# Applications of Heegaard Floer

Let $Y^3$ be $\mathbb{Q}$-$S^3$, with pointed Heegaard diagram $(\Sigma, \alpha, \beta, z)$. Also fix a spin$^c$-structure $t$ and let $\widehat{CF}(\alpha, \beta, t)$ be free abelian group gen. by $T_\alpha \cap T_\beta$, $s_z(x) = t$.
We get relative $\mathbb{Z}$-grading $gr(x, y) = \mu(\varphi) - 2n_z(\varphi)$ for $\varphi \in \pi_2(x, y)$ ($gr$ is indep. of $\varphi$).

Then boundary map $\partial x = \sum \#(\mathcal{M}(\varphi)/\mathbb{R})\, y$ for $y \in T_\alpha \cap T_\beta$, $\varphi \in \pi_2(x, y)$, $s_z(y) = t$, $n_z(\varphi) = 0$, $\mu(\varphi) = 1$.
Homology is $\widehat{HF}(Y, t)$, and $\widehat{HF}(Y) = \bigoplus_t \widehat{HF}(Y, t)$.
Other varieties: $CF^\infty(\alpha, \beta, t)$ gen. by $[x, i] \in T_\alpha \cap T_\beta \times \mathbb{Z}$ with relative grading $gr([x, i], [y, j]) = gr(x, y) + 2i - 2j$, and

$$\partial[x, i] = \sum_{y \in T_\alpha \cap T_\beta,\ \varphi \in \pi_2(x,y),\ s_z(y) = t} \#(\mathcal{M}(\varphi)/\mathbb{R})\, [y, i - n_z(\varphi)]$$

Check that this decreases grading by one, and formal action

$$U \cdot [x, i] = [x, i-1]$$

decreases grading by two.

Thm [O-Sz.] $HF^\infty(Y, t) \cong \mathbb{Z}[U, U^{-1}]$

Then $HF^-$, $HF^+$ come from following complexes.

$CF^-(\alpha, \beta, t) := \{\text{subgroup of } CF^\infty \text{ w/ } i<0\}$

$CF^+ = CF^\infty / CF^-$

(Key fact: $\partial$, $U$-map preserves $CF^-$)

This can be expressed as

$$0 \to CF^- \xrightarrow{\text{incl}} CF^\infty \xrightarrow{\pi} CF^+ \to 0$$

and similarly

$$0 \to \widehat{CF} \xrightarrow{\hat{\iota}} CF^+ \xrightarrow{U} CF^+ \to 0$$

so we get induced LES

$$\cdots \to HF^-(Y,t) \to HF^\infty(Y,t) \longrightarrow HF^+(Y,t) \to \cdots$$

$$\cdots \to \widehat{HF}(Y,t) \to HF^+(Y,t) \to HF^+(Y,t)$$

Key point: Relative $\mathbb{Z}$-grading can be lifted to absolute $\mathbb{Q}$-grading $\widetilde{gr}$ characterized by

① $\iota, \hat{\iota}, \pi$ all preserve $\widetilde{gr}$

② $\widehat{HF}(S^3)$ supported in $\widetilde{gr} = 0$

③ If $W$ is cobordism $Y_1 \to Y_2$ and $y \in HF^\infty(Y_1, t_1)$ then

$$\widetilde{gr}(F_{W,s}(y)) - \widetilde{gr}(y) = \frac{c_1(s)^2 - 2\chi(W) - 3\sigma(W)}{4}$$

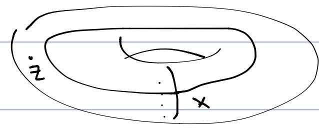
where $\mathfrak{s}|_{Y_i} = \mathfrak{t}_i$.

<u>Def</u> The correction term / $d$-invariant $d(Y,\mathfrak{t})$ is the minim grading ($\widetilde{gr}$) of any non-torsion element in $HF^+$ coming from $HF^\infty$.

<u>Ex</u>: ($S^3$) $gr([x,i],[x,i-1]) = \overset{=0}{gr(x,x)} + 2i - 2(i-1)$

$= 2$, so $\partial = 0$

$\Sigma$




Thus $HF^+(S^3) = \mathbb{Z}[u,u^{-1}]/\mathbb{Z}[u] \Rightarrow d(S^3) = 0$

<u>Ex</u>: $HF^+(\Sigma(2,3,5)) \cong \mathbb{Z}[u,u^{-1}]/\mathbb{Z}[u] \cong HF^+(S^3)$

$\vdots$

$\mathbb{Z}$ $\partial u$ Tower represents coeff. of (positive) power of $u^{-1}$.

