Topological Invariance of HF

Last time: defined HF

Given: ① a pointed Heegaard diagram

$$\mathcal{H} = (\Sigma, \underline{\alpha}, \underline{\beta}, z) \text{ for } Y^3 \text{ closed}$$

② Choice of cx structure on Σ

This data determines a chain complex $\widehat{CF}(\mathcal{H}, J)$, generated by intersection points $T_\alpha \cap T_\beta$ of Heegaard tori. Differential is

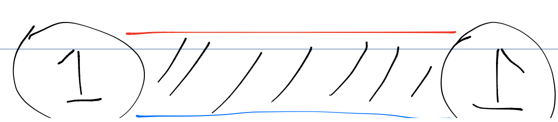
$$\partial(x) = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\substack{\phi \in \hat{\pi}_2(x,y) \\ \mu(\phi) = 1}} \#\left(\mathcal{M}(\phi)/\mathbb{R}\right) y$$

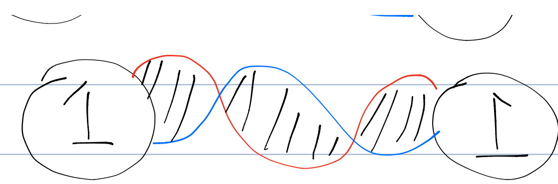
where $\hat{\pi}_2(x,y) = \{$htpy classes of Whitney discs not intersecting base point $z\}$

Rmk: $\exists$ more general constructions $CF^{\pm}$, CF^{∞} with

- Different coefficients (e.g. adjoining formal variable u to give module structure), or
- counts of discs intersecting basepoint

Note: If $b_1(Y) > 0$ we have to be careful, restricting to Heegaard diagrams that are "admissible";

Bad! 

Good: 

Thm $\widehat{HF} := H_*(\widehat{CF})$ is an invariant of Y.

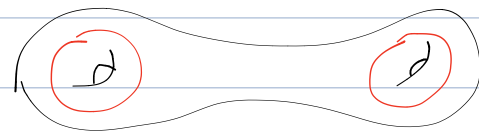
① $\widehat{HF}$ doesn't depend on J, complex structure, via std arguments in Lagrangian Floer theory.

Fact: any two Heegaard diagrams for the same 3-mfld are related by a finite seq. of Heegaard moves:

① Isotopy of $\mathbb{T}_\alpha, \mathbb{T}_\beta$
② (de) stabilization
③ handleslides

Prop: Any two pointed Heegaard Heegaard diagrams for the same Y^3 are related by a sequence of pointed Heegaard moves, i.e. supported in the complement of z.

pf: Suppose we have isotopy $\beta_1 \simeq \beta_1'$ passes through the basepoint. Then β_1' can be obtained from β_1 by a sequence of handleslides. If we surger Σ along $\beta_2, \dots, \beta_g$ we get a torus T with $2g-2$ marked points, and $\beta_1' \simeq \beta_1$ in $T \setminus \{z\}$. So isotopy over $z \rightsquigarrow$ handleslide.

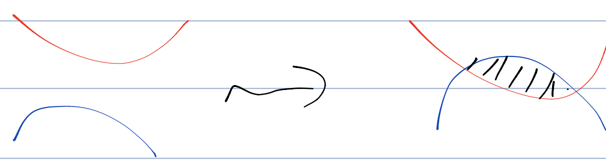


<u>Note</u>: Can also check that $\exists$ a sequence of Heegaard moves through admissible diagrams.

(2) $\widehat{HF}$ is invariant under isotopy of $\underline{\alpha}, \underline{\beta}$.

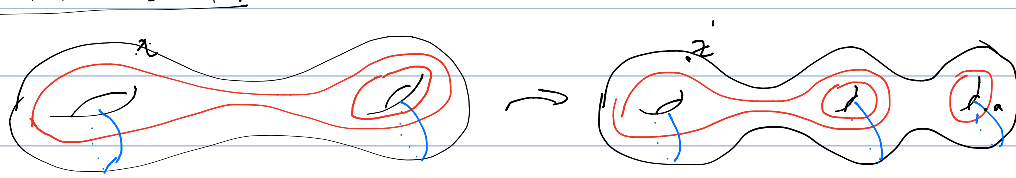
<u>pf</u>: If isotopy doesn't give more/less $\cap$ points, generators of $\widehat{CF}$ doesn't change. Differential may change, but this is equiv. to changing metric.

Another type of isotopy: adding small bi-gon.



Such isotopies are realized by Hamiltonian isotopies of $\mathbb{T}_\alpha, \mathbb{T}_\beta$, so continuation map argument gives chain homotopy.

