Problem 1. Suppose $A$ is a positive integer, and that the digits of its base 10 representation are $a_na_{n-1} \ldots a_1a_0$. That is, that each $a_i$ is in $\{0, 1, 2, \ldots, 9\}$, $a_0$ is the 1's digit, $a_1$ is the 10's digit, etc. Prove that $A \equiv \sum_{i=1}^{n} a_i \pmod{9}$. (Some of you will already know the computational trick that 9 divides a positive integer if and only if it divides the sum of its digits. This problem tells you that, and also tells you how what the remainder is in the case that 9 doesn’t divide the sum of the digits.)

Problem 2. If $a, b, c \in \mathbb{Z}$, then $a \mid b$ if and only if $ac \mid bc$. (Actually this is wrong. Give a counterexample, then correct the statement and prove it.)

Problem 3. For each of the following pairs of numbers, find the gcd, and find Bezout numbers for them.

(a) $(51, 87)$
(b) $(105, 300)$
(c) $(981, 1234)$
(d) $(3709, 10313)$

Problem 4. Find Bezout numbers for the following, which you might remember from last homework.

(a) $(6k + 7, 3k + 4)$, where $k$ is an integer.
(b) $(n + 1, n^2 - n + 1)$, when $n$ is an integer. (Your answer will have to have several cases.)

Problem 5. Prove the following theorem that I stated in class but did not prove: if $a, b \in \mathbb{Z}$ and $d \in \mathbb{Z}$ divides both of them, then $\gcd(a, b) = d \gcd(a/d, b/d)$. (For example, $\gcd(125, 100) = 25 \gcd(5, 4) = 25 \cdot 1 = 25$.)

Problem 6. Suppose $s$ and $t$ are coprime integers, both odd, and that $s > t \geq 1$. Show that the greatest integer dividing all three of the numbers $a = st$, $b = (s^2 - t^2)/2$ and $c = (s^2 + t^2)/2$ is 1.

This looks like just a random result. The point is in the next problem.

Problem 7. We continue in the situation of the previous problem.

(a) show that $a^2 + b^2 = c^2$. Triples of integers $(a, b, c)$ satisfying this equation are called Pythagorean triples. So you now have a way to generate Pythagorean triples. The point of the previous problem is that there is no number dividing all three of the terms in this equation. For example the triple $(6, 8, 10)$ doesn’t come up even though $6^2 + 8^2 = 10^2$. We count this as a kind of a boring “repeat” of $3^2 + 4^2 = 5^2$.

(b) Explicitly write out the Pythagorean triple corresponding to $s = 15$, $t = 13$, and check on your calculator that it really works.

(c) The most famous Pythagorean triple is $(3, 4, 5)$. What are the $s, t$ corresponding to it?

(d) I once stumbled across the Pythagorean triple $(99, 20, 101)$ accidentally while laying out a small deck outside my home. Repeat (d) for this triple.

(In fact it turns out that all Pythagorean triples arise this way. And the formulas for $a$, $b$ and $c$ in terms of $s$ and $t$ did not just fall from the sky, even though it looks like that. There is lovely geometry behind this, which we might cover if the class is
interested. Truth in advertising: my use of the word “all” requires some “without loss of generality” preparation.)