$\mathbb{Z}$ $\partial u$

$gr=2$ $\to$ $\mathbb{Z}$ Thus absolute grading distinguishes $HF^+(\Sigma(2,3,5))$ and $HF^+(S^3)$.

<u>Ex</u>: $HF^+(\Sigma(2,3,7))$ is same isom. class of module, but

$\vdots$

$\mathbb{Z}$ $d(\Sigma(2,3,7)) = 0$

$\downarrow$

$\mathbb{Z}$

$\downarrow$

$gr=0$ $\to$ $\mathbb{Z}$

Assume $Y$ is $\mathbb{Z}$-$HS^3$. Then

Thm (OS) If $Y$ bounds a smooth 4-mfld w/ pos. def. intersection form $Q_X$ of rk $n$, then for any characteristic vector $\eta$ (meaning $\eta \cdot \alpha = \alpha \cdot \alpha \bmod 2$, $\alpha \in H_2(X)$) we have

$\eta \cdot \eta \geq n + 4\, d(Y)$ ⊛

Thm (Elkies) Any unimodular pos. def bilinear form $Q$ of rk $n$ has a char. vector $\eta$ s.t. $\eta \cdot \eta \leq n$, w/ equality unless $Q = n\langle 1 \rangle$.

This plus ⊛ above $\Rightarrow$ Donaldson: if $X^4$ smooth closed and pos. def rk $n$, $X^\circ := X^4 - B^4$ bounds $S^3$. By ⊛, $\eta \cdot \eta \geq n$, but by by Elkies $n > \eta \cdot \eta$ unless $Q_X = n\langle 1 \rangle$, diagonal.

If we combine ⊛ w/ a similar ineq. for negative def, we get

Thm (OS) If $Y$ bounds rational ball, $d(Y) = 0$.

These also play nicely w/ knots and concordance.

Def Two knots $K_1, K_2$ are concordant if $K_1 \# K_2$ bounds a disc in $B^4$ ("slice".) Equiv. classes of knots form a group under $\#$

Def Integral homology spheres form a group under equiv relatio of cobordism, operation connected sum. Denote this by $\Theta^3_{\mathbb{Z}}$.

<u>Rmks:</u> ① These are infinitely generated.

② Highlights a difference between Top and Smooth: The kernel of $\mathcal{C} \to \mathcal{C}_{top}$ contains a $\mathbb{Z}^3$-summand <u>and</u> a subgrp isom to $(\mathbb{Z}/2\mathbb{Z})^\infty$ $(\mathbb{Z}^\infty?)$. OTOH by Freedman $\Theta^3_{\mathbb{Z},top} = 0$.

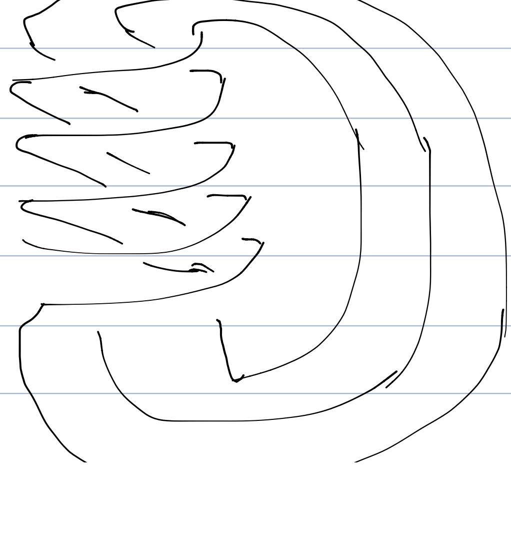
Questions: ① Is there other torsion in $\mathcal{C}$ besides 2-torsion?

② Torsion in $\Theta^3_{\mathbb{Z}}$? Manolescu: no $\mathbb{Z}HS^3$ w/ Rohklin inv $=1$ is 2-torsion in $\Theta^3_{\mathbb{Z}}$.

The correction terms obstruct triviality of elements of these groups ($\mathcal{C}$, $\Theta^3_{\mathbb{Z}}$, $\Theta^3_{\mathbb{Q}}$)

For knots, use fact that double branched cover of slice knot (branched cover of $B^4$ over slice disc) bounds rational $B^4$.

<u>Ex:</u> Branched cover of $T_{3,5}$ is $\Sigma(2,3,5)$, so $T_{3,5}$ (torus knot) not slice, as $d(\Sigma(2,3,5)) = 2$.