(3) <u>Stabilization</u>



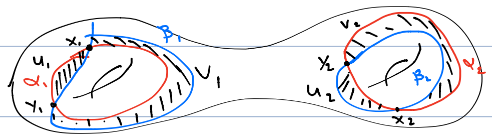
$\mathcal{H} = (\Sigma, \underline{\alpha}, \underline{\beta}, z)$ $\qquad$ $\mathcal{H}' = (\Sigma', \underline{\alpha}', \underline{\beta}', z')$

There is a clear bijection $\mathbb{T}_{\alpha'} \cap \mathbb{T}_{\beta'} \to \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $(x_1, \ldots, x_n)$

Also, given $\hat{\pi}_2(x,y)$, clearly if we don't involve $(x_1,\dots,x_g,a)$ a, classes are same. Can see that $\mathcal{M}(\phi)\cong\mathcal{M}(\phi')$, since $\phi'\in\hat{\pi}_2(x',y')$ can't pass through puncture...

④ Handleslide Invariance

Ex: $(\#^2 S^1\times S^2)$



Generators: $x_1x_2,\ x_1y_2,\ y_1x_2,\ y_1y_2$

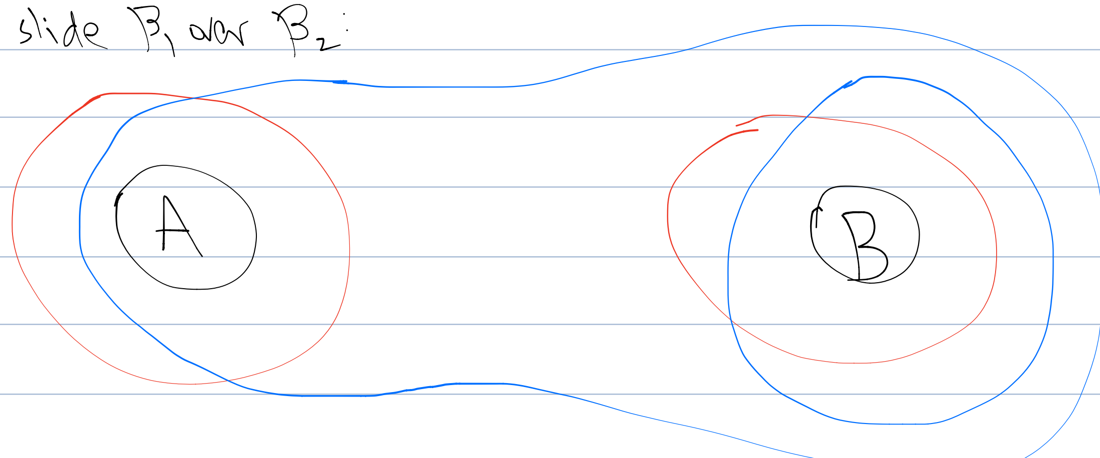
$$\begin{tikzcd}[column sep=small]
 & x_1x_2 \arrow[dl, shift right, "u_1\ +"'] \arrow[dl, shift left, "v_1\ -"] \arrow[dr, shift right, "-"'] \arrow[dr, shift left, "+\ v_2"] & \\
y_1x_2 \arrow[dr, shift right, "-"'] \arrow[dr, shift left, "+"] & & x_1y_2 \arrow[dl, shift right, "+"'] \arrow[dl, shift left, "-"] \\
 & y_1y_2 &
\end{tikzcd}$$

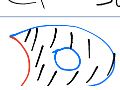
etc.

Prop: $\widehat{HF}(\#^g S^1\times S^2)\cong H_*(T^g)\cong \underbrace{H_*(S^1)\otimes\cdots\otimes H_*(S^1)}_{g\text{ times}}$

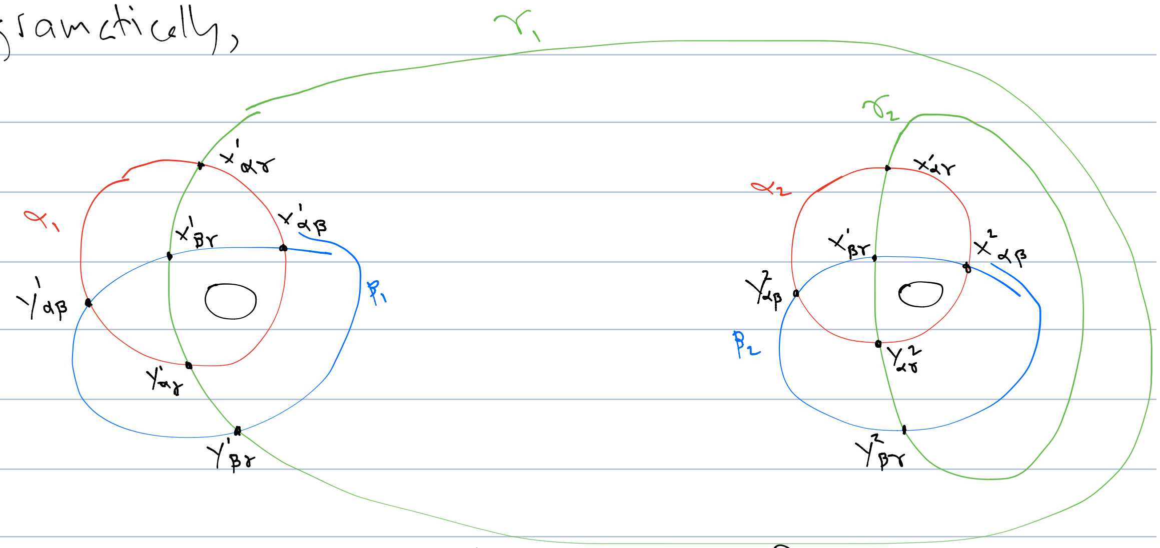
We say $x_1x_2\dots x_g$ is "top dimensional generator"

