in a manner respecting isomorphisms. Intuitively, S arbitrates among elements of X, declaring alternatives in $S(T) \subseteq X$ to be "better" than those in $X \setminus S(T)$. If $x \in S(T)$, write "x is an S-winner of T." $S(T|_Y)$ is similarly defined for non-empty sub-tournaments $T|_Y$, $Y \subseteq X$. Define $S(\emptyset) = \emptyset$. When T is understood, simply write S instead of $S(T)$.

[3] More formally, $R|_Y = R \cap (Y \times Y)$, which can be considered as the subgraph of R on Y.

[4] This is a non-standard definition of a tournament solution, some version of the Condorcet criterion usually being required in addition. We purposely treat two versions of the Condorcet criteria for tournament solutions separately.

Usually one imposes one of the following Condorcet criteria as part of the definition, but we keep them separate so that we can discuss both cases. A *Condorcet winner* of a tournament $T = (X, \succ)$ is an element of X that beats all other elements of X, writing $Cond(T) = \{x \in X \text{ such that } x \succ y\}$. We say that a solution S satisfies the *weak* (respectively. *strong*) *Condorcet criteria* if whenever a tournament $T = (X, \succ)$ has a Condorcet winner x, then x is among the S-winners (respectively. is the only S-winner) in T.

2.2 Binary Relations

We will also consider some binary relations that don't satisfy any specific axioms, but carry a suggestion that one alternative is "better" than another. To be able to define some terms, we will suppose $\rhd$ is some binary relation on a set X, pronounced "challenges".

Definition 2.2 Let $\rhd$ be a binary relation on a set X and let $Y \subseteq X$. Then Y is **retentive for** $\rhd$ (or is "$\rhd$-retentive") if and only iff

$$Y \neq \emptyset \tag{1}$$

$$\nexists (x \in X \setminus Y \text{ and } y \in Y) \text{ such that } x \rhd y. \tag{2}$$

A retentive subset Y is **minimal retentive** for $\rhd$ if $\nexists Z \subset Y$ such that Z is $\rhd$-retentive.

Retentive sets consist of alternatives that are not challenged by anything not in the set. It is well-known that if Y_1 and Y_2 are two $\rhd$-retentive subsets such that $Y_1 \cap Y_2 \neq \emptyset$ then $Y_1 \cap Y_2$ is $\rhd$-retentive as well. As a consequence, two minimal retentive subsets are either equal or disjoint. When $\rhd$ is intransitive, maximality may be generalized as follows.

Definition 2.3 The **top-set** of a binary relation $\rhd$ is the union of the minimal retentive subsets of $\rhd$ and is denoted $TS(\rhd)$.

The top-set generalizes the the concept of the top-cycle when $\rhd$ is incomplete. The top-set of any binary relation is always nonempty since there exists a nonempty retentive set (X itself) and therefore a minimal such set.

2.3 Tournament Equilibrium Set

Given a solution S, Schwartz [1990] introduces a binary relation obtained from S he called *bears on*[5] as follows. If S is an solution, $T = (X, \succ)$ is a tournament and $x, y \in X$, then we say that x bears on y if x is an S-winner among those alternatives that beat y and write $x \mathbb{B}_{S,T} y$. Less briefly, consider the elements of X that beat y, $T^{-1}(y)$. If this is empty then nothing bears on y. Otherwise, $T^{-1}(y) \subseteq X$ becomes a tournament by restricting $\succ$ to it. Since it is a tournament, S specifies one or more winners, and these are the alternatives that bear on y.

