Computing $\widehat{HF}$

Based on Sarkar-Wang.

Outline: ① Describe "nice" pointed Heegaard diagrams (HD)
② How to compute $\widehat{HF}$ for nice HD
③ Algorithm to obtain "nice" HD
④ Examples
⑤ Why it works

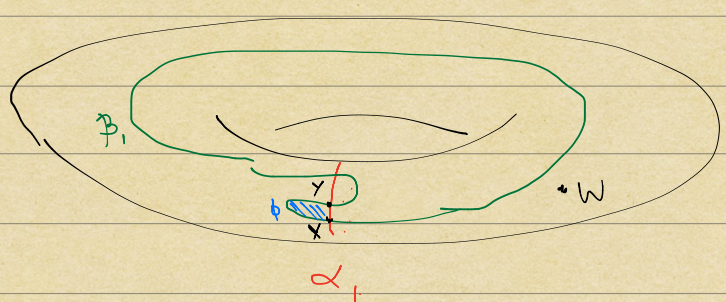
Definitions

Def Let $H = (\Sigma_g, \underline{\alpha}, \underline{\beta}, z)$ be pointed HD for Y closed, oriented. A connected component of $\Sigma_g \setminus (\underline{\alpha} \cup \underline{\beta})$ is called a region. A formal sum of regions with $\mathbb{Z}$-coef. is called a 2-chain.

Given $\underline{x}, \underline{y} \in T_\alpha \cap T_\beta$, let $\pi_2(\underline{x}, \underline{y})$ be the collection of 2-chains ϕ with $\partial((\partial\phi)|_{\underline{\alpha}}) = \underline{y} - \underline{x}$.

Such a 2-chain is called a domain.

Ex:



So $\partial(\partial\phi)|_{\underline{\alpha}}) = y - x$.

Given $p \in \Sigma \setminus \{\underline{\alpha} \cup \underline{\beta}\}$, let n_p be the coefficient of the region containing p in ϕ. A domain is positive if $n_p(\phi) \geq 0$ for all p.

Let $\pi_2^0(x,y) := \{\phi \in \pi_2(\underline{x},\underline{y}) \mid n_w(\phi)=0\}$

$\mathcal{H}$ is admissible if for all $\underline{x} \in T_\alpha \cap T_\beta$, every positive domain in $\pi_2^0(\underline{x},\underline{x})$ is trivial.

$\mathcal{H}$ is nice if every region not containing the base point w is a bi-gon or square.

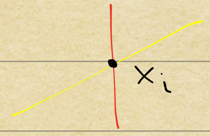
§2: Computing $\widehat{HF}$ for nice HD

Lipschitz's formula for Maslov index: let $\phi \in \pi_2(x,y)$. The Maslov index is given by

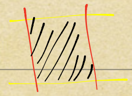
$$\mu(\phi) = e(\phi) + \mu_{\underline{x}} + \mu_{\underline{y}}$$

where i) $\mu_{\underline{x}}(\phi) = \sum \mu_{x_i}(\phi)$, $\mu_{x_i}(\phi)$ is the average of the coeff. of the four regions around it,

ii) Given a region D, $e(D) = \chi(D) - m/4$ where m = # corners (w/ multiplicity).



Ex: ① D  $e(D) = 1 - \frac{2}{4} = \frac{1}{2}$

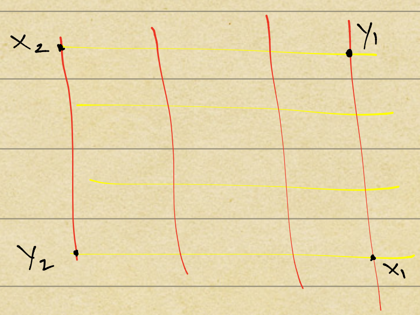
②  $e(D) = 1 - 4/4 = 0$

Extend e linearly to 2-chains.