**Note:** Meier computed correction terms for double branched covers of family of top. doubly slice to show they aren't smoothly doubly slice.

## Surgery Exact Sequence

For framed knot $K$ (longitude $\lambda$), let $Y_0$ denote $\lambda$-surgery on $Y$ and $Y_1$ denote $(\mu+\lambda)$-surgery. We call $(Y, Y_0, Y_1)$ a *triad*.

**Ex:** $(S^3, S^3_n(K), S^3_{n+1}(K))$

**Thm** (OS) If $(Y, Y_0, Y_1)$ are triad, we get

$$\cdots \to \widehat{HF}(Y) \to \widehat{HF}(Y_0) \to \widehat{HF}(Y_1) \to \cdots$$
$$\cdots \to HF^+(Y) \to HF^+(Y_0) \to HF^+(Y_1) \to \cdots$$

Recall that a $\mathbb{Q}HS^3$ is an L-space if $\widehat{HF}(Y)$ free abelian of rank $|H^2(Y)|$ (e.g. lens spaces.)

**Exercise:** If a triad $(Y, Y_0, Y_1)$ ordered so that $|H_1(Y)| = |H_1(Y_0)| + |H_1(Y_1)|$, then $Y_0, Y_1$ L-spaces $\Rightarrow$ $Y$ is L-space.

**Corollary:** If $S^3_n(K)$ is an L-space, $S^3_m(K)$ is an L-space if $m > n$ $(m, n \in \mathbb{Q})$.

Ex: $\mathcal{D}_5 = L(5,1)$ so $S^3_r(RHT)$ is an L-space for $r \geq 5$.

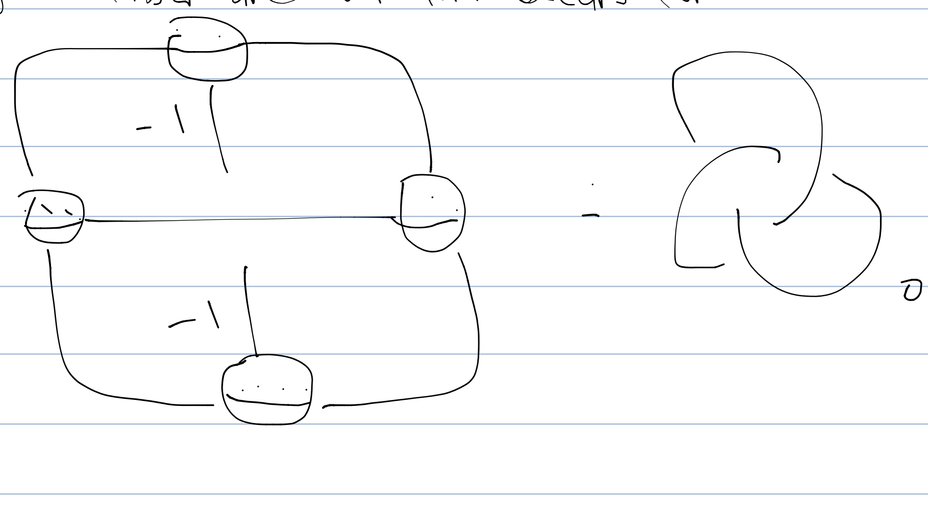
Thm (Kron-Mrowka-OS) Let $U$ be unknot, $K$ any knot. If $S^3_r(K) \cong S^3_r(U)$ then $K = U$.

Thm (OS, Ghiggini) Let $T$ be trefoil. If $S^3_r(K) \cong S^3_r(T)$ then $K = T$. Similarly for figure eight.

Conjecture: Any simply connected 4 mfld admits a handle decomp w/o 1 or 3-handles. (Implies poincaré)

Note that this fails for 4-mflds with boundary. (Akbulut)

If $C$ is a cusp nbhd then it admits an elliptic fibration whose regular fibers are tori. This occurs for




Remove a nbhd $T^2 \times D^2$ of a regular fiber, and reglue $S^3 \setminus \nu K \times S^1$ so longitude of $K \to \{pt\} \times \partial D^2$. This is called Fintushel-Stern

Knot surgery. Can show that 4-mfld $C_K$ is homeom. but not diffeom. to $C$.

Claim A handle decomp. of $C_K$ must have 1- or 3-handles.
pf: If not, $C_K$ is obtained by attaching 0-framed 2-handle to $B^4$ along some $K'$...