Definition 2.4 Let S be a tournament solution and T be a tournament on X. Define the **contestation relation with respect to** S, as the binary relation $\mathbb{B}_{S,T}$ on X by

$$\forall (x,y) \in X^2,\ x \mathbb{B}_{S,T} y \text{ iff } x \in S(T|_{T^{-1}(y)}).$$

Given an tournament solution S, consider the sequence of tournament solutions defined by $S^{(0)} := S$ and $S^{(k+1)}(T) := TS\Big(\mathbb{B}_{S^{(k)},T}\Big)$, $k \geq 1$ where T is any tournament and $\mathbb{B}_{S^{(k)},T}$ is the binary relation "bears on with respect to S^k." Brandt et al. [2010] showed that this process converges in the sense that $S^{(k-1)}$ and $S^{(k)}$ agree on any tournament with no more than k alternatives.[6] So the definition of $S^{(\infty)}(T)$ as $S^{(k)}(T)$ for any $k \geq |T|$ is unambiguous. Furthermore, if S and S' are any tournament solutions then $S^{(\infty)} = S'^{(\infty)}$, by Brandt

[5] called *contests* in subsequent literature [Brandt, 2011; Laslier, 1997]

[6] In this construction, each iteration involves the top-set of the contestation relation. For the interested reader, a study isolating contestation from the top-set operator, and the limits thereof, is given in Moser [2012].

et al. [2010, Theorem 2]. Hence, there is a single limiting tournament solution based on the iterated application of the top-set to the contestation relation. Combining Schwartz [1990] and Brandt et al. [2010], this iterated tournament solution is also the unique fixed point of $TS(\mathbb{B}_{\cdot,T})$ for any tournament T, and it is called TEQ:

$$TEQ(T) = TS(\mathbb{B}_{TEQ,T})$$

for all $T \in \mathcal{T}$.

Our goal is to identify ways to modify the construction, to challenge the uniqueness of TEQ, while keeping its appealing recursive nature. Obviously the place to begin is the definition of $\mathbb{B}$, which we generalize as follows.

3 Binary Relations from Tournament Solutions

We now define a class of "challenges" relations, one that can be defined by only considering proper subsets of X. Define a *neighborhood map* to be a function N that assigns to each alternative x in each tournament $(X, \succ)$ a proper subset $N_{X,\succ,x}$ of X, in a manner respecting isomorphisms. For easy of reading, denote $\tilde{T}(x) := T|_{N_{X,\succ,x}}$ so that $\tilde{T}(x)$ is the beats-relation among the neighbors of x. We write just N_x when $(X, \succ)$ is understood. In the TEQ case N_x is the set of alternatives that beat x, $T^{-1}(x)$.

Definition 3.1 Given a neighborhood map N and a tournament solution S, we define a binary relation $\rhd_{N,S}$ on each tournament $T = (X, \succ)$ by

$$x \rhd_{N,S} y \quad \text{just if} \quad x \in S\big(\tilde{T}(y)\big).$$

Because S and N respect isomorphisms of tournaments, so does $\rhd_{N,S}$.

When N and S are understood we write just $x \rhd y$. We say that a binary relation arising by this construction (for some S) is *definable by neighborhoods*.

4 Solutions from Binary Relations

In this section we study the usefulness of $\rhd$ for collective choice.

4.1 Iterated Top-Sets

Motivated by Brandt [2011], who examines the repeated application of the top-set of the contestation relation of general tournament solutions, we consider the repeated application of the top-set to general binary relations arising from tournament solutions.

Now we fix a neighborhood map, N, and mimic the Brandt et al. [2010] construction of TEQ: if S is any tournament solution, we consider the sequence of tournament solutions defined by $S_N^{(0)} := S$ and $S_N^{(k+1)}(T) := TS\left(\rhd_{N,S_N^{(k)}}\right) \subseteq X$, for $k \geq 1$ where $T = (X, \succ)$ is any tournament. We will suppress the subscript when N is understood.

4.2 Fixed Solutions

Alternatively, one may examine fixed solutions. For given tournament solution, S, and a $\rhd$ that is definable by neighborhoods, define a *fixed solution*, S_N^* as

$$S_N^*(T) := TS\left(\rhd_{N,S_N^*}\right) \tag{3}$$

for all $T \in \mathcal{T}(X)$. In words, $S_N^*(T)$ is defined to be a fixed-point of the operator $TS(\rhd_{N,\cdot})$. For example, if $\rhd$ is taken to be the contestation relation, so that $N_{T,x} = T^{-1}(x)$, we have $S_N^*(T) = TEQ(T)$ for all $T \in \mathcal{T}$ [Schwartz, 1990].