A domain $\phi \in \pi_2^0(\underline{x}, \underline{y})$ with coeff. 0 or 1 is called an empty embedded $2n$-gon if it is topologically a disc with $2n$ marked points on its boundary, such that at each vertex v, $\mu_v(\phi) = 1/4$ and it does not contain any x_i or y_i in its interior.

Ex: 

2 marked points.



4 marked pts.

Thm (Sarkar-Wang) Let $\mathcal{H} = (\Sigma, \underline{\alpha}, \underline{\beta}, w)$ be (admissible) nice pointed HD for Y^3 closed, oriented. Let $\widehat{CF} = \mathbb{Z}^{T_\alpha \cap T_\beta}$. Then for $\underline{x} \in T^\alpha \cap T^\beta$, let

$$\partial(\underline{x}) = \sum_{\underline{y}} \sum_{\substack{\phi \in \pi_2^0(\underline{x}, \underline{y}) \\ \mu(\phi) = 1}} c(\phi) \underline{y}$$

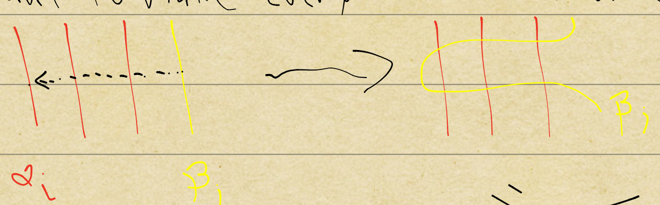
where $c(\phi) = 1$ if ϕ is an empty embedded bi-gon or empty embedded square, and $c(\phi) = 0$ otherwise.
Then the homology of this complex is $\widehat{HF}(Y)$ as defined previously.

§3: An Algorithm to obtain a nice HD

Start with $\mathcal{H} = (\Sigma, \underline{\alpha}, \underline{\beta}, w)$. We will (isotope/handleslide <u>only</u> the β curves.

Step 1: Killing non-disc regions.

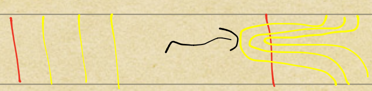
Want to make every α-curve intersect some β-curve, and vice-versa.



Make α_i intersect some β_j

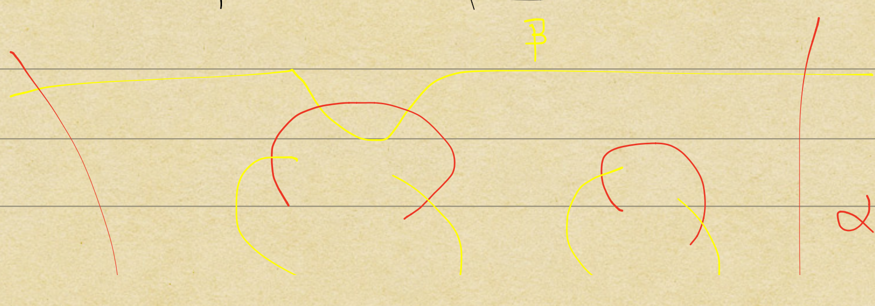
"Finger moves"

or,



Make β_j intersect some α_i.

$\Sigma \setminus \underline{\alpha}$ is a punctured sphere, so every region lives in $\Sigma \setminus \underline{\alpha} \subset S^2$, hence it's <u>planar</u>.



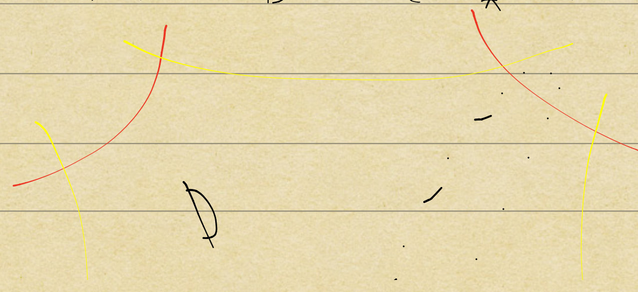
Finger move outside β-curves into inside α-curves.
Now every region is a disc.

Step 2: Make all but one region bi-gons or squares.

