

A BANACH SPACE DETERMINED BY THE WEIL HEIGHT

DANIEL ALLCOCK AND JEFFREY D. VAALER

ABSTRACT. The absolute logarithmic Weil height is well defined on the quotient group $\overline{\mathbb{Q}}^\times / \text{Tor}(\overline{\mathbb{Q}}^\times)$ and induces a metric topology in this group. We show that the completion of this metric space is a Banach space over the field \mathbb{R} of real numbers. We further show that this Banach space is isometrically isomorphic to a co-dimension one subspace of $L^1(Y, \mathcal{B}, \lambda)$, where Y is a certain totally disconnected, locally compact space, \mathcal{B} is the σ -algebra of Borel subsets of Y , and λ is a certain measure satisfying an invariance property with respect to the absolute Galois group $\text{Aut}(\overline{\mathbb{Q}}/\mathbb{Q})$.

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1. INTRODUCTION

Let k be an algebraic number field of degree d over \mathbb{Q} , v a place of k and k_v the completion of k at v . We select two absolute values from the place v . The first is denoted by $\|\cdot\|_v$ and defined as follows:

- (i) if $v|\infty$ then $\|\cdot\|_v$ is the unique absolute value on k_v that extends the usual absolute value on $\mathbb{Q}_\infty = \mathbb{R}$,
- (ii) if $v|p$ then $\|\cdot\|_v$ is the unique absolute value on k_v that extends the usual p -adic absolute value on \mathbb{Q}_p .

The second absolute value is denoted by $|\cdot|_v$ and defined by $|x|_v = \|x\|_v^{d_v/d}$ for all x in k_v , where $d_v = [k_v : \mathbb{Q}_v]$ is the local degree. If $\alpha \neq 0$ is in k then these absolute values satisfy the product formula

$$(1.1) \quad \prod_v |\alpha|_v = 1.$$

Let $\overline{\mathbb{Q}}$ be an algebraic closure of \mathbb{Q} and $\overline{\mathbb{Q}}^\times$ the multiplicative group of nonzero elements in $\overline{\mathbb{Q}}$. The *absolute, logarithmic Weil height* (or simply the *height*)

$$h : \overline{\mathbb{Q}}^\times \rightarrow [0, \infty)$$

is defined as follows. Let α be a nonzero algebraic number, we select an algebraic number field k containing α , and then

$$(1.2) \quad h(\alpha) = \sum_v \log^+ |\alpha|_v,$$

where the sum on the right of (1.2) is over all places v of k . It can be shown that $h(\alpha)$ is well defined because the right hand side of (1.2) does not depend on the

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field k . By combining (1.1) and (1.2) we obtain the useful identity

$$(1.3) \quad 2h(\alpha) = \sum_v |\log |\alpha|_v|,$$

where $|\cdot|$ (an absolute value without a subscript) is the usual archimedean absolute value on \mathbb{R} .

Let $\text{Tor}(\overline{\mathbb{Q}}^\times)$ denote the torsion subgroup of $\overline{\mathbb{Q}}^\times$ and write

$$\mathcal{G} = \overline{\mathbb{Q}}^\times / \text{Tor}(\overline{\mathbb{Q}}^\times)$$

for the quotient group. If ζ is a point in $\text{Tor}(\overline{\mathbb{Q}}^\times)$, then it is immediate from (1.2) that $h(\alpha) = h(\zeta\alpha)$ for all points α in $\overline{\mathbb{Q}}^\times$. Thus h is constant on each coset of the quotient group \mathcal{G} , and so we may regard the height as a map

$$h : \mathcal{G} \rightarrow [0, \infty).$$

The height has the following well known properties (see [1, Section 1.5]):

- (i) $h(\alpha) = 0$ if and only if α is the identity element in \mathcal{G} ,
- (ii) $h(\alpha^{-1}) = h(\alpha)$ for all α in \mathcal{G} ,
- (iii) $h(\alpha\beta) \leq h(\alpha) + h(\beta)$ for all α and β in \mathcal{G} .

These conditions imply that the map $(\alpha, \beta) \mapsto h(\alpha\beta^{-1})$ defines a metric on the group \mathcal{G} and therefore induces a metric topology. Our objective in this paper is to determine the completion of \mathcal{G} with respect to this metric.

Let r/s denote a rational number, where r and s are relatively prime integers and s is positive. If α is in $\overline{\mathbb{Q}}^\times$ and ζ_1 and ζ_2 are in $\text{Tor}(\overline{\mathbb{Q}}^\times)$, then all roots of the two polynomial equations

$$x^s - (\zeta_1\alpha)^r = 0 \quad \text{and} \quad x^s - (\zeta_2\alpha)^r = 0$$

belong to the same coset in \mathcal{G} . If we write $\alpha^{r/s}$ for this coset, we find that

$$(r/s, \alpha) \mapsto \alpha^{r/s}$$

defines a scalar multiplication in the abelian group \mathcal{G} . This shows that \mathcal{G} is a vector space (written multiplicatively) over the field \mathbb{Q} of rational numbers. Moreover, we have (see [1, Lemma 1.5.18])

$$(1.4) \quad h(\alpha^{r/s}) = |r/s|h(\alpha).$$

Therefore the map $\alpha \mapsto h(\alpha)$ is a norm on the vector space \mathcal{G} with respect to the usual archimedean absolute value $|\cdot|$ on its field \mathbb{Q} of scalars. From these observations we conclude that the completion of \mathcal{G} is a Banach space over the field \mathbb{R} of real numbers. It remains now to give an explicit description of this Banach space.

Let Y denote the set of all places y of the field $\overline{\mathbb{Q}}$. Let $k \subseteq \overline{\mathbb{Q}}$ be an algebraic number field such that k/\mathbb{Q} is a Galois extension. At each place v of k we write

$$(1.5) \quad Y(k, v) = \{y \in Y : y|v\}$$

for the subset of places of Y that lie over v . Clearly we can express Y as the disjoint union

$$(1.6) \quad Y = \bigcup_v Y(k, v),$$

where the union is over all places v of k . If y is a place in $Y(k, v)$ we select an absolute value $\|\cdot\|_y$ from y such that the restriction of $\|\cdot\|_y$ to k is equal to $\|\cdot\|_v$.

As the restriction of $\|\cdot\|_v$ to \mathbb{Q} is one of the usual absolute values on \mathbb{Q} , it follows that this choice of the normalized absolute value $\|\cdot\|_y$ does not depend on k .

In section 2 we show that each subset $Y(k, v)$ can be expressed as an inverse limit of finite sets. This determines a totally disconnected, compact, Hausdorff topology in $Y(k, v)$. Then (1.6) implies that Y is a totally disconnected, locally compact, Hausdorff space. Again the topology in Y does not depend on the field k . We also show that the absolute Galois group $\text{Aut}(\overline{\mathbb{Q}}/k)$ acts transitively and continuously on the elements of each compact, open subset $Y(k, v)$.

