Problem 1. Find the prime factorizations of the following numbers.
(a) 111,111
(b) 196,560
(c) 16!
(d) \( \binom{30}{10} \) (meaning the binomial coefficient \( 30!/20!10! \))

Problem 2. Find the gcd and lcm of each pair. Careful—there exists a trap. (Remember that when a mathematician says “there exists a...” he or she means “there exists at least one...”.)
(a) \( 2 \cdot 3^2 \cdot 5^3 \) and \( 2^7 \cdot 3^5 \cdot 5^3 \cdot 7^2 \)
(b) \( 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \) and \( 17 \cdot 19 \cdot 21 \cdot 23 \cdot 29 \)
(c) \( 2^3 \cdot 5^7 \cdot 11 \cdot 13 \) and \( 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \)
(d) \( 47^{11} \cdot 79^{11} \cdot 101^{1001} \) and \( 41^{11} \cdot 83^{11} \cdot 101^{1000} \)

Problem 3. (a) You showed on the first homework that the rational numbers \( \mathbb{Q} \) contain no square root of 2. Prove this again by applying Theorem 3.18 to the polynomial \( f(x) = x^2 - 2 \).
(b) Show that \( \sqrt{5} \notin \mathbb{Q} \).
(c) Show that \( \sqrt{2} + \sqrt{3} \notin \mathbb{Q} \). (This one is harder. Hint: write \( x_0 = \sqrt{2} + \sqrt{3} \) and find a suitable polynomial \( f(x) \) with \( f(x_0) = 0 \). You have to go beyond quadratic polynomials.)

Problem 4. Use uniqueness of prime factorization to prove that if \( p \) is a prime number then the only solutions to \( x^2 = 1 \) in \( \mathbb{Z}_p \) are \( x = \pm 1 \). (A problem on a previous homework, about \( \mathbb{Z}_8 \) and \( \mathbb{Z}_{15} \), was meant to show you that there really is something special about prime numbers.)

Problem 5. Let \( m \) be a positive integer, and let \( \mathbb{Z}_m^* \) mean the set of elements of \( \mathbb{Z}_m \) that have inverses. That is, an element \( x \) of \( \mathbb{Z}_m \) lies in \( \mathbb{Z}_m^* \) just if there exists \( y \in \mathbb{Z}_m \) such that \( xy = 1 \in \mathbb{Z}_m \).

We saw in class that inverses are unique when they exist, so \( f(x) = x^{-1} \) is a function \( \mathbb{Z}_m^* \rightarrow \mathbb{Z}_m^* \). (a) Show that it is a bijection. (b) Show that when \( m \) is prime then \( x = \pm 1 \) are the only elements of \( \mathbb{Z}_m^* \) satisfying \( f(x) = x \).

The previous homework had a problem that I postponed. I meant to postpone it to this homework, but starting Hensel’s lemma on the day before spring break is silly. Also, I promised to make this homework set shorter on account of posting it late. So: you do NOT have to solve that postponed problem.