Now slide β_1 over β_2:



Can check: $\widehat{CF}$ same as before, even though we get more domains, e.g. .

Diagramatically,



Idea: define a map $\widehat{HF}(\mathcal{H}_{\alpha\beta}) \to \widehat{HF}(\mathcal{H}_{\alpha\gamma})$ by counting holomorphic triangles.

Triangles: Given symplectic (M,ω) L_0, L_1, L_2 Lagrangians, $x \in L_0 \cap L_1$, $y \in L_1 \cap L_2$, $z \in L_0 \cap L_2$, we can consider

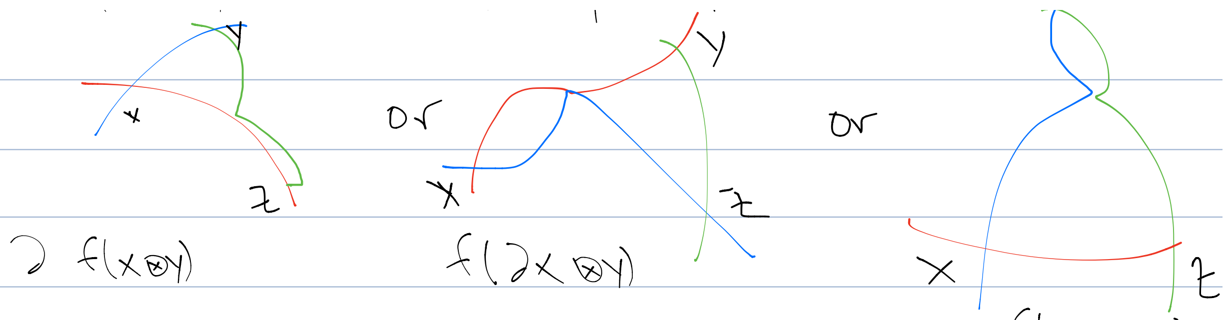
$$\pi_2(x,y,z) = \{\text{htpy classes of maps } u: \Delta \to M\}$$

called "Whitney triangles". We then define a map

$$f(x \otimes y) = \sum_{z \in L_0 \cap L_2} \sum_{\substack{\psi \in \pi_2(x,y,z) \\ \mu(\psi)=0}} \#(\mathcal{M}(\psi)) \cdot z$$

Claim: f is a chain map.

Idea: Consider $\psi \in \pi_2(x,y,z)$ with $\mu(\psi) = 1$. Then $\mathcal{M}(\psi)$ is 1 dim'l, so compactifying gives $\overline{\mathcal{M}}(\psi)$, oriented 1-mfld w/ ∂. The boundary looks like



$\partial f(x \otimes y)$ $\quad$ $f(\partial x \otimes y)$ $\quad$ $f(x \otimes \partial y)$

where f is as above. We get a chain map

$$f_{\alpha\beta\gamma}: \widehat{CF}(\mathcal{H}_{\alpha\beta}) \otimes \widehat{CF}(\mathcal{H}_{\beta\gamma}) \to \widehat{CF}(\mathcal{H}_{\alpha\gamma})$$

So we define the induced map on homology

$$F_{\alpha\beta\gamma}: \widehat{HF}(\mathcal{H}_{\alpha\beta}) \to \widehat{HF}(\mathcal{H}_{\alpha\gamma})$$
$$x \longmapsto f_{\alpha\beta\gamma}(x \otimes \Theta_{\beta\gamma})$$

"top dim'l element"

Claim: This determines a canonical isomorphism.

"Pf": Show that the $F_{\alpha\beta\gamma}$ maps are associative by looking at holomorphic rectangles. Then, if α fixed,

$$\beta \xrightarrow[\text{handle slide}]{} \gamma \xrightarrow[\text{undoing handleslide isotopic to } \beta]{} \delta$$

and need to show associativity:

$$f_{\alpha\gamma\delta}(f_{\alpha\beta\gamma}(-\otimes-)\otimes -) \simeq f_{\alpha\beta\delta}(- \otimes f_{\beta\gamma\delta}(-\otimes-))$$

Map counts e.g. shaded triangles: