Symplectic Topology/Geometry

Some linear algebra: Let V be finite dim'l real vector space, ω a skew-symmetric nondeg. bilinear form.

Ex: If $V=\mathbb{R}^{2n}$ w/ $\omega(v_1,v_2)=v_1^T A^T J A v_2$ and

$$J=\begin{bmatrix} 0 & -1 & & & & & \\ 1 & 0 & & & & & \\ & & 0 & -1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & -1 \\ & & & & & 1 & 0 \end{bmatrix} \quad \text{cx structure}$$

Diagonalization of skew-symmetric forms: for any skew-symmetric form $\exists$ a basis of V

$$\{\underbrace{e_1,\dots,e_i}_{\text{"real" coord.}},\underbrace{f_1,\dots,f_i}_{\text{imag. coord}},\underbrace{g_1,\dots,g_k}_{\text{degenerate}}\}$$

so if ω nondeg. then V even dim'l. For a subspace $V_1 \subseteq (V,\omega)$ define $V_1^{\perp}=\{x\in V \mid \omega(x,V_1)=0\}$. We say a subspace of V Lagrangian if $V_1^{\perp}=V_1$. This cuts out a submfld of the Grassmannian $Gr_n(\mathbb{R}^{2n})$ which we denote Λ.

Fact: $H_1(\Lambda)\cong\mathbb{Z}$ so we get a map $\pi_1(\Lambda)\to\mathbb{Z}$, called the Maslov index.

Def A symplectic mfld M is a smooth mfld with a nondeg

closed 2-form ω. A submfld L is Lagrangian if $TL \subset TM$ is an inclusion of Lagrangian subspaces.
A map $\phi: M \to M$ is a symplectomorphism if ϕ diffeo and $\phi^*(\omega)=\omega$.

Ex: ① If Σ orientable surface, any volume form is symplectic
② $\mathbb{R}^{2n}$ or $\mathbb{C}^n$ w/ $\sum dz_i \wedge d\bar{z}_i$ $(\sum dx_i \wedge dy_i)$
③ T^*M for M smooth, also $T(T^*M) \cong p^*(T^*M) \oplus p^*(TM)$ where $p: T^*M \to M$.
④ On $\mathbb{CP}^n$, the Hopf map $H: S^{2n+1} \to \mathbb{CP}^n$, fiber S^1.
Round metric induces ω_{FS}.

Almost cx structures

Def An almost cx structure on M smooth is $J \in End(TM)$ with $J^2 = -Id$.
Such a J is ω-tame for ω symplectic if $\omega(v,Jv)>0$ and ω-compatible if $\omega(v,Jw)$ is a Riem. metric, or equiv. $\omega(v,w) = \omega(Jv,Jw)$.

Thm (Gromov, cf. Wendl) Let The space $\mathcal{J}(M)$ of a.c. structures on M sympl. is contractible (in p^tic nonempty) if we restrict to tame/compatible

Idea (following Oh) Pick a metric on M, find A such that $\omega(-,-) =$

$g(A(-), -)$. This has SVD w/ unitary... and take
$J = A(A^T A)^{-1/2}$.

<u>Fact</u>: The almost complex structures coming from std almost cx structures on $\mathbb{C}^n, \mathbb{CP}^n$ are compatible with sympl. forms given earlier (Kähler).

<u>Lemma</u>: If $N \subseteq M$ with $J(TN) = TN$ we say that N is pseudoholomorphic, and if ω tame/compatible then N also sympl. submfld.

<u>pf</u>: $\omega(-, J-)$ is positive definite, so $\omega|_{TN + J(TN)}$ is nondeg. on TN. Closed-ness follows from naturality of d, exterior derivative. ☐

More examples: ① Cx submflds of $\mathbb{C}^n, \mathbb{CP}^n$
② Affine/Projective Varieties
③ Stein mflds
④ Blowups of above are all (pseudo)holomorphic

<u>Lemma</u> If $N \subseteq M$ with $J(TN) \cap TN = 0$ and $J(TN) \oplus TN = TM$ then N is <u>totally real</u>.