In section 4 we establish the existence of a regular measure λ , defined on the Borel subsets of Y , that is positive on open sets, finite on compact sets, and satisfies $\lambda(\tau E) = \lambda(E)$ for all automorphisms τ in $\text{Aut}(\overline{\mathbb{Q}}/k)$ and all Borel subsets E of Y . The restriction of the measure λ to each subset $Y(k, v)$ is unique up to a positive multiplicative constant. We construct λ so that

$$(1.7) \quad \lambda(Y(k, v)) = \frac{[k_v : \mathbb{Q}_v]}{[k : \mathbb{Q}]}$$

for each Galois extension k of \mathbb{Q} and each place v of k . It follows from our construction that λ does not depend on the number field k . In particular, if l is any finite, Galois extension of \mathbb{Q} , if w is place of l and

$$Y(l, w) = \{y \in Y : y|w\},$$

then

$$\lambda(Y(l, w)) = \frac{[l_w : \mathbb{Q}_w]}{[l : \mathbb{Q}]}.$$

Next we consider the real Banach space $L^1(Y, \mathcal{B}, \lambda)$, where \mathcal{B} denotes the σ -algebra of Borel subsets of Y . Let

$$(1.8) \quad \mathcal{X} = \left\{ F \in L^1(Y, \mathcal{B}, \lambda) : \int_Y F(y) \, d\lambda(y) = 0 \right\},$$

so that \mathcal{X} is a co-dimension one linear subspace of $L^1(Y, \mathcal{B}, \lambda)$. For each point α in \mathcal{G} we define a map $f_\alpha : Y \rightarrow \mathbb{R}$ by

$$(1.9) \quad f_\alpha(y) = \log \|\alpha\|_y.$$

If k is a finite Galois extension of \mathbb{Q} that contains α , then $y \mapsto \log \|\alpha\|_y$ is constant on each compact, open set $Y(k, v)$. And the value of this map on each set $Y(k, v)$ is nonzero for only finitely many places v of k . It follows that $f_\alpha(y)$ is a continuous function on Y with compact support. Using (1.7) and the product formula (1.1), we find that

$$(1.10) \quad \begin{aligned} \int_Y f_\alpha(y) \, d\lambda(y) &= \sum_v \int_{Y(k, v)} \log \|\alpha\|_y \, d\lambda(y) \\ &= \sum_v \frac{[k_v : \mathbb{Q}_v]}{[k : \mathbb{Q}]} \log \|\alpha\|_v \\ &= \sum_v \log |\alpha|_v = 0. \end{aligned}$$

This shows that $\alpha \mapsto f_\alpha(y)$ maps \mathcal{G} into the subspace \mathcal{X} . It follows easily that

$$f_{\alpha\beta}(y) = f_\alpha(y) + f_\beta(y) \quad \text{and} \quad f_{\alpha^{r/s}}(y) = (r/s)f_\alpha(y),$$

and therefore the map $\alpha \mapsto f_\alpha(y)$ is a linear map from the vector space \mathcal{G} into \mathcal{X} . The L^1 -norm of each function f_α is given by

$$\begin{aligned}
 \int_Y |f_\alpha(y)| \, d\lambda(y) &= \sum_v \int_{Y^{(k,v)}} |\log \|\alpha\|_y| \, d\lambda_v(y) \\
 &= \sum_v \frac{[k_v : \mathbb{Q}_v]}{[k : \mathbb{Q}]} |\log \|\alpha\|_v| \\
 &= \sum_v |\log |\alpha|_v| \\
 &= 2h(\alpha).
 \end{aligned}
 \tag{1.11}$$

This shows that the map $\alpha \mapsto f_\alpha$ is a linear isometry from the vector space \mathcal{G} with norm determined by $2h$ into the subspace \mathcal{X} with the L^1 -norm. Let

$$\mathcal{F} = \{f_\alpha(y) : \alpha \in \mathcal{G}\}
 \tag{1.12}$$

denote the image of \mathcal{G} under this linear map. Then $\alpha \mapsto f_\alpha$ is a linear isometry from the vector space \mathcal{G} (written multiplicatively) onto the vector space \mathcal{F} (written additively). Now the completion of \mathcal{G} is determined by finding the closure of \mathcal{F} in \mathcal{X} .

Theorem 1. *Let \mathcal{X} be the co-dimension one subspace of $L^1(Y, \mathcal{B}, \lambda)$ defined by (1.8). Then \mathcal{F} is dense in \mathcal{X} .*

It is immediate from Theorem 1 that there exists an isometric isomorphism from the completion of the vector space \mathcal{G} with respect to the height $2h$ onto the real Banach space \mathcal{X} .

The functions in the vector space \mathcal{F} belong to the real vector space $C_c(Y)$ of continuous functions with compact support. Hence \mathcal{F} belongs to the space $L^p(Y, \mathcal{B}, \lambda)$ for $1 \leq p \leq \infty$. Theorem 1 asserts that the closure of \mathcal{F} in $L^1(Y, \mathcal{B}, \lambda)$ is the co-dimension one subspace \mathcal{X} . We also determine the closure of \mathcal{F} with respect to the other L^p -norms.

Theorem 2. *If $1 < p < \infty$ then \mathcal{F} is dense in $L^p(Y, \mathcal{B}, \lambda)$.*

Let $C_0(Y)$ denote the Banach space of continuous real valued functions on Y which vanish at infinity equipped with the sup-norm. As $\mathcal{F} \subseteq C_c(Y) \subseteq C_0(Y)$, it is clear that the closure of \mathcal{F} with respect to the sup-norm is a subspace of $C_0(Y)$.

Theorem 3. *The vector space \mathcal{F} is dense in $C_0(Y)$.*

It follows from the classification of separable L^p -spaces (see [4, pp. 14-15]) that the Banach space $L^1(Y, \mathcal{B}, \lambda)$ has a Schauder basis, or simply a basis. As $\mathcal{X} \subseteq L^1(Y, \mathcal{B}, \lambda)$ is a closed subspace of co-dimension one, it is easy to show that \mathcal{X} also has a basis. Then it follows from a well known result of Krein, Milman and Rutman [5] that a basis for \mathcal{X} can be selected from the dense subset \mathcal{F} . Thus there exists a sequence of distinct elements $\alpha_1, \alpha_2, \dots$ in \mathcal{G} such that the corresponding collection of functions

$$\{f_{\alpha_1}(y), f_{\alpha_2}(y), \dots\}
 \tag{1.13}$$

is a basis for the Banach space \mathcal{X} . That is, for every function F in \mathcal{X} there exists a *unique* sequence of real numbers x_1, x_2, \dots such that

$$F(y) = \lim_{N \rightarrow \infty} \sum_{n=1}^N x_n f_{\alpha_n}(y)$$

in L^1 -norm. While these remarks establish the existence of such a basis, it would be of interest to construct an explicit example of a sequence $\alpha_1, \alpha_2, \dots$ in \mathcal{G} such that the corresponding sequence of functions (1.13) forms a basis for \mathcal{X} .