5 Results

In this section we study the existence and uniqueness of fixed solutions, and the convergence of iterated solutions. Importantly, the two coincide when $\rhd$ is definable by neighborhoods and yields a family of new tournament solutions.

Proposition 1 (Convergence) *Suppose N is any neighborhood map and $k \geq 1$. Then for any tournament solution S and any tournament T with $|T| \leq k$, $S_N^{(k)}(T) = S_N^{(k-1)}(T)$.*

Proof: By induction on k. The case $k = 1$ is trivial: the top-set of any binary relation is nonempty, hence (in the case $|T| = 1$) the set of all alternatives. Now take $k > 1$ and suppose S and $T = (X, \succ)$ are as stated. Then for all $x \in X$ we have $|N_x| < |X|$ by the definition of a neighborhood map, so by induction we have $S^{(k-1)}(N_x, \succ) = S^{(k-2)}(N_x, \succ)$. It follows that the binary relations $\rhd_{N,S^{(k-1)}}$ and $\rhd_{N,S^{(k-2)}}$ coincide on X. Since $S^{(k)}(T)$ and $S^{(k-1)}(T)$ are defined as the top-sets of these binary relations, they are equal. ∎

We write $S^{(\infty)}$ for the limiting tournament solution : $S^{(\infty)}(T) := S^{|T|}(T)$. This definition makes clear what the notation suggests: applying the construction to $S^{(\infty)}$ yields $S^{(\infty)}$ again.

Corollary 5.1 Let $\rhd$ be definable by neighborhoods with neighborhood map N. Then $S_N^{(\infty)}(T)$ exists for any $S \in \mathcal{S}(X)$, $T \in \mathcal{T}$.

Proposition 2 (Uniqueness) *Suppose N is any neighborhood map and S, A are tournament solutions. Then $S_N^{(\infty)}(T) = A_N^{(\infty)}(T)$ for any $T \in \mathcal{T}$.*

Proof: We must show that $S^{(\infty)}(T) = A^{(\infty)}(T)$ for any tournament $T = (X, \succ)$, and we proceed by induction on $|X|$. The argument is a slight variation on the previous one. For all $x \in X$ we have $|N_x| < |X|$ (by definition of neighborhood map), so by induction we have

$S^{(\infty)}(N_x, \succ) = A^{(\infty)}(N_x, \succ)$. It follows that the binary relations $\rhd_{N,S^{(\infty)}}$ and $\rhd_{N,A^{(\infty)}}$ on X coincide. Since $S^{(\infty)}(T)$ and $A^{(\infty)}(T)$ are the top-sets of these binary relations, they are equal. ∎

We have shown that the choice of neighborhood mapping N defines a tournament solution, and we will write S^*_N for it. So we have $S^*_N = TS(\rhd_{N,S^*})$ for any tournament.

An immediate corollary of the above two propositions (stated below and without proof) is that if $\rhd$ is definable by neighborhoods then there is one and only one fixed solution (i.e., satisfying equation 3) and it is precisely the iterated top-set, $S^{(\infty)}_N$ of any $S \in \mathcal{S}$ (and hence does not depend on S).

Corollary 5.2 Let $\rhd$ be definable by neighborhoods and N be the corresponding neighborhood map. Then $S^*_N(T) = A^{(\infty)}_N(T)$ for any $A \in \mathcal{S}$, $T \in \mathcal{T}$.

5.1 Properties of Fixed Solutions and Iterated Top-Sets

Note that the while S^*_N exists for any neighborhood map, N, it might not be a proper tournament solution as there is no guarantee it satisfy the Condorcet criterion. For example, $S^{(1)}_N$ might not satisfy the strong Condorcet criterion, as there is no guarantee that $a \in Cond(T)$ implies $\tilde{T}(a) = \emptyset$. In this section, we identify conditions on a neighborhood map, N, for which S^*_N is a proper tournament solution. To do so, we introduce a version of the Condorcet criteria for binary relations.

