Definition of $\hat{F}$

1. Definition & well-definedness (not topological, init yet)
2. Extra structure

Goal: Start w/ $(\xi, \alpha, \beta, \omega)$ based Heegaard diagram, do Lag Floer in $\text{Sym}^g(\xi, \beta \omega)$, with Heegaard tori $T_A, T_B$. Then

$$\hat{F} := \mathbb{Z} < T_A \cap T_B >, \quad \forall x \in \mathbb{Z}, \quad \frac{\#(M(\phi)/R)}{\text{YES}} = \frac{\#(M(\phi)/R)}{\text{YES}}$$

Basepoint: $\text{Sym}^g(\xi, \beta) \equiv \text{Sym}^g(\xi) - R_x, \quad R_x \equiv \mathbb{Z}^2 \times \text{Sym}^g(\xi)$

We consider maps $u: \mathbb{T}^2 \to \text{Sym}^g(\xi)$, which have an associated intersection

$$N_{\hat{F}}(u) := \#(R_x \cap u^*(R_x))$$

Facts:
1. $N_{\hat{F}}(u_1, u_2) = N_{\hat{F}}(u_1) + N_{\hat{F}}(u_2)$
2. $N_{\hat{F}}: \mathbb{C}^2(\mathbb{C}, \mathbb{C}) \to \mathbb{Z}$
3. $\phi \in \mathbb{C}^2(\mathbb{C}, \mathbb{C})$ has holom rep $\equiv \eta_{\hat{F}}(\phi) \in \mathbb{N}$

In particular, if $\phi$ has holom rep, has representative disjoint from $R_x$.

Also, if maps $\phi_i$ have same index and converge (in $\text{C}^0$ or Sobolev), but are same hom class, then limit has same index $\eta_{\hat{F}}(\phi) \in \mathbb{N}$

Assume:
1. $T_A, T_B$ comp. La
g
2. $\text{Sym}^g(\xi, \beta)$ convex at $\infty$ (choose right sympl form)
3. Can take generic $J$ so all $J$-holom. disc, sphere bubbles
here are $\mathcal{O}$ and moduli are cut out transversely

Bound on $E(\ker M)$

Facts:
1. $\Pi_2(Sym^3(X^2)) = 0$
2. $\Pi_2(Sym^3(X^2), T_2) = 0$
3. If $H^1(M) = 0$, $\Pi_2(X^x) = 0$ (in punctured $X^\text{pr}$) but if $b > 0$, need to do more.

Easy examples:

1. $S^3$

So $\mathbb{Z}\langle x \rangle = \mathbb{C}F$ and $\beta = 0$, so $HF(\mathbb{S}^3) = \mathbb{Z}$.

2. $L(3,1)$ lens space

So $\mathbb{C}F = \mathbb{Z}\langle x_1, x_2, x_3 \rangle$

and $\beta = 0$ by incompatibility of $\beta$ conditions as pictured above. Thus $rk HF(L(3,1)) = 3$. 
Suppose \( x, y \in T_{\alpha} \cap T_{\beta} \), and choose \( (a, b) : \{0, 1\} \to T_{\alpha}, T_{\beta} \). Then
\[
[a - b] \in H_1(\Sigma^{\alpha \beta}) / H_1(T_{\alpha} \cap T_{\beta}) \cong H_1(\Sigma^{\alpha \beta}) / \langle a, b \rangle.
\]
Define \( \mathcal{E}(a, b) \) as follows:
\[
\begin{align*}
(x, y) &\quad \text{in } \Pi \lambda \\
(1, y) &\quad \text{in } \Pi \rho \quad \Rightarrow \quad \mathcal{E}(a, b) \in H_1(\Sigma^{\alpha \beta}) / \langle a, b \rangle \\
&\quad \Rightarrow \quad \mathcal{E}(a, b) \cong H_1(\Sigma^{\alpha \beta}).
\end{align*}
\]

Claim: \( \pi_2(x, y) \neq \phi \Rightarrow \mathcal{E}(x, y) \neq 0 \)

Let \( z_1 = \phi(x_1, y_1^5) \) and \( z_2 = \phi(x_2, y_2^3) \), both in \( T_{\alpha} \cap T_{\beta} \). Then the blue loop is \( \mathcal{E}(z_1, z_2) \), nontrivial.

If \( \mathcal{E} \neq 0 \), no discs, so we don't worry about matrix and of \( \langle x, y \rangle \).

Spin\(^c\)-Structure

Det Two vector fields \( v_i \) on \( Y^3 \) are homologous if...
homotopic outside of some $\partial$.

$\text{Det} \, \text{Spin}^c(\mathbb{Y}) := 2$ homology classes of nonvanishing v.f. on $\mathbb{Y}^3$

For $\mathbb{Y}$ a trivialization of $\mathbb{Ty}$, so we can identify

$$\delta_z: \text{Spin}^c(\mathbb{Y}) \to H^1(\mathbb{Y}) \cong H_1(\mathbb{Y})$$

Recall how to get Heegaard diagram from self-indexing surgery.

Given $x = \gamma_1, x_2, 3$ intersections of $\mathbb{Y}$, get (9H) flow lines, from index 2 to index 1. Away from $\bar{\mathbb{U}}(\mathbb{Y} \cup \mathbb{V})$, $\gamma_3$ of hander, $\gamma_1, \gamma_2$ of not.

We extend v.f. across what in nonvanishing way (?).

to get a $\text{Spin}^c$-structure. Call this $S_3^c(x) \notin \text{Spin}^c(\mathbb{Y})$

Using argument about $\text{Det}(\mathbb{Y})$. We see
\[ CF(Y) = \bigoplus_{\text{simplicial}} CF_s(Y) \]

In general, need bound on \( E(\ker m) \) energy on index zero. Using tautological correspondence to branched cover (or just intersection with \( R_\alpha \)), we have for \( \phi \in \pi_2(x,Y) \) the domain, defined as

\[ D(\phi) = \Sigma \pi_2(\phi) R_\alpha, \quad \Sigma \exists \beta \R_\beta = \bigcup R_\alpha, \quad \alpha \in \beta \]

Claim \( \ker m : \pi_2(x,Y) \to \mathbb{Z} \), if \( \exists D(\alpha) \) is a disjoint union of curves in \( x, \beta \). Such \( \alpha \) are called periodic.

Det \( (\Sigma, \alpha, \beta, \tau) \) is weakly admissible if it is a signed area form on \( \Sigma \) such that \( A \) periodic domain disjoint from \( \tau \), \( E(\tau) = 0 \)

\[ \text{Maslov Index} \]

Since \( m : \pi_2(x,Y) \to \mathbb{Z}, \text{Ind}(m) = N_x \mathbb{Z} \), e.g. if \( Y \setminus \{Q = \mathbb{H}^3 \} \) then \( N_x = 0 \) (minimal chain number).

If \( \exists \phi \in \pi_2(x,Y) \) then \( m(x,Y) : = m(\pi_2(x,Y) \to \mathbb{Z}_{N_x} \)

gives relative grading so we can write differential as

\[ \partial X = \sum_{\phi \neq 0} \# (M(\phi)) R \phi \]
Fact that lifts to an absolute \( \mathbb{N}_5 \mathbb{Z} \)-algebra.