2. PRELIMINARY LEMMAS

We have stated Theorem 1 for the Weil height on algebraic number fields. However, many of the arguments can be given in the more general setting of a field K with a proper set of absolute values satisfying a product formula. We now describe this situation.

Let K be a field and let v be a place of K . That is, v is an equivalence class of nontrivial absolute values on K . We write K_v for the completion of K at the place v . If L/K is a finite extension of fields then there exist finitely many places w of L such that $w|v$. In general we have

$$\sum_{w|v} [L_w : K_v] \leq [L : K],$$

where L_w is the completion of L at w . We say that v is *well behaved* if the identity

$$\sum_{w|v} [L_w : K_v] = [L : K]$$

holds for all finite extensions L/K , (see [6, Chapter 1, section 4]).

Let \mathcal{M}_K be a collection of distinct places of K and at each place v in \mathcal{M}_K let $\|\cdot\|_v$ denote an absolute value from v . We say that the collection of absolute values

$$(2.1) \quad \{\|\cdot\|_v : v \in \mathcal{M}_K\}$$

is *proper* if it satisfies the following conditions:

- (i) each place v in \mathcal{M}_K is well behaved,
- (ii) if α is in K^\times then $\|\alpha\|_v \neq 1$ for at most finitely many places v in \mathcal{M}_K ,
- (iii) if α is in K^\times then the absolute values in (2.1) satisfy the product formula

$$\prod_{v \in \mathcal{M}_K} \|\alpha\|_v = 1.$$

Now suppose that (2.1) is a proper set of absolute values on K and L/K is a finite extension of fields. Let \mathcal{M}_L be the collection of places of L that extend the places in \mathcal{M}_K . That is, if $W_v(L/K)$ is the finite set of places w of L such that $w|v$, then

$$\mathcal{M}_L = \bigcup_{v \in \mathcal{M}_K} W_v(L/K).$$

At each place w in $W_v(L/K)$ we select an absolute value $\|\cdot\|_w$ that extends the absolute value $\|\cdot\|_v$ on K . Then we define an equivalent absolute value $|\cdot|_w$ from the place w by setting

$$\log |\alpha|_w = \frac{[L_w : K_v]}{[L : K]} \log \|\alpha\|_w$$

for all α in L^\times . In general $\|\cdot\|_w$ and $|\cdot|_w$ are distinct but equivalent absolute values on L . And we note that $|\cdot|_w$ is an absolute value because

$$0 < \frac{[L_w : K_v]}{[L : K]} \leq 1.$$

Then it follows, as in [6, Chapter 2, section 1], that

$$(2.2) \quad \{|\cdot|_w : w \in \mathcal{M}_L\}$$

is a proper set of absolute values on L . In particular, if α is in L^\times then the absolute values in (2.2) satisfy the product formula

$$\prod_{w \in \mathcal{M}_L} |\alpha|_w = 1.$$

We assume that $K \subseteq N$ are fields, that N/K is a (possibly infinite) Galois extension, and we write $\text{Aut}(N/K)$ for the corresponding Galois group. We give $\text{Aut}(N/K)$ the Krull topology, and we briefly recall how this is defined. Let \mathcal{L} denote the set of intermediate fields L such that $K \subseteq L \subseteq N$ and L/K is a finite Galois extension. Obviously \mathcal{L} is partially ordered by set inclusion. If L and M are in \mathcal{L} then the composite field LM is in \mathcal{L} , $L \subseteq LM$, $M \subseteq LM$, and therefore \mathcal{L} is a directed set. For each L in \mathcal{L} let $\text{Aut}(L/K)$ denote the Galois group of automorphisms of L that fix K . If $L \subseteq M$ are both in \mathcal{L} , we define $\pi_L^M : \text{Aut}(M/K) \rightarrow \text{Aut}(L/K)$ to be the map that restricts the domain of an automorphism in $\text{Aut}(M/K)$ to the subfield L . Then each map π_L^M is a surjective homomorphism of groups and π_L^L is the identity map. It follows that

$$\{\text{Aut}(L/K), \pi_L^M\}$$

is an inverse system, and $\text{Aut}(N/K)$ can be identified with the inverse (or projective) limit:

$$\text{Aut}(N/K) = \varprojlim_{L \in \mathcal{L}} \text{Aut}(L/K).$$

Thus $\text{Aut}(N/K)$ is a profinite group, and therefore is a totally disconnected, compact, Hausdorff, topological group. We write

$$\pi_L : \text{Aut}(N/K) \rightarrow \text{Aut}(L/K)$$

for the canonical map associated with each L in \mathcal{L} . Then π_L is continuous and the collection of open sets

$$(2.3) \quad \{\pi_L^{-1}(\tau) : L \in \mathcal{L} \text{ and } \tau \in \text{Aut}(L/K)\}$$

is a basis for the Krull topology in $\text{Aut}(N/K)$.

Next we assume that v is a place of the field K . That is, v is an equivalence class of nontrivial absolute values on K . If L is in \mathcal{L} we write $W_v(L/K)$ for the set of places w of L such that $w|v$. As L/K is a finite extension, it follows that $W_v(L/K)$ is a finite set. If $L \subseteq M$ belong to \mathcal{L} we define connecting maps

$$\psi_L^M : W_v(M/K) \rightarrow W_v(L/K)$$

as follows: if w_M belongs to $W_v(M/K)$ then $\psi_L^M(w_M)$ is the unique place w_L in $W_v(L/K)$ such that $w_M|w_L$. If $L \subseteq M$ are in \mathcal{L} then each absolute value on L extends to M and therefore each connecting map ψ_L^M is surjective. We give

each finite set $W_v(L/K)$ the discrete topology so that each map ψ_L^M is continuous. Clearly ψ_L^L is the identity map. We find that

$$\{W_v(L/K), \psi_L^M\}$$

is an inverse system of finite sets. Let

$$Y(K, v) = \varprojlim_{L \in \mathcal{L}} W_v(L/K)$$

denote the inverse limit and write $\psi_L : Y(K, v) \rightarrow W_v(L/K)$ for the canonical continuous map associated to each L in \mathcal{L} . It follows, as in [3, Appendix 2, section 2.4], that $Y(K, v)$ is a nonempty, totally disconnected, compact, Hausdorff space. Moreover, see [3, Appendix 2, section 2.3], the collection of open sets

$$(2.4) \quad \{\psi_L^{-1}(w) : L \in \mathcal{L} \text{ and } w \in W_v(L/K)\}$$

is a basis for the topology of $Y(K, v)$. Clearly each subset in the collection (2.4) is also compact, and for each field L in \mathcal{L} we can write

$$Y(K, v) = \bigcup_{w \in W_v(L/K)} \psi_L^{-1}(w)$$

as a disjoint union of open and compact sets.