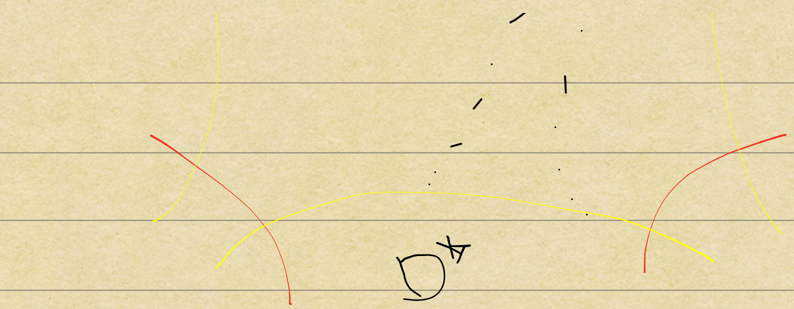
For a region D, pick $w' \in D$. The distance $d(D)$ is the minimum number of intersections of an arc connecting w' to w in the complement of the α-curves.
For a $2n$-gon region D, the badness of D is $b(D) = \max\{n-2, 0\}$.

The distance $d(H)$ of a diagram $\mathcal{H}$ is the maximum distance of a bad region.

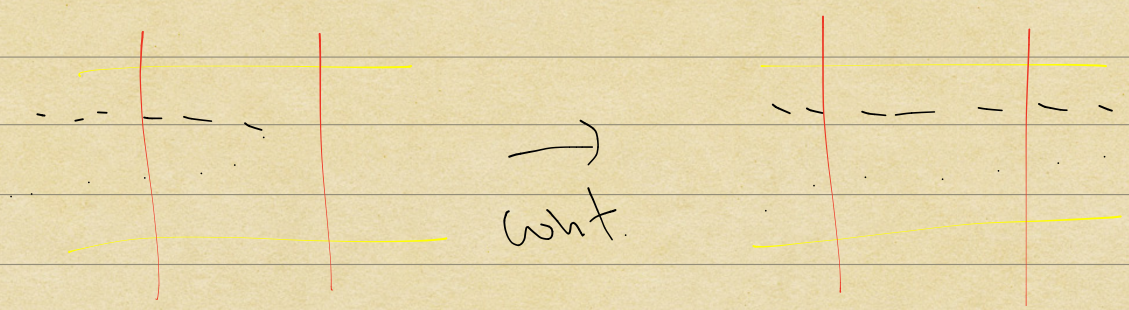
Algorithm: Let D be a distance $d = d(H)$ bad region with minimal badness, and let D^* be an adjacent distance $d-1$ region with a common β curve b_* with D.



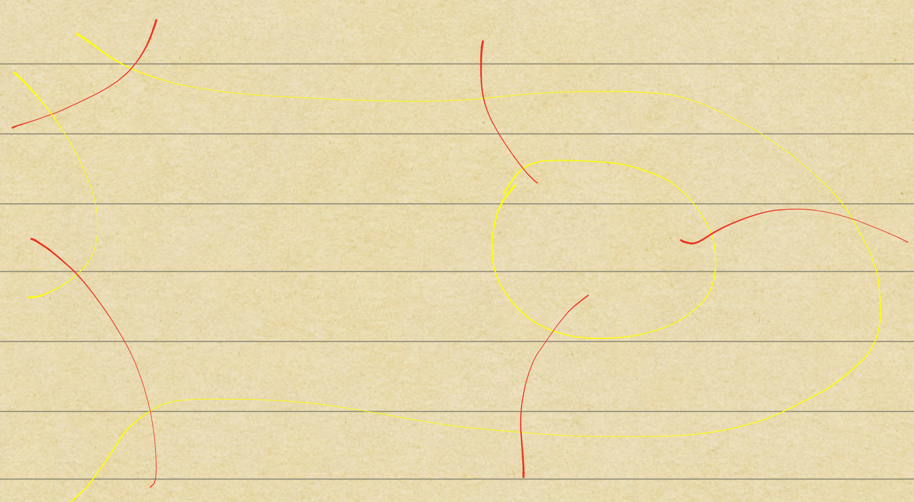
Push in



Do a finger move of b_* into a_2. If region we push into is a square, we can continue:

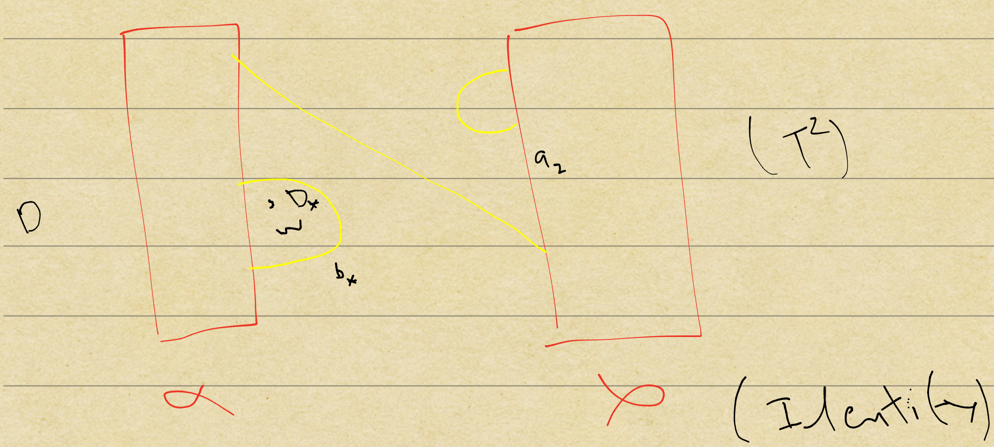


Push as far as possible until
a) we reach a bi-gon, done.
b) we reach a region of dist $\leq d-1$
c) A bad region of dist. d, which is not D, done.
d) We come back to D:

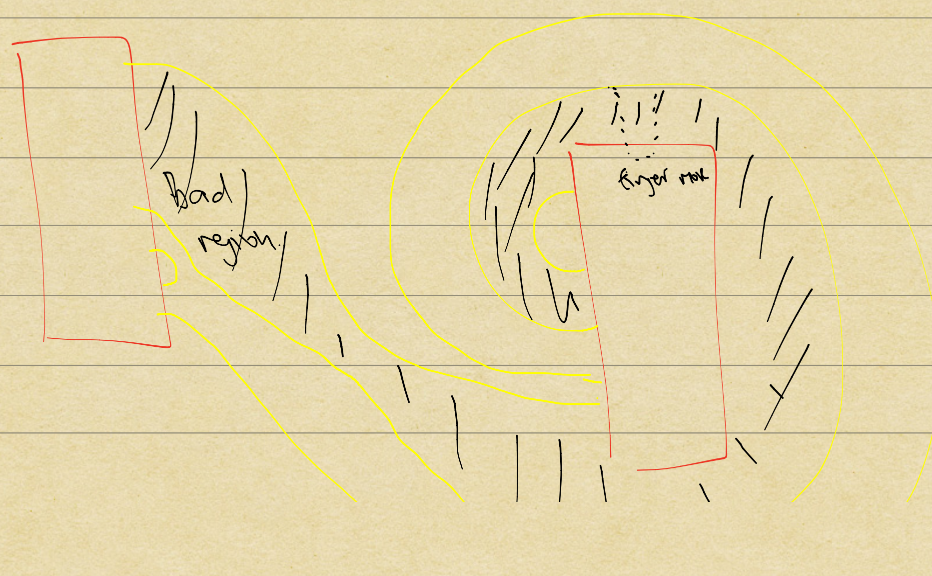


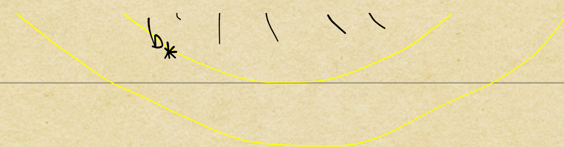
Now either (a) we returned to D through adjacent region a_1 or a_3 and we see all of β_i, so we do a handleslide over β_i, or (b) we return to D through non-adjacent α-curve. In this case do a finger move through a_3 instead of a_2; if we reach steps 1-4(a), stop the finger move. If a_3 doesn't work, go thru a_4, etc.