Definition 5.3 (Condorcet) The neighborhood map, N, satisfies **weak Condorcet** (wC) iff $N_{X,\succ,x}$ is empty whenever x is a Condorcet winner in $(X, \succ)$.

N satisfies **strong Condorcet** (sC) iff $N_x = \emptyset$ and $x \in N_y$ for all $y \in X$ other than x, whenever x is a Condorcet winner in $(X, \succ)$.

As the results below show, the above is a useful notion of Condorcet criteria in this setting.

Proposition 3 (Weak Condorcet condition) *Suppose N is a neighborhood map satisfying weak Condorcet (wC). Then S_N^* is a weak Condorcet tournament solution.*

Proof: Suppose x is a Condorcet winner of $(X, \succ)$. Then there is no $y \in X$ with $y \rhd_{N,S^*} x$ since $N_x = \emptyset$. So $\{x\}$ is a minimal $\rhd_{N,S^*}$-retentive subset of X, hence part of $TS(\rhd_{N,S^*}) = S_N^*(X, \succ)$. ∎

Proposition 4 (Strong Condorcet condition) *Suppose N is neighborhood mapping such that $N_x = \emptyset$ and $x \in N_y$ for all $y \in X$ other than x, whenever x is a Condorcet winner in $(X, \succ)$. Then S_N^* is a strong Condorcet tournament solution.*

Proof: First assume N satisfies the stated conditions; we must show that for any tournament $T = (X, \succ)$ with a Condorcet winner x, we have $S_N(T) = \{x\}$. We use induction on $|X|$. Suppose $y \in X - \{x\}$. Since $N_x = \emptyset$ we cannot have $y \rhd_N x$. And by hypothesis on N we have $x \in N_y$. Obviously x is a Condorcet winner in N_y. Since $|N_y| < |X|$, our inductive hypothesis says $S_N(N_y) = \{x\}$. So $x \rhd_N y$ for all $y \neq x$. It is now obvious that the top-set of $\rhd_N$ is $\{x\}$. That is, $S_N(T) = \{x\}$, as desired. ∎

Note that the converse is not true: the strong Condorcet criterion is not necessary for S_N^* to satisfy strong Condorcet property, as the following example shows. Let $X = \{1, 2, 3\}$ and $\succ$ being $>$. On this tournament define $N(1) = \{2\}$, $N(2) = \{3\}$ and $N(3) = \emptyset$. Then N does not satisfy sC since $3 \notin N(1)$. But $S_N^*(T) = 1$ is the same as $TEQ(T)$.

Notice that the requirement for S_N^* to satisfy the strong Condorcet criteria does not depend on the properties of $A \in \mathcal{S}$, even though $S_N^* = A_N^{(\infty)}$. Indeed, starting with a solution A that does not satisfy even the weak Condorcet criteria, we may nonetheless recover a proper tournament solution, S_N^*. An example of this phenomena can be seen in the following example.

Example 5.4 Let $T = (X, \succ)$ be a tournament and let $N_{X,\succ,x} = T^{-1}(x)$. Consider S to be the "anti-Copeland" rule, $S(T) = \underset{a \in T}{\mathrm{argmin}}\{|T(a)|\}$, selecting alternatives majority preferred to the least number of other alternatives. Clearly, S does not satisfy SCC nor WCC. Equally clearly, N satisfies sC. Consider the tournament T on $X = \{w, a, b, c,\}$ defined as:

- wTa, wTb, wTc
- aTb
- bTc
- cTa.

  That is, w is a Condorcet winner, and T is intransitive. For example, $N_a = \{w, c\}$ and $c \rhd a$. The $\rhd$ relation is seen to be $c \rhd a$, $a \rhd b$, $b \rhd c$. Hence, $S^{(1)}(T) = TS(\rhd) = X$. Continuing, it can be seen that $w \rhd_{S^{(1)}} a$, $w \rhd_{S^{(1)}} b$, $w \rhd_{S^{(1)}} c$ so that $S^{(2)}(T) := TS(\rhd_{S^{(1)}}) = w = S^{(3)}(T)$. Hence, $S_N^* = w$.