We recall that a map $g : Y(K, v) \rightarrow \mathbb{R}$ is *locally constant* if at each point y in $Y(K, v)$ there exists an open neighborhood of y on which g is constant.

Lemma 1. *Let $g : Y(K, v) \rightarrow \mathbb{R}$ be locally constant. Then there exists L in \mathcal{L} such that for each place w in $W_v(L/K)$ the function g is constant on the set $\psi_L^{-1}(w)$.*

Proof. At each point y in $Y(K, v)$ there exists a field $L^{(y)}$ in \mathcal{L} and a place $w^{(y)}$ in $W_v(L^{(y)}/K)$ such that y is contained in $\psi_{L^{(y)}}^{-1}(w^{(y)})$ and g is constant on the open set $\psi_{L^{(y)}}^{-1}(w^{(y)})$. By compactness there exists a finite collection of fields $L^{(1)}, L^{(2)}, \dots, L^{(J)}$ in \mathcal{L} , and for each integer j a corresponding place $w^{(j)}$ in $W_v(L^{(j)}/K)$, such that

$$Y(K, v) \subseteq \bigcup_{j=1}^J \psi_{L^{(j)}}^{-1}(w^{(j)}),$$

and g is constant on each open set $\psi_{L^{(j)}}^{-1}(w^{(j)})$. Let $L = L^{(1)}L^{(2)} \dots L^{(J)}$ be the composite field, which is obviously in \mathcal{L} . If w is a place of L then there exists an integer j such that

$$\psi_L^{-1}(w) \cap \psi_{L^{(j)}}^{-1}(w^{(j)})$$

is not empty. As L is a finite extension of $L^{(j)}$, we conclude that $\psi_L^{-1}(w) = \psi_{L^{(j)}}^{-1}(w^{(j)})$, and therefore

$$(2.5) \quad \psi_L^{-1}(w) \subseteq \psi_{L^{(j)}}^{-1}(w^{(j)}).$$

Then (2.5) implies that g is constant on $\psi_L^{-1}(w)$. □

Let $C(Y(K, v))$ denote the real Banach algebra of continuous functions

$$F : Y(K, v) \rightarrow \mathbb{R}$$

with the supremum norm. Let $LC(Y(K, v)) \subseteq C(Y(K, v))$ denote the subset of locally constant functions.

Lemma 2. *The subset $LC(Y(K, v))$ is a dense subalgebra of $C(Y(K, v))$.*

Proof. It is obvious that $LC(Y(K, v))$ is a subalgebra of $C(Y(K, v))$, and that $LC(Y(K, v))$ contains the constant functions. Now suppose that y_1 and y_2 are distinct points in $Y(K, v)$. Let U_1 be an open neighborhood of y_1 , and U_2 an open neighborhood of y_2 , such that U_1 and U_2 are disjoint. Then there exists a field L in \mathcal{L} and a place w in $W_v(L/K)$ such that

$$y_1 \in \psi_L^{-1}(w), \quad \text{and} \quad \psi_L^{-1}(w) \subseteq U_1.$$

As $\psi_L^{-1}(w)$ is both open and compact, the characteristic function of the set $\psi_L^{-1}(w)$ is a locally constant function that separates the points y_1 and y_2 . Then it follows from the Stone-Weierstrass theorem that the subalgebra $LC(Y(K, v))$ is dense in $C(Y(K, v))$. \square

We select an absolute value from the place v of K and denote it by $\| \cdot \|_v$. If L is in \mathcal{L} and w is a place in $W_v(L/K)$, we select an absolute value $\| \cdot \|_w$ from w such that the restriction of $\| \cdot \|_w$ to K is equal to $\| \cdot \|_v$. As

$$N = \bigcup_{L \in \mathcal{L}} L,$$

it follows that each point (w_L) in $Y(K, v)$ determines a unique absolute value on the field N . That is, each point (w_L) in $Y(K, v)$ determines a unique place y of N such that $y|v$.

Now suppose y is a place of N such that $y|v$. Select an absolute value $\| \cdot \|_y$ from y such that the restriction of $\| \cdot \|_y$ to the subfield K is equal to $\| \cdot \|_v$. If L is in \mathcal{L} then the restriction of $\| \cdot \|_y$ to L must equal $\| \cdot \|_{w_L}$ for a unique place w_L in $W_v(L/K)$. Thus each place y of N with $y|v$ determines a unique point (w_L) in the product

$$\prod_{L \in \mathcal{L}} W_v(L/K)$$

such that $y|w_L$ for each L . It is trivial to check that

$$\psi_L^M(w_M) = w_L$$

whenever $L \subseteq M$ are in \mathcal{L} . Therefore each place y of N with $y|v$ determines a unique point (w_L) in the inverse limit $Y(K, v)$. In view of these remarks we may identify $Y(K, v)$ with the set of all places y of N that lie over the place v of K . In this way we determine a totally disconnected, compact, Hausdorff, topology in the set of all places y of N that lie over the place v of K .

3. GALOIS ACTION ON PLACES

Next we recall that the Galois group $\text{Aut}(N/K)$ acts on the set $Y(K, v)$ of all places of N that lie over v . More precisely, if τ is in $\text{Aut}(N/K)$ and y is in $Y(K, v)$, then the map

$$(3.1) \quad \alpha \mapsto \|\tau^{-1}\alpha\|_y$$

is an absolute value on N , and the restriction of this absolute value to K is clearly equal to $\| \cdot \|_v$. Therefore (3.1) determines a unique place τy in $Y(K, v)$. That is, the identity

$$(3.2) \quad \|\tau^{-1}\alpha\|_y = \|\alpha\|_{\tau y}$$

holds for all α in N , for all τ in $\text{Aut}(N/K)$, and for all places y in $Y(K, v)$. It is immediate that $1y = y$ and $(\sigma\tau)y = \sigma(\tau y)$ for all σ and τ in $\text{Aut}(N/K)$. Thus

$(\tau, y) \mapsto \tau y$ defines an action of the group $\text{Aut}(N/K)$ on the set $Y(K, v)$. Moreover, $\text{Aut}(N/K)$ acts transitively on $Y(K, v)$, (see [8, Chapter II, Proposition 9.1].)

