Definition of $\widehat{HF}$

① Definition & well-definedness (not topological inv't yet)
② Extra Structure

I) Goal: Start w/ $(\Sigma_g, \underline{\alpha}, \underline{\beta}, z)$ based Heegaard diagram, do Lag. Floer in $Sym^g(\Sigma_g \setminus \{z\})$, with Heegaard tori T_α, T_β. Then

$$\widehat{CF} := \mathbb{Z}\langle T_\alpha \cap T_\beta \rangle, \quad \partial x = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1}} \#(\mathcal{M}(\phi)/\mathbb{R})$$

Basepoint: $Sym^g(\Sigma \setminus z) \cong Sym^g(\Sigma) - R_z$, $R_z := \{z\} \times Sym^{g-1}(\Sigma)$

We consider maps $u: \mathbb{D}^2 \to Sym^g(\Sigma)$, which have an associated intersection #

$$n_z(u) := \#(R_z \cap u^{-1}(R_z))$$

Facts: ① $n_z(u_1 \cdot u_2) = n_z(u_1) + n_z(u_2)$

② $n_z: \pi_2(x,y) \to \mathbb{Z}$

③ $\phi \in \pi_2(x,y)$ has holom. rep $\Rightarrow n_z(\phi) \geq 0$.

- In particular, if ϕ has holom. rep., has representative disjoint from R_z
- Also if maps $\{\phi_n\}$ have same index and converge (in C^∞ or appropriate Sobolev topology) but are same htpy class, then limit has same index & n_z.

Assume: ① T_α, T_β comp. $\pitchfork$ Lag

② $Sym^g(\Sigma - z)$ convex at ∞ (choose right sympl. form.)

③ Can take generic J so all J-holom. disc, sphere bubbles have area 0 and moduli are cut out transversely

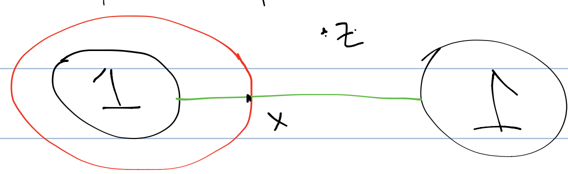
④ Bound on $E(\ker \mu)$

Facts: ① $\pi_2(\mathrm{Sym}^g(\Sigma - z)) = 0$

② $\pi_2(\mathrm{Sym}^g(\Sigma - z), T_\alpha) = 0$

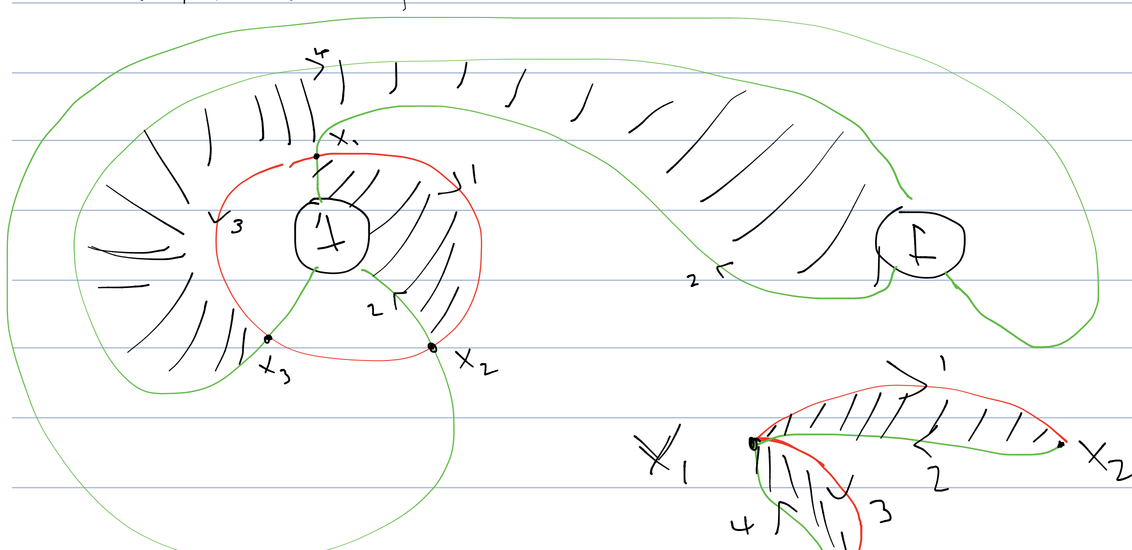
③ If $H^1(M) = 0$, $\hat{\pi}_2(x,x) = 0$ (in punctured Sym) but if $b_1 > 0$, need to do more.

Easy examples: ① S^3.



So $\mathbb{Z}\langle x\rangle = \widehat{CF}$ and $\partial = 0$, so $\widehat{HF}(S^3) \cong \mathbb{Z}$.

② $L(3,1)$ lens space.



So $\widehat{CF} = \mathbb{Z}\langle x_1, x_2, x_3\rangle$

and $\partial = 0$ by incompatibility of ∂ conditions as pictured above. Thus $\mathrm{rk}\, \widehat{HF}(L(3,1)) = 3$.

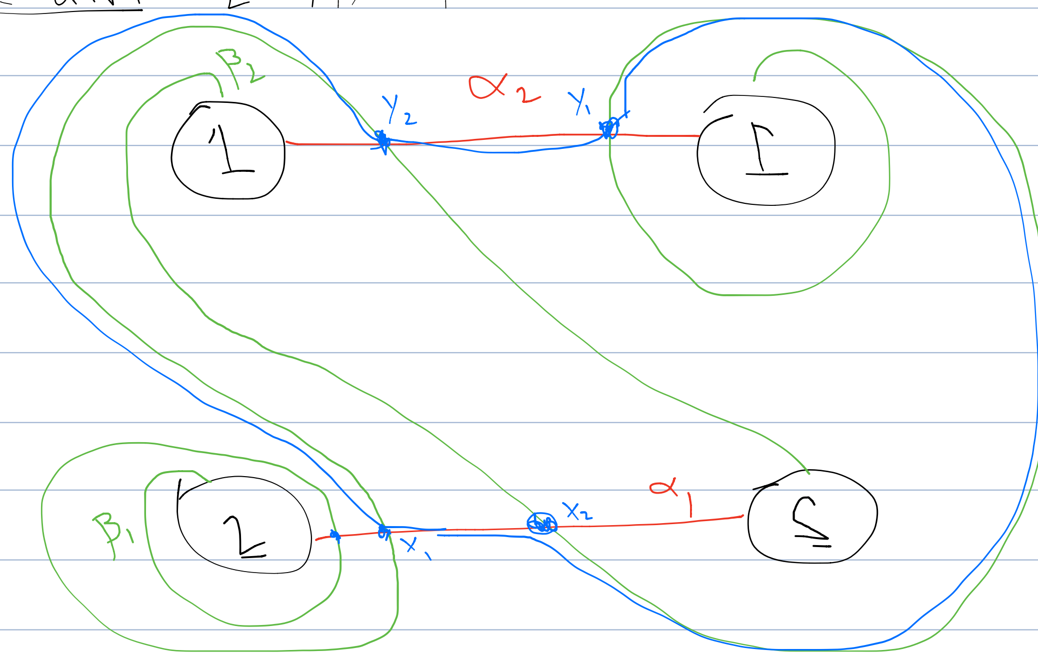
Suppose $x, y \in T_\alpha \cap T_\beta$, and choose $a, b : [0,1] \to T_\alpha, T_\beta$.
Then

$$[a-b] \in \frac{H_1(\mathrm{Sym}^g)}{H_1(T_\alpha) \oplus H_1(T_\beta)} \cong \frac{H_1(\Sigma)}{\langle \underline{\alpha}, \underline{\beta} \rangle}$$

Define $\varepsilon(a,b)$ as follows.

$\begin{cases} x = \{x_1, \ldots x_g\} \\ y = \{y_1, \ldots, y_g\} \end{cases}$ and $x_i \to y_{\sigma(i)}$ in $\amalg \alpha$, $y_j \to x_{\tau(j)}$ $\amalg \beta$ $\Big\}$ $\varepsilon(a,b) \in H_1(\Sigma)/\langle \underline{\alpha}, \underline{\beta} \rangle \cong H_1(Y)$

<u>Claim:</u> $\pi_2(x,y) \neq \phi \Rightarrow \varepsilon(x,y) = 0$



Let $z_1 = \{x_1, y_1\}$ and $z_2 = \{x_2, y_2\}$ both in $T_\alpha \cap T_\beta$. Then blue loop is $\varepsilon(z_1, z_2)$, nontrivial.

If $\varepsilon \neq 0$, no discs, so we don't worry about matrix coeff. of $\langle \partial x, y \rangle$.

<u>Spin^c-structures</u>

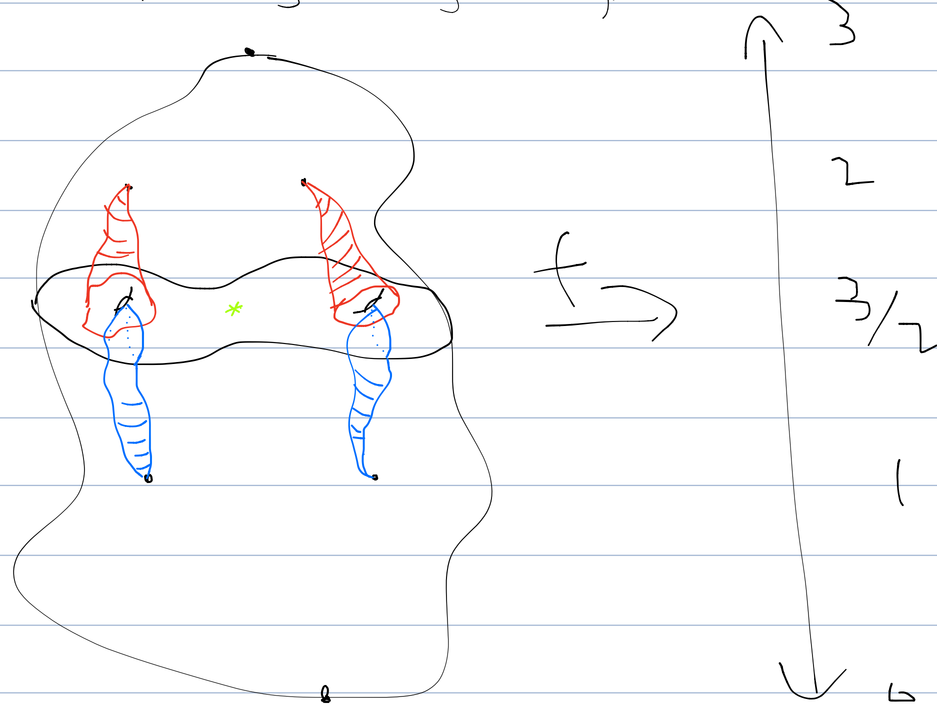
<u>Def</u> Two vector fields v_i on Y^3 are <u>homologous</u> if there

homotopic outside of some B^3.

Def $Spin^c(Y^3) := \{$homology classes of nonvanishing v.f. on $Y^3\}$
For τ a trivialization of TY, so we can identify

$$\delta_\tau : Spin^c(Y) \to H^2(Y) \cong H_1(Y)$$

Recall how to get Heegaard diagram from self-indexing Morse fn.



Given $x = \{x_1, .. x_g\}$ intersections of $\underline{\alpha}, \underline{\beta}$, get $(g+1)$ flow lines, from index 2 to index 1 and traj. * from max_f to min_f. Away from ν(flowlines), grad $f \neq 0$. We extend v.f. across nbhds in nonvanishing way (?) to get a $Spin^c$-structure. Call this $s_z(x) \in Spin^c(Y)$.

Using argument about $\Sigma(a,b)$.. we see

$$\widehat{CF}(Y) = \bigoplus_{s \in spin^c} \widehat{CF}_s(Y)$$

In general: need bound on $E(\ker \mu)$, energy on index zero. Using tautological correspondence to branched cover (or just intersection w/ R_z), we have for $\phi \in \pi_2(x,x)$ the <u>domain</u>, defined as

$$D(\phi) = \sum n_{z_i}(\phi) R_i, \quad \Sigma \setminus \{\underline{\alpha}, \underline{\beta}\} = \coprod R_i, \; z_i \in R_i$$

<u>Claim</u> $u \in \ker \mu : \pi_2(x,x) \to \mathbb{Z}$ iff $\partial(D(u))$ is a disjoint union of curves in $\underline{\alpha}, \underline{\beta}$. Such u are called <u>periodic</u>.

<u>Def</u> $(\Sigma, \underline{\alpha}, \underline{\beta}, z)$ is <u>weakly admissible</u> if $\exists$ a signed area form on Σ such that $\forall$ periodic domain disjoint from z, $E(D) = 0$

Maslov Index

Since $\mu : \hat{\pi}_2(x,x) \to \mathbb{Z}$, $\operatorname{Im}(\mu) = N_x \mathbb{Z}$, e.g. if Y $\mathbb{Q}$-HS^3 then $N_x = 0$ (minimal chern number).

If $\exists \phi \in \hat{\pi}^2(x,y)$ then $\mu(x,y) := \mu_{x,y}(\phi) : \pi_2(x,y) \to \mathbb{Z}/N_x$ gives relative grading. So we can write differential as

$$\partial x = \sum_{\substack{y \in T_\alpha \cap T_\beta \\ s = s(y)}} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi) = 1}} \#(M(\phi)/\mathbb{R})\, y$$

Fact Can lift to a absolute $N, \mathbb{Z}$-grdg.