6 Illustration

There is a a surprisingly rich family of tournament solutions that may be defined via the method we present. The role of the neighborhood map N and relations of S_N^* to existing tournament solutions are illustrated in this section.

For example, the trivial solution, returning all alternatives, may be obtained by taking the neighborhood map to be all alternatives except one, and S^0 to be the trivial solution.

Claim 1 *Let S^0 be the trivial solution ($S^0(T) = X$ for all $T \in \mathcal{T}(X)$). Fix $T := (X, \succ)$ and let $N_x = X \setminus x$ for all $x \in X$. Then $S_N^{(\infty)}(T) = X$.*

The proof is trivial: for all alternatives $y \neq x$, $y \rhd_{N,S^0} x$, for all $x \in X$. So $S^{(1)}(T) = X$. Continuing, $y \in S^{(1)}(T_{X\backslash x})$ for all $y \neq x$, for all $x \in X$, so $S^{(2)}(T) = X$. Clearly, then, $y \in S^{(k)}(T|_{X\backslash x})$ for all $y \neq x$, for all $x \in X$ and the result obtains. Intuitively, neither the neighborhood map nor the method of arbitrating depends on the details of the $\succ$ relation and hence iterating cannot help discriminate among alternatives in X.

As a more interesting example, define a neighborhood map by

$$N_x = \{y : (yTx) \vee (\exists z \;\; s.t. \;\; xTyTzTx)\}, \tag{4}$$

for all $x \in X$ (we call this the 'inclusive better neighborhood map'). Define $\rhd : \mathcal{S} \times \mathcal{T} \to \mathcal{R}$ as $a \rhd_{N,S} b$ iff $a \in S(\tilde{T}(b))$ and abbreviate $\rhd$ as (N,S). As $x \notin N_x$, for any $x \in X$, $\rhd_N$ is definable by neighborhoods. Hence, by Proposition 2 S^*_N exists, is unique and by Proposition 1 can be computed by taking $A(T) = \{X\}$ and calculating $A_N^{(|T|)}$.

Equation 4 satisfies the antecedents of Lemma 4 and so $S_N^{(\infty)}$ satisfies strong Condorcet. Indeed, more can be said: it is a subset of the uncovered set [Miller, 1980]. Recall that for $T \in \mathcal{T}$, alternative *a covers* alternative b in T iff $T(a) \supseteq \{b \cup T(b)\}$. In words, a covers b in T if a beats b, and beats everything that b beats. The *uncovered set* of T, denoted $UC(T)$, is the set of alternatives not covered in T. The uncovered set has been studied, among other places, in Miller [1980], Shepsle and Weingast [1984], Cox [1987], Feld et al. [1987], Epstein [1997], Miller [2007].

Proposition 6.1 Let $T \in \mathcal{T}(X)$ be irreducible. Let N be as in equation 4 and define $\rhd_{N,S}$ as: $a \rhd_{N,S} b$ iff $a \in S(\tilde{T}(b))$ for any $S \in \mathcal{S}(X)$. Then $S_N^{(\infty)}(T) \subseteq UC(T)$.

Claim 2 *Let $T \in \mathcal{T}(X)$ be irreducible and let $S \in \mathcal{S}$. Let x cover c in T, and $d \in T(c)$. If $c \rhd_{N,S^{(\infty)}} d$, then $x \rhd_{N,S^{(\infty)}} d$.*

Proof of claim: Clearly, $x \in N_d$. That $x \in S_N^{(|T|)}(N_d)$ follows from induction on $|T|$. ∎

Claim 3 *Let $T \in \mathcal{T}(X)$ be irreducible and let $S \in \mathcal{S}$. Let x cover c, and $y \in T^{-1}(c)$. If $c \rhd_{N,S^{(\infty)}} y$, then $x \rhd_{N,S^{(\infty)}} y$.*

Proof of claim: If T is irreducible, then there exists $y \in T^{-1}(c)$ and $d \in T(c)$ with $cTdTyTc$. Hence, $c \in N_y$. Further, $x \in N$ as yTx implies $yTxTdTy$. By induction on the order of T, the claim follows. ∎

Proof of Proposition 6.1: Let $S \in \mathcal{S}$. We prove $S_N^{(|T|)}(T) \subseteq UC(T)$ by induction on $|T|$. The $|T| = 1$ case is trivial, so assume the inductive hypothesis $S_N^{(|T|)}(T) \subseteq UC(T)$ for all T with $|T| \leq n$.