<u>Def</u> A map from a Riemann surface
$$u : (\Sigma, j) \to (M, \omega, J)$$
is pseudohol. if $du \circ j = J \circ du$.

Let $A \in H_2(M)$. A J-curve $u: S^2 \to M$ is <u>simple</u> if no nontrivial branched cover of S^2 factors thru it:

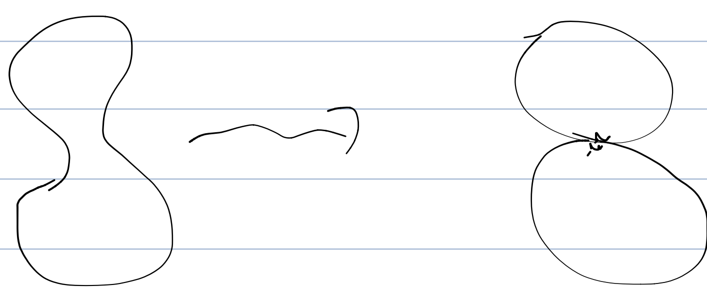
$$u = f \circ b \Rightarrow b = Id, u = f$$

Define $\mathcal{M}^*(A,J) := \{ u \in C^\infty(S^2, M) \mid u \text{ is J-curve, simple and } [u] = A \}$

<u>Thm</u> There is a "generic" (Baire second category) set $\mathcal{J}_{reg}$ in $\mathcal{J}_{tame/compatible}$ so that $\mathcal{M}^*(A,J)$ smooth, dimension (real) $2n + 2\langle c_1(TM), A \rangle$

<u>Note</u>: Action of $PSL(2;\mathbb{C})$ noncompact means $\mathcal{M}^*$ not cpt. But quotient $\mathcal{M}^*/PSL(2;\mathbb{C})$ can be.

<u>Thm</u> [Mc-S] Assume there are no spherical homology classes B w/ $0 < \omega(B) < \omega(A)$, then $\overline{\mathcal{M}}(A,J)$ compact.



"Bubbling"

<u>Evaluation Maps</u>

There is a natural map $ev: S^2 \times \mathcal{M}^*(A,J) \to M$

$$(x, u) \to u(x)$$

which we can use to constrain images of J-curves so moduli are smaller. ("Marking points")

In certain cases can get evaluation to be smooth map which we can use to pull back fundamental class, or more specifically, if same dimension, use degree.

Thm For J_0, J_1 cx structures, $\exists$ a path $\{J_\lambda\}_{\lambda \in [0,1]}$ induces a cobordism of moduli spaces

$$\bigcup_{\lambda \in [0,1]} \mathcal{M}^*(A, J_\lambda)$$

so in particular $\deg_{\mathbb{Z}/2\mathbb{Z}} ev$ does not depend on J_0, J_1.

Consequences

If $\exists$ a symplectic embedding $\iota : B_r^{2n} \to B_R^2 \times \mathbb{R}^{2n-2}$ then $r \leq R$. (Gromov nonsqueezing)

Energy of a J-curve

If $u : \Sigma \to M$ smooth, then the energy $E(u)$ is

$$E(u) = \frac{1}{2}\int_\Sigma |du|_J^2 \, d\mathrm{vol}$$, use ω and J to get metric.

Lemma If J ω-compatible, then $\forall$ smooth $u : \Sigma \to M$

$$E(u) = \int_\Sigma |\bar{\partial}_J(u)|_J^2 \, d\mathrm{vol} + \int_\Sigma u^*(\omega) d\mathrm{vol}$$

so in ptic if u is J-holom then energy only depends on homology class, and J-curves are energy minimizing in homology class.

Thm (Gromov) If $\pi_2(M) = 0$ and J is ω-compatible then a sequence of J-holom. curves w/ bdd energy then $\exists$ subseq. converging to a J-holom curve.

For Lagrangian Floer homology: consider maps of discs w/ ∂ on Lagrangians which limit to $\cap$ points of Lag. on each side.