Lemma 3. *The function $(\tau, y) \mapsto \tau y$ from $\text{Aut}(N/K) \times Y(K, v)$ onto $Y(K, v)$ is continuous.*

Proof. Let L be in \mathcal{L} and w in $W_v(L/K)$. In view of (2.4) we must show that

$$\{(\tau, y) \in \text{Aut}(N/K) \times Y(K, v) : \tau y \in \psi_L^{-1}(w)\}$$

is open in $\text{Aut}(N/K) \times Y(K, v)$ with the product topology. For w in $W_v(L/K)$ we define

$$E_w = \{(\sigma, z) \in \text{Aut}(L/K) \times W_v(L/K) : \sigma z = w\}.$$

Then we have

$$\begin{aligned} & \{(\tau, y) \in \text{Aut}(N/K) \times Y(K, v) : \tau y \in \psi_L^{-1}(w)\} \\ &= \{(\tau, y) \in \text{Aut}(N/K) \times Y(K, v) : \pi_L(\tau)\psi_L(y) = w\} \\ &= \bigcup_{(\sigma, z) \in E_w} \{(\tau, y) \in \text{Aut}(K/k) \times Y(K, v) : \pi_L(\tau) = \sigma \text{ and } \psi_L(y) = z\} \\ &= \bigcup_{(\sigma, z) \in E_w} \pi_L^{-1}(\sigma) \times \psi_L^{-1}(z), \end{aligned}$$

which is obviously an open subset of $\text{Aut}(N/K) \times Y(K, v)$. \square

4. THE INVARIANT MEASURE

In this section it will be convenient to write $G = \text{Aut}(N/K)$. Let μ denote a Haar measure on the Borel subsets of the compact topological group G normalized so that $\mu(G) = 1$. If F is in $C(Y(K, v))$ and z_1 is a point in $Y(K, v)$ then it follows from Lemma 3 that $\tau \mapsto F(\tau z_1)$ is a continuous function on G with values in \mathbb{R} . Let z_2 be a second point in $Y(K, v)$. Because G acts transitively on $Y(K, v)$, there exists η in G so that $\eta z_2 = z_1$. Then using the translation invariance of Haar measure we get

$$(4.1) \quad \int_G F(\tau z_1) \, d\mu(\tau) = \int_G F(\tau \eta z_2) \, d\mu(\tau) = \int_G F(\tau z_2) \, d\mu(\tau).$$

It follows that the map $I_v : C(Y(K, v)) \rightarrow \mathbb{R}$ given by

$$(4.2) \quad I_v(F) = \int_G F(\tau z_v) \, d\mu(\tau)$$

does not depend on the point z_v in $Y(K, v)$.

Let \mathcal{M}_K be a collection of distinct places of K and at each place v in \mathcal{M}_K let $\|\cdot\|_v$ denote an absolute value from v . We assume that

$$\{\|\cdot\|_v : v \in \mathcal{M}_K\}$$

is a proper collection of absolute values. Again we assume that N/K is a (possibly infinite) Galois extension of fields. Let Y be defined by the disjoint union

$$(4.3) \quad Y = \bigcup_{v \in \mathcal{M}_K} Y(K, v).$$

Thus Y is the collection of all places y of N such that $y|v$ for some place v in \mathcal{M}_K . It follows that Y is a nonempty, totally disconnected, locally compact, Hausdorff space.

Let $C_c(Y)$ denote the real vector space of continuous functions $F : Y \rightarrow \mathbb{R}$ having compact support. If F belongs to $C_c(Y)$ then there exists a finite subset $S_F \subseteq \mathcal{M}_K$ such that F is supported on the compact set

$$\bigcup_{v \in S_F} Y(K, v).$$

In particular we have $I_v(F) = 0$ for almost all places v of \mathcal{M}_K . Therefore we define $I : C_c(Y) \rightarrow \mathbb{R}$ by

$$(4.4) \quad I(F) = \sum_{v \in \mathcal{M}_K} \int_G F(\tau z_v) \, d\mu(\tau),$$

where z_v is a point in $Y(K, v)$ for each place v in \mathcal{M}_K . By our previous remarks the value of each integral on the right of (4.4) does not depend on z_v , and only finitely many integrals on the right of (4.4) are nonzero. Hence there is no question of convergence in the sum on the right of (4.4).

Theorem 4. *There exists a σ -algebra \mathcal{Y} of subsets of Y , that contains the σ -algebra \mathcal{B} of Borel sets in Y , and a unique, regular measure λ defined on \mathcal{Y} , such that*

$$(4.5) \quad I(F) = \int_Y F(y) \, d\lambda(y)$$

for all F in $C_c(Y)$. Moreover, the measure λ satisfies the following conditions:

(i) *If η is in G and F is in $L^1(Y, \mathcal{Y}, \lambda)$ then*

$$(4.6) \quad \int_{Y(K, v)} F(\eta y) \, d\lambda(y) = \int_{Y(K, v)} F(y) \, d\lambda(y)$$

at each place v in \mathcal{M}_K .

(ii) *If E is in \mathcal{Y} then*

$$\lambda(E) = \inf\{\lambda(U) : E \subseteq U \subseteq Y \text{ and } U \text{ is open}\}.$$

(iii) *If E is in \mathcal{Y} then*

$$\lambda(E) = \sup\{\lambda(V) : V \subseteq E \text{ and } V \text{ is compact}\}.$$

(iv) *If E is in \mathcal{Y} and $\lambda(E) = 0$ then every subset of E is in \mathcal{Y} .*

Proof. Clearly (4.4) defines a positive linear functional on $C_c(Y)$. By the Riesz representation theorem, (see [9, Theorem 2.14 and Theorem 2.17]), there exists a σ -algebra \mathcal{Y} of subsets of Y , containing the σ -algebra \mathcal{B} of Borel sets in Y , and a regular measure λ defined on \mathcal{Y} , such that

$$(4.7) \quad I(F) = \int_Y F(y) \, d\lambda(y)$$

for all F in $C_c(Y)$. If η is in G and F is in $C_c(Y)$, then by the translation invariance of the Haar measure μ we have

$$(4.8) \quad \begin{aligned} \int_{Y(K, v)} F(\eta y) \, d\lambda(y) &= \int_G F(\eta \tau z) \, d\mu(\tau) \\ &= \int_G F(\tau z) \, d\mu(\tau) \\ &= \int_{Y(K, v)} F(y) \, d\lambda(y) \end{aligned}$$

at each place v in \mathcal{M}_K . Initially (4.8) holds for all functions F in $C_c(Y)$. As $C_c(Y)$ is dense in the $L^1(Y, \mathcal{Y}, \lambda)$, (see [9, Theorem 3.14]), it follows in a standard manner that (4.8) holds also for functions F in $L^1(Y, \mathcal{Y}, \lambda)$.

The properties (ii), (iii) and (iv) attributed to λ are all consequences of the Riesz theorem. \square

Because the Haar measure μ satisfies $\mu(G) = 1$, it is immediate from (4.2) and (4.5) that $\lambda(Y(K, v)) = 1$ at each place v in \mathcal{M}_K . As the places in \mathcal{M}_K are well behaved, we obtain a further identity for the λ -measure of basic open sets in each subset $Y(K, v)$.