Consider T with $|T| = n$. We prove the contrapositive. Let $c \notin UC(T)$. Then there exists an a such that aTc and aTx for all $x \in T(c)$. Since aTc, $a \in N_c$. Further, $c \notin N_a$ as there is no b such that $aTbTc$. Suppose, by way of obtaining a contradiction, that $c \in S_N^{(\infty)}(T)$. Letting M be a minimal $\rhd_{N,S^{(\infty)}}$-retentive set containing c, either

i $c \in M$ is a singleton or

ii M forms a $\rhd_{N,S^{(\infty)}}$ cycle.

Case (i) leads to a contradiction, as $N(c)$ is non-empty and $S_N^{(k)}$ is nonempty for all k. Case (ii) implies there exists $m \in M$ with $c \rhd_{N,S^{(\infty)}} m$. By Claims 1 and 2, $a \rhd_{N,S^{(\infty)}} m$ which in turn implies $a \in M$ (as $m \in M$, M is $\rhd_{N,S^{(\infty)}}$-retentive, and $a \rhd_{N,S^{(\infty)}} m$. But this implies M is not minimal as $\neg(c \rhd_{N,S^{(\infty)}} a)$ because $c \in N(a)$. Hence, $M \setminus \{c \cap \{q : c \rhd_{N,S^{(\infty)}} q\}\}$ is retentive contradicting, the supposition. ∎

As the next two examples show, however, $S_N^{(\infty)}$ is distinct from both UC as well as the idempotent uncovered set, UC^∞. Further, the novelty of $S_N^{(\infty)}$ can be seen in Example 6.2 in which $S_N^{(\infty)}(T) \neq TEQ(T)$.

Example 6.2 Figure 1 gives a tournament T in which $S_N^{(\infty)}(T) \neq UC^{\infty}(T)$. For the interested reader, $TEQ(T) = \{x_1, x_2, x_3\}$. Here, $S_N^{(\infty)}(T) = \{x_1, x_2, x_3, x_4, x_5, x_6\} = UC(T)$. As $UC(T)$ forms a T-cycle, the top-cycle of $UC(T)$ is equal to $UC(T)$.

Example 6.3 Figure 2 shows a tournament T in which $S_N^{(\infty)}(T) \neq UC(T)$. Here, $S_N^{(\infty)}(T) = \{v_1, v_2, v_3\} = UC^{\infty}(T)$. The uncovered set, $UC(T) = \{v_1, v_2, v_3, v_4\}$, does not form a T-cycle. The top-cycle of $UC(T)$ is equal to $\{v_1, v_2, v_3\}$. For the interested reader, we note that $TEQ(T) = \{v_1, v_2, v_3\}$.

Proposition 6.1 along with the above two examples, lead us to conjecture that $S_N^{(\infty)}(T)$ is equal to the top-cycle of the uncovered set of T, when N is defined as in Equation 4.

7 Discussion

We model the process of group choice by cooperative reconsideration of alternatives. The intuitive process of group choice we model can be described as follows. A group starts with a criteria by which alternatives are arbitrated "better" or "worse", A. The group then considers each alternative, finding "better" alternatives to each, using whatever scheme A uses to arbitrate *only those alternatives in their (respective) neighborhoods*. That is, if alternative x is under consideration as a potential group choice, then only alternatives in the neighborhood of x are even possible challenges to it. Then $A^{(1)}$ declares an alternative x_0 unfit for collective choice (displaced from consideration) if there is a x_1 that challenges it, unless every chain $x_0, \ldots, x_m$, with each x_i challenging x_{i-1}, can be extended to such a chain $x_0, \ldots, x_m, x_{m+1}, \ldots, x_n$ with $x_n = x_0$. If one accepts that A is a "reasonable" way to evaluate alternatives in tournaments smaller than T, then one is forced to accept that $A^{(1)}$ is a "reasonable" way to evaluate alternatives in a tournament T. Continuing, if a group starts with a criteria of arbitrating "challengers" from non-challengers, A, and alternatives are compared/ reconsidered according to the neighborhood map, N, then the group is logically committed to choosing alternatives only in $A_N^{(\infty)}$.