§4: Example.

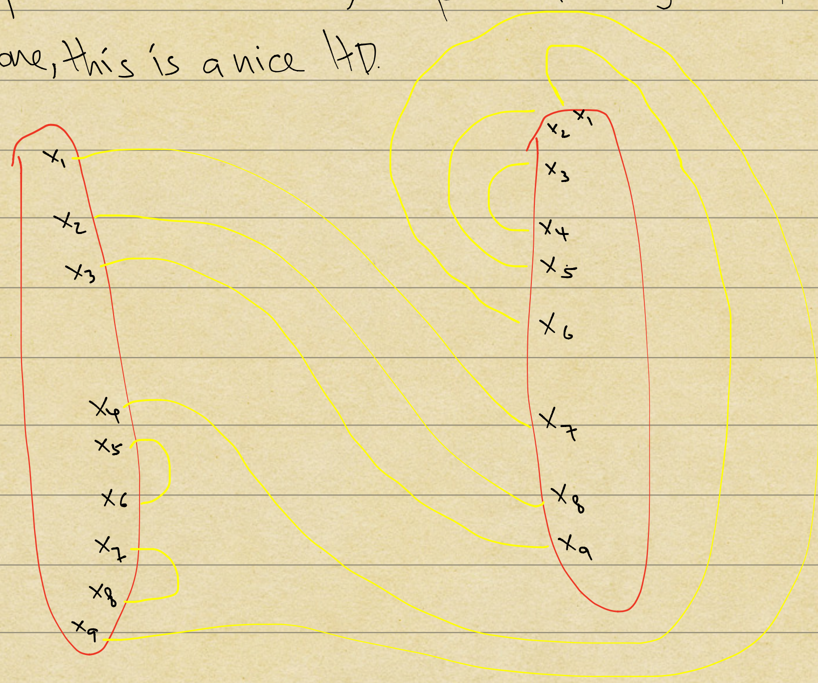


Only outside region is bad, has distance 1, since there's only 2 regions. Do finger move pushing into a_2:



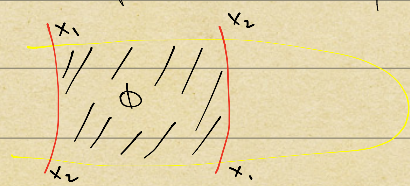


Continue process once more, we push into region w/ base point, so we're done, this is a nice HD.



So $\widehat{CF} = \mathbb{Z}^9$. We compute boundary map as follows.

$\partial(x_1) = x_2$



Similarly, $\partial(x_2) = 0$ since regions containing it also contain w.

⋮

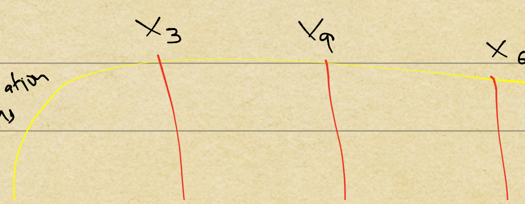
$\partial(x_3) = 0$

$\partial(x_4) = x_3$

$\partial(x_5) = x_6$

$\partial(x_6) = 0$ (not x_7 for orientation reasons)

$\partial(x_7) = x_6 + x_8$ ⟶



$\partial(x_8)=0$

$\partial(x_9)=0$

$\Rightarrow \text{Im}(\partial) = \mathbb{Z}_2\text{-span}\{x_2, x_3, x_6, x_8\}$

$\ker \partial = \mathbb{Z}_2\text{-span}\{x_2, x_3, x_6, x_8, x_9\}$

So $\widehat{HF} \cong \mathbb{Z}_2$, since Y here is S^3.

(Sarkar-Wang) Done for Poincaré homology S^3.

<u>Rmk:</u> Can be done for multiple basepoints as well.