Theorem 5. *If L is in \mathcal{L} and w is a place in $W_v(L/K)$, then*

$$(4.9) \quad \lambda(\psi_L^{-1}(w)) = \frac{[L_w : K_v]}{[L : K]}.$$

Proof. Let τ be in G . Then we have

$$(4.10) \quad \begin{aligned} \tau\psi_L^{-1}(w) &= \{\tau y \in Y(K, v) : \psi_L(y) = w\} \\ &= \{y \in Y(K, v) : \pi_L(\tau^{-1})\psi_L(y) = w\} \\ &= \{y \in Y(K, v) : \psi_L(y) = \pi_L(\tau)w\} \\ &= \psi_L^{-1}(\pi_L(\tau)w). \end{aligned}$$

Now let w_1 and w_2 be distinct places in $W_v(L/K)$. Select τ in G so that $\pi_L(\tau)w_2 = w_1$. Then (4.10) implies that

$$\tau\psi_L^{-1}(w_2) = \psi_L^{-1}(w_1),$$

and using (4.6) we find that

$$\lambda\{\psi_L^{-1}(w_2)\} = \lambda\{\psi_L^{-1}(w_1)\}.$$

Because

$$(4.11) \quad Y(K, v) = \bigcup_{w \in W_v(L/K)} \psi_L^{-1}(w)$$

is a disjoint union of $|W_v(L/K)|$ distinct sets, the sets on the right of (4.11) all have equal λ -measure, and $\lambda(Y(K, v)) = 1$, we conclude that

$$(4.12) \quad \lambda(\psi_L^{-1}(w)) = |W_v(L/K)|^{-1}.$$

As v is well behaved we have

$$(4.13) \quad [L : K] = \sum_{w \in W_v(L/K)} [L_w : K_v].$$

Because L/K is a Galois extension, all local degrees $[L_w : K_v]$ for w in $W_v(L/K)$ are equal, and we conclude from (4.13) that

$$(4.14) \quad |W_v(L/K)| = \frac{[L : K]}{[L_w : K_v]}.$$

The identity (4.9) follows now from (4.12) and (4.14). \square

Let $LC_c(Y)$ be the algebra of locally constant, real valued functions on Y having compact support. Clearly we have $LC_c(Y) \subseteq C_c(Y)$.

Lemma 4. *Let g belong to $LC_c(Y)$. Then there exists L in \mathcal{L} such that for each place w in \mathcal{M}_L the function g is constant on the set $\psi_L^{-1}(w)$.*

Proof. Let $S_g \subset \mathcal{M}_K$ be a finite set of places of K such that the support of g is contained in the compact set

$$V_g = \bigcup_{v \in S_g} Y(K, v).$$

For each place v in S_g we apply Lemma 1 to the restriction of g to $Y(K, v)$. Thus there exists a field $L^{(v)}$ in \mathcal{L} such that for each place w' in $W_v(L^{(v)}/K)$, the function g is constant on $\psi_{L^{(v)}}^{-1}(w')$. Let L be the compositum of the finite collection of fields

$$\{L^{(v)} : v \in S_g\}.$$

Clearly L belongs to \mathcal{L} .

Let w be a place in \mathcal{M}_L . If $w|v$ and $v \notin S_g$, then g is identically zero on $\psi_L^{-1}(w)$, and in particular it is constant on this set. If $w|v$ and $v \in S_g$, then $w|w'$ for a unique place w' in $W_v(L^{(v)}/K)$. Because

$$\psi_L^{-1}(w) \subseteq \psi_{L^{(v)}}^{-1}(w')$$

and g is constant on $\psi_{L^{(v)}}^{-1}(w')$, it is obvious that g is constant on $\psi_L^{-1}(w)$. \square

Lemma 5. *For $1 \leq p < \infty$ the set $LC_c(Y)$ is dense in $L^p(Y, \mathcal{B}, \lambda)$. And $LC_c(Y)$ is dense in $C_0(Y)$ with respect to the sup-norm.*

Proof. Let $1 \leq p < \infty$. Because $C_c(Y)$ is dense in $L^p(Y, \mathcal{B}, \lambda)$, it suffices to show that if F is in $C_c(Y)$ and $\epsilon > 0$, then there exists a function g in $LC_c(Y)$ such that

$$\left\{ \int_Y |F(y) - g(y)|^p d\lambda(y) \right\}^{1/p} < \epsilon.$$

Let $S_F \subseteq \mathcal{M}_K$ be a nonempty, finite set of places such that F is supported on the compact set

$$V_F = \bigcup_{v \in S_F} Y(K, v).$$

For each v in S_F we apply Lemma 2 to the restriction of F to $Y(K, v)$. Thus there exists a locally constant function $g_v : Y(K, v) \rightarrow \mathbb{R}$ such that

$$(4.15) \quad \sup \{|F(y) - g_v(y)| : y \in Y(K, v)\} < |S_F|^{-1/p} \epsilon.$$

Now define $g : Y \rightarrow \mathbb{R}$ by

$$(4.16) \quad g(y) = \begin{cases} g_v(y) & \text{if } y \in Y(K, v) \text{ and } v \in S_F, \\ 0 & \text{if } y \in Y(K, v) \text{ and } v \notin S_F. \end{cases}$$

Then g is locally constant and supported on the compact set V_F . Therefore g belongs to $LC_c(Y)$. As $\lambda(Y(K, v)) = 1$ at each place v in \mathcal{M}_K , we get

$$\begin{aligned} \left\{ \int_Y |F(y) - g(y)|^p d\lambda(y) \right\}^{1/p} &= \left\{ \sum_{v \in S_F} \int_{Y(K, v)} |F(y) - g_v(y)|^p d\lambda(y) \right\}^{1/p} \\ &< \left\{ \sum_{v \in S_F} |S_F|^{-1} \epsilon^p \right\}^{1/p} \leq \epsilon. \end{aligned}$$

This proves the first assertion of the lemma.

As $C_c(Y)$ is dense in $C_0(Y)$ with respect to the sup-norm, the second assertion of the lemma follows by the same argument. In this case we select the locally constant functions $g_v : Y(K, v) \rightarrow \mathbb{R}$ so that

$$\sup \{|F(y) - g_v(y)| : y \in Y(K, v)\} < \epsilon.$$

Then we define $g : Y \rightarrow \mathbb{R}$ as in (4.16). Again we find that g belongs to $LC_c(Y)$, and the inequality

$$\sup \{|F(y) - g(y)| : y \in Y\} < \epsilon$$

is obvious. \square

5. THE COMPLETION OF \mathcal{G}

In this section we return to the situation considered in the introduction. We let $K = \mathbb{Q}$, $N = \overline{\mathbb{Q}}$, and we let $\mathcal{M}_{\mathbb{Q}}$ be the set of all places of \mathbb{Q} . Then Y is the set of all places of $\overline{\mathbb{Q}}$, and Y is a nonempty, totally disconnected, locally compact, Hausdorff space. By Theorem 4 there exists a σ -algebra \mathcal{Y} of subsets of Y , containing the σ -algebra \mathcal{B} of Borel sets in Y , and a measure λ on \mathcal{Y} , satisfying the conclusions of that result. The basic identity (1.7) is verified by Theorem 5. Then the map

$$(5.1) \quad \alpha \rightarrow f_\alpha(y)$$

defined by (1.9) is a linear map from the \mathbb{Q} -vector space

$$\mathcal{G} = \overline{\mathbb{Q}}^\times / \text{Tor}(\overline{\mathbb{Q}}^\times)$$

(written multiplicatively) into the vector space $C_c(Y)$. The identity (1.10) implies that each function $f_\alpha(y)$ belongs to the closed subspace $\mathcal{X} \subseteq L^1(Y, \mathcal{B}, \lambda)$ defined by (1.8). It follows from basic properties of the height, and in particular (1.4), that

$$\alpha \rightarrow 2h(\alpha),$$

defines a norm on \mathcal{G} with respect to the usual archimedean absolute value on \mathbb{Q} . Then (1.11) shows that (5.1) defines a linear isometry of \mathcal{G} into the subspace \mathcal{X} .