Perhaps surprisingly, we showed that this set is does not depend on the initial "challenging" criteria, A, but rather does depend on set of *potential* challengers (on which alternatives one is compared to), N. Indeed, if we restrict attention to binary relations that are definable by neighborhoods, then Propositions 1 and 2 establish a family of tournament solutions, parameterized only by a neighborhood map, N. By way of example, it is shown that the family contains well-know tournaments solutions (for example, the trivial solution, as well as TEQ) and some solutions distinct from the extant solutions TEQ, uncovered set and the idempotent uncovered set (Section 6).

The majority preference relation has long held a central role in the theory and study of collective choice. General, extra-majoritarian relations among alternatives play an increasing role in the study of group choice, however [Schwartz, 1974; Miller, 1980; Schwartz, 1986; Patty, 2008]. The results of Moser [2012] together with Brandt [2011] suggest an examination of binary relations derived from tournament solutions, and the top-sets thereof. We introduce a framework for doing so and showed its potential usefulness for the study of choice from tournaments. Indeed, the essential component in the construction of tournament solutions in Schwartz [1990] and Brandt [2011] lies in the set of alternatives that are *potential* challenges to a possible collective choice.

Several questions remain regarding this family of tournament solutions, most obviously those of inheritance. Does $S_N^{(k)}$ inherit any properties from $S_N^{(k-1)}$? That is, if $S_N^{(k-1)}$ satisfies the strong superset property, monotonicity,etc., does $S_N^{(k)}$? Further, exactly how rich is this family of solutions? That is: what tournament solutions are expressible as the limit of the process of group choice modeled here? We have shown by example some preliminary findings but deeper study is warranted. At a broader level, the conceptualization of group choice here quite general, but is only one of many models of group choice. For example, Banks [1985] formulates a solution from a model of group choice based on a non-cooperative voting game. Examining the relationship between of tournament solutions obtained from different models of group choice could prove fruitful for future study.

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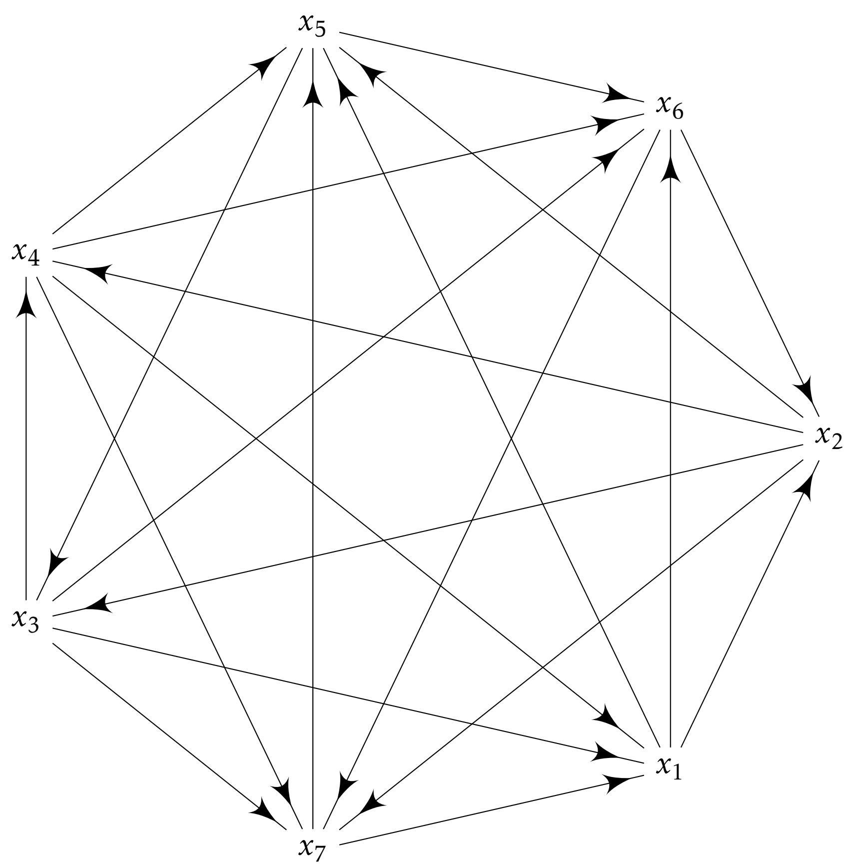

Figure 1: A tournament T in which $S_N^{(\infty)}(T) \neq UC^{\infty}(T)$. Here, $S_N^{(\infty)}(T) = \{x_1, x_2, x_3, x_4, x_5, x_6\} = UC(T)$. As $UC(T)$ forms a T-cycle, the top-cycle of $UC(T)$ is equal to $UC(T)$. For the interested reader, we note that $TEQ(T) = \{x_1, x_2, x_3\}$.

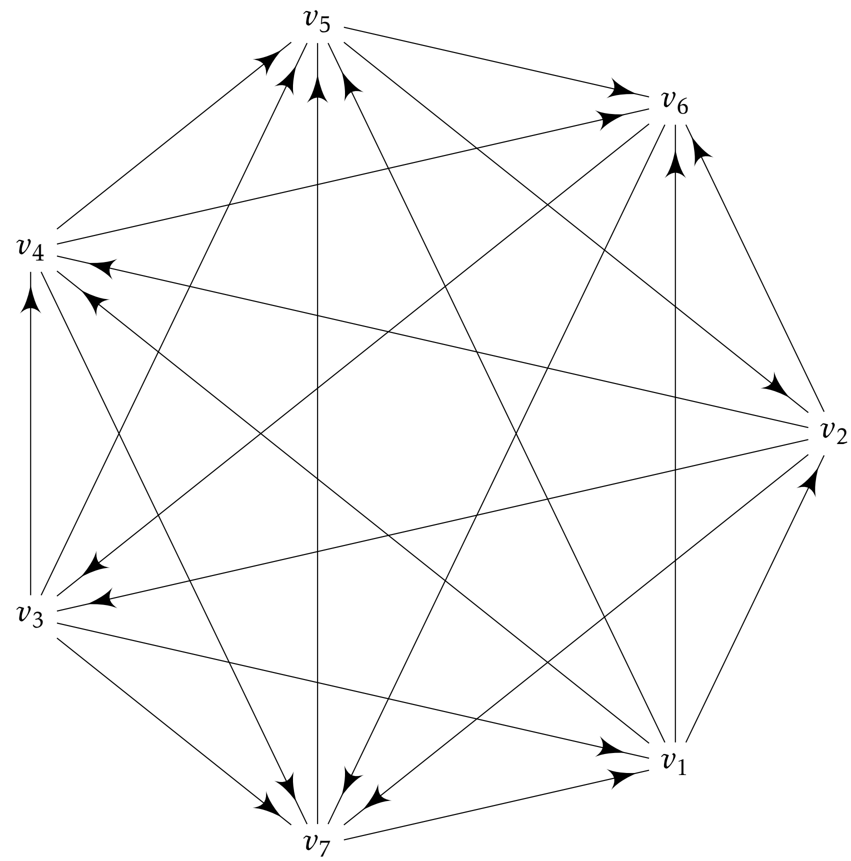

Figure 2: A tournament T in which $S_N^{(\infty)}(T) \neq UC(T)$. Here, $S_N^{(\infty)}(T) = \{v_1, v_2, v_3\} = UC^{\infty}(T)$. The uncovered set, $UC(T) = \{v_1, v_2, v_3, v_4\}$, does not form a T-cycle. The top-cycle of $UC(T)$ is equal to $\{v_1, v_2, v_3\}$. For the interested reader, we note that $TEQ(T) = \{v_1, v_2, v_3\}$.