Lemma 6. *Let k be an algebraic number field and let $v \rightarrow t_v$ be a real valued function defined on the set of all places v of k . If*

$$(5.2) \quad \sum_v t_v \log |\alpha|_v = 0$$

for all α in $k^\times / \text{Tor}(k^\times)$, then the function $v \rightarrow t_v$ is constant.

Proof. Let S be a finite set of places of k containing all archimedean places, and assume that the cardinality of S is $s \geq 2$. We write \mathbb{R}^s for the s -dimensional real vector space of column vectors $\mathbf{x} = (x_v)$ having rows indexed by places v in S . In particular, we write $\mathbf{t} = (t_v)$ for the column vector in \mathbb{R}^s formed from the values of the function $v \rightarrow t_v$ restricted to S . And we write $\mathbf{u} = (u_v)$ for the column vector in \mathbb{R}^s such that $u_v = 1$ for each v in S .

Let

$$U_S(k) = \{\eta \in k : |\eta|_v = 1 \text{ for all } v \notin S\}$$

denote the multiplicative group of S -units in k . By the S -unit theorem (stated as [7, Theorem 3.5]), there exist multiplicatively independent elements $\xi_1, \xi_2, \dots, \xi_{s-1}$ in $U_S(k)$ which form a fundamental system of S -units. Write

$$M = ([k_v : \mathbb{Q}_v] \log \|\xi_r\|_v)$$

for the associated $(s-1) \times s$ real matrix, where $r = 1, 2, \dots, s-1$ indexes rows and v in S indexes columns. As the S -regulator does not vanish, the matrix M has rank $(s-1)$. Hence the null space

$$\mathcal{N} = \{\mathbf{x} \in \mathbb{R}^s : M\mathbf{x} = \mathbf{0}\}$$

has dimension 1. From the product formula we have $M\mathbf{u} = \mathbf{0}$. Therefore \mathcal{N} is spanned by the vector \mathbf{u} . By hypothesis we have $M\mathbf{t} = \mathbf{0}$, and it follows that \mathbf{t} is a scalar multiple of \mathbf{u} . That is, the function $v \rightarrow t_v$ is constant on S . As S is arbitrary the lemma is proved. \square

We now prove Theorem 1. Let \mathcal{E}_1 denote the closure of \mathcal{F} in \mathcal{X} . As \mathcal{F} is a vector space over the field \mathbb{Q} , it follows that \mathcal{E}_1 is a vector space over \mathbb{R} , and therefore \mathcal{E}_1 is a closed linear subspace of \mathcal{X} . If \mathcal{E}_1 is a proper subspace then it follows from the Hahn-Banach theorem (see [10, Theorem 3.5]) that there exists a continuous linear functional $\Phi : \mathcal{X} \rightarrow \mathbb{R}$ such that Φ vanishes on \mathcal{E}_1 , but Φ is not the zero linear functional on \mathcal{X} . We will show that such a Φ does not exist, and therefore we must have $\mathcal{E}_1 = \mathcal{X}$.

Let $\Phi : \mathcal{X} \rightarrow \mathbb{R}$ be a continuous linear functional that vanishes on \mathcal{E}_1 , but Φ is not the zero linear functional on \mathcal{X} . It follows from (1.8) that $\mathcal{X}^\perp \subseteq L^\infty(Y, \mathcal{B}, \lambda)$ is the one dimensional subspace spanned by the constant function 1. As the dual space \mathcal{X}^* can be identified with the quotient space $L^\infty(Y, \mathcal{B}, \lambda)/\mathcal{X}^\perp$, there exists a function $\varphi(y)$ in $L^\infty(Y, \mathcal{B}, \lambda)$ such that $\varphi(y)$ and the constant function 1 are linearly independent, and

$$\Phi(F) = \int_Y F(y)\varphi(y) \, d\lambda(y)$$

for all F in \mathcal{X} . Because Φ vanishes on \mathcal{E}_1 we have

$$(5.3) \quad \int_Y f_\alpha(y)\varphi(y) \, d\lambda(y) = 0$$

for each function f_α in \mathcal{F} .

Now let k be a number field in \mathcal{L} and let α be in $k^\times / \text{Tor}(k^\times) \subseteq \mathcal{G}$. From (4.9) and (5.3) we find that

$$(5.4) \quad \begin{aligned} 0 &= \sum_v \left\{ \int_{\psi_k^{-1}(v)} \log \|\alpha\|_y \varphi(y) \, d\lambda(y) \right\} \\ &= \sum_v \left\{ \int_{\psi_k^{-1}(v)} \varphi(y) \, d\lambda(y) \right\} \log \|\alpha\|_v \\ &= \sum_v \left\{ \lambda(\psi_k^{-1}(v))^{-1} \int_{\psi_k^{-1}(v)} \varphi(y) \, d\lambda(y) \right\} \log |\alpha|_v. \end{aligned}$$

It follows from Lemma 6 that the function

$$v \rightarrow \lambda(\psi_k^{-1}(v))^{-1} \int_{\psi_k^{-1}(v)} \varphi(y) \, d\lambda(y)$$

is constant on the set of places v of k . We write $c(k)$ for this constant.

Let $k \subseteq l$ be number fields in \mathcal{L} , and let v be a place of k . Using (4.9) and (4.14) we have

$$\lambda(\psi_k^{-1}(v)) = |W_v(l/k)| \lambda(\psi_l^{-1}(w))$$

for all places w in the set $W_v(l/k)$. This leads to the identity

$$\begin{aligned}
(5.5) \quad c(l) &= |W_v(l/k)|^{-1} \sum_{w \in W_v(l/k)} \left\{ \lambda(\psi_l^{-1}(w))^{-1} \int_{\psi_l^{-1}(w)} \varphi(y) \, d\lambda(y) \right\} \\
&= \lambda(\psi_k^{-1}(v))^{-1} \sum_{w \in W_v(l/k)} \left\{ \int_{\psi_l^{-1}(w)} \varphi(y) \, d\lambda(y) \right\} \\
&= \lambda(\psi_k^{-1}(v))^{-1} \int_{\psi_k^{-1}(v)} \varphi(y) \, d\lambda(y) \\
&= c(k).
\end{aligned}$$

Thus there exists a real number C such that $C = c(k)$ for all fields k in \mathcal{L} .

Let g belong to $LC_c(Y)$. By Lemma 4 there exists a number field l in \mathcal{L} such that g is constant on $\psi_l^{-1}(w)$ for each place w of l . Therefore we have

$$\begin{aligned}
(5.6) \quad \int_Y g(y)\varphi(y) \, d\lambda(y) &= \sum_w \left\{ \int_{\psi_l^{-1}(w)} g(y)\varphi(y) \, d\lambda(y) \right\} \\
&= C \sum_w \left\{ \lambda(\psi_l^{-1}(w))g(\psi_l^{-1}(w)) \right\} \\
&= C \sum_w \left\{ \int_{\psi_l^{-1}(w)} g(y) \, d\lambda(y) \right\} \\
&= C \int_Y g(y) \, d\lambda(y).
\end{aligned}$$

By Lemma 5 the set $LC_c(Y)$ is dense in $L^1(Y, \mathcal{B}, \lambda)$, and we conclude from (5.6) that

$$\int_Y F(y)\varphi(y) \, d\lambda(y) = C \int_Y F(y) \, d\lambda(y)$$

for all F in $L^1(Y, \mathcal{B}, \lambda)$. This shows that $\varphi(y) = C$ in $L^\infty(Y, \mathcal{B}, \lambda)$, and so contradicts our assumption that $\varphi(y)$ and the constant function 1 are linearly independent. Hence the continuous linear functional Φ does not exist, and therefore $\mathcal{E}_1 = \mathcal{X}$.

6. PROOF OF THEOREM 2 AND THEOREM 3

We suppose that $1 < p < \infty$ and write \mathcal{E}_p for the closure of \mathcal{F} in $L^p(Y, \mathcal{B}, \lambda)$. As before, \mathcal{E}_p is a closed linear subspace. By the Hahn-Banach theorem it suffices to show that if $\Phi : L^p(Y, \mathcal{B}, \lambda) \rightarrow \mathbb{R}$ is a continuous linear functional that vanishes on \mathcal{E}_p , then in fact Φ is identically zero on $L^p(Y, \mathcal{B}, \lambda)$.

Let $p^{-1} + q^{-1} = 1$, and let $\varphi(y)$ be an element of $L^q(Y, \mathcal{B}, \lambda)$ such that

$$\Phi(F) = \int_Y F(y)\varphi(y) \, d\lambda(y)$$

for all F in $L^p(Y, \mathcal{B}, \lambda)$. We assume that Φ vanishes on \mathcal{E}_p , and then we have

$$(6.1) \quad \int_Y f_\alpha(y)\varphi(y) \, d\lambda(y) = 0$$

for each function f_α in \mathcal{F} .

Let k be a number field in \mathcal{L} and let α be in $k^\times / \text{Tor}(k^\times) \subseteq \mathcal{G}$. As before we apply (4.9) and (5.3) to obtain the identity (5.4). Then Lemma 6 implies that the function

$$(6.2) \quad v \rightarrow \lambda(\psi_k^{-1}(v))^{-1} \int_{\psi_k^{-1}(v)} \varphi(y) \, d\lambda(y)$$

is constant on the set of places v of k . Now, however, we apply Hölder's inequality and find that

$$\begin{aligned} & \sum_v \left| \lambda(\psi_k^{-1}(v))^{-1} \int_{\psi_k^{-1}(v)} \varphi(y) \, d\lambda(y) \right|^q \\ & \leq \sum_v \left\{ \lambda(\psi_k^{-1}(v))^{-1} \int_{\psi_k^{-1}(v)} |\varphi(y)|^q \, d\lambda(y) \right\} \\ & \leq [k : \mathbb{Q}] \int_Y |\varphi(x)|^q \, d\lambda(y) < \infty. \end{aligned}$$

This shows that the constant value of the function (6.2) is zero. Thus we have

$$\int_{\psi_k^{-1}(v)} \varphi(y) \, d\lambda(y) = 0$$

for all k in \mathcal{L} and for all places v of k . It follows using Lemma 4 that

$$\int_Y g(y) \varphi(y) \, d\lambda(y) = 0$$

for all g in $LC_c(Y)$. By Lemma 5 the set $LC_c(Y)$ is dense in $L^p(Y, \mathcal{B}, \lambda)$, and we conclude that the continuous linear functional Φ is identically zero. This completes the proof of Theorem 2.

Next we suppose that \mathcal{E}_∞ is the closure of \mathcal{F} in $C_0(Y)$. Again it suffices to show that if $\Phi : C_0(Y) \rightarrow \mathbb{R}$ is a continuous linear functional that vanishes on \mathcal{E}_∞ , then Φ is identically zero on $C_0(Y)$. If Φ is such a linear functional, then by the Riesz representation theorem (see [9, Theorem 6.19]) there exists a regular signed measure ν , defined on the σ -algebra \mathcal{B} of Borel sets in Y , such that

$$\Phi(F) = \int_Y F(y) \, d\nu(y)$$

for all F in $C_0(Y)$. Moreover, we have $\|\Phi\| = \|\nu\|$, where $\|\Phi\|$ is the norm of the linear functional Φ and $\|\nu\|$ is the total variation of the signed measure ν . We assume that Φ vanishes on \mathcal{E}_∞ , and therefore

$$\int_Y f_\alpha(y) \, d\nu(y) = 0$$

for each function f_α in \mathcal{F} . By arguing as in the proof of Theorem 2, we conclude that for each number field k in \mathcal{L} the function

$$(6.3) \quad v \rightarrow \lambda(\psi_k^{-1}(v))^{-1} \nu(\psi_k^{-1}(v)),$$

defined on the set of all places v of k , is constant. As

$$\begin{aligned} \sum_v \left| \lambda(\psi_k^{-1}(v))^{-1} \nu(\psi_k^{-1}(v)) \right| \\ \leq [k : \mathbb{Q}] \sum_v |\nu(\psi_k^{-1}(v))| \\ \leq [k : \mathbb{Q}] \|\nu\| < \infty, \end{aligned}$$

we conclude that the value of the constant function (6.3) is zero. This shows that

$$\nu(\psi_k^{-1}(v)) = 0$$

for all k in \mathcal{L} and for all places v of k . It follows as before that

$$\Phi(g) = \int_Y g(y) \, d\nu(y) = 0$$

for all g in $LC_c(Y)$. As $LC_c(Y)$ is dense in $C_0(Y)$ by Lemma 5, we find that Φ is identically zero on $C_0(Y)$. This proves Theorem 3.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS, AUSTIN, TEXAS 78712 USA
E-mail address: `allcock@math.utexas.edu`
E-mail address: `vaaler@math.utexas.edu`