Knot Floer Homology

Outline: ① Input Data
② Chain cx & diff'l
③ Properties
④ Computation

References: O - Oszvath
S - Szabo
M - Manolescu
R - Rasmussen

Notation: K = knot (oriented) in S^3

§1: Data: 3-mfld M^3, closed

$(\Sigma, \underline{\alpha}, \underline{\beta}, z) \longrightarrow HF(M)$, Heegaard Floer homology.

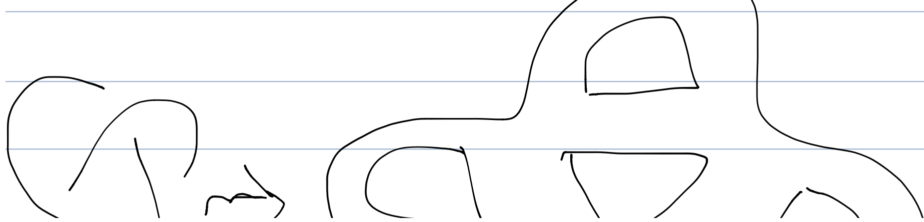
Σ: Heegaard surface
$\underline{\alpha}, \underline{\beta}$: ∂(compression discs)
z: basept

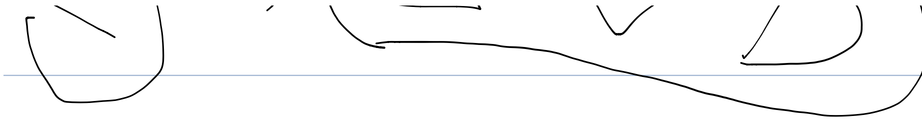
For knots in S^3, will use

$(\Sigma, \underline{\alpha}, \underline{\beta}, z, w) \longrightarrow HFK$, knot Floer homology.

Σ: splitting of S^3
z, w: basepts

Ex: (OS) ① Take tubular nbhd of K
② Forget crossings, project to plane.

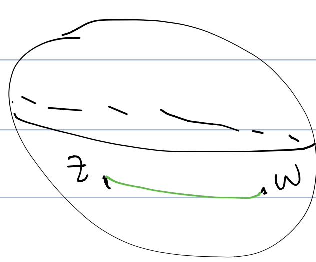




③ Plug w/ "pringles"™ (4-valent crossing)

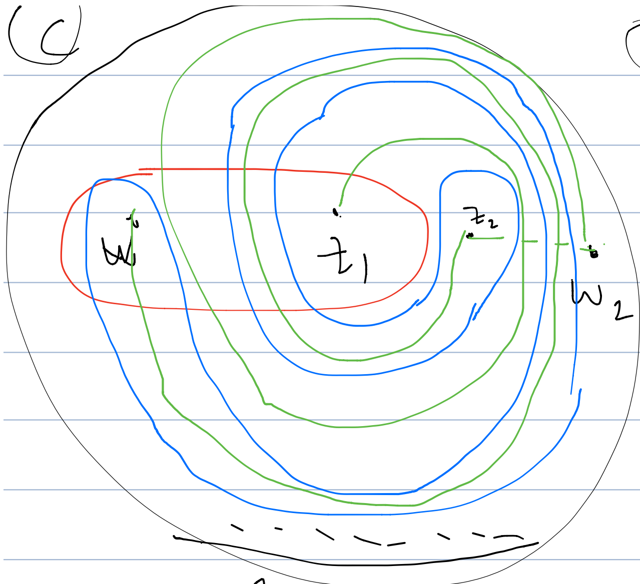
④ Color 2 of bdd regions in blue, pick basept z on one side of longitude μ of K, w on other side:

Can see that this determines Heegaard splitting of S^3. Recover knot by considering $B^3 \cup B^3 = S^3$; green curve above is



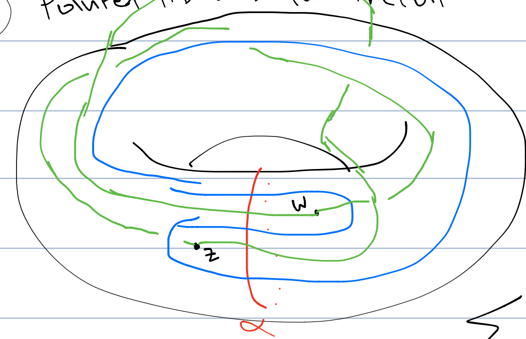
and rest of knot is in other ball.

(C)



$\Sigma = S^2$

(B) Pointed HD of S^3 for trefoil



So we use this process to generate Lagrangians in $\mathrm{Sym}^d(\Sigma)$, $d = g+k-1$ (can check that $\underline{\alpha}$ is exactly d curves, so this gives $\frac{1}{2}$ dim'l torus).

As with HF, we consider $T_\alpha \cap T_\beta$, and letting $v = \{w_1, \ldots, w_k, v_1, \ldots, v_k\}$, we consider $D_v = \{v\} \times \mathrm{Sym}^{d-1}(\Sigma)$ for grading.

Given $x, y \in T_\alpha \cap T_\beta$, $\phi \in \pi_2(x,y)$ (including those intersecting D_z!), we have

- $n_{w_i}(\phi) = |\phi \cap R_{w_i}|$
- $M(\phi)/_{\mathbb{R}\text{-action}} = \widehat{M}(\phi)$
- Maslov grading

(Relative) $\mu(x) - \mu(y) = \mu(\phi) - 2\sum_{i=1}^{k} n_{w_i}$

which can be made absolute over $\mathbb{Z}$.

Rmk: Related to Alexander grading.

$\Delta_L(T) :=$ Alexander-Conway polynomial

$$A(x)-A(y)=\sum n_{z_i}(\phi)-\sum n_{w_i}(\phi), \text{ and}$$

$$\sum_{x\in T_\alpha\cap T_\beta}(-1)^{M(x)}q^{A(x)}=(1-q^{-1})^{k-1}\Delta_K(q)$$

<u>Ex</u>: In diagram (B) above, $d=1$, $T_\alpha\cap T_\beta=\{a,b,c\}$

Letting $\phi\in\pi_2(a,b)$ and $\psi\in\pi_2(b,c)$,

$$\mu(\phi)=\mu(\psi)$$

$$\#\hat{\mathcal{M}}(\phi)=\#\hat{\mathcal{M}}(\psi)=1$$

$$n_w(\phi)=0=n_z(\psi)$$

$$n_z(\phi)=1=n_w(\psi)$$

For gradings, $M(a)-M(b)=1=M(b)-M(c)$, $M(a)=2$

<u>Fact</u>: $\Delta_K(q)=q^{-1}-1+q$

$$A(a)+A(b)=1=A(b)-A(c),\quad A(c)=-1$$

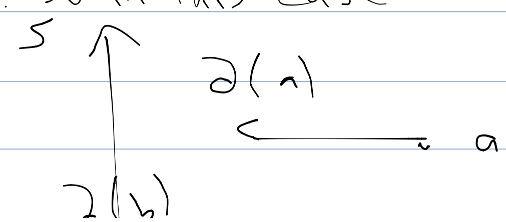
We obtain these indices using diagram; in (B) $d=1$, so no symmetric product. ($Sym^1(\Sigma)=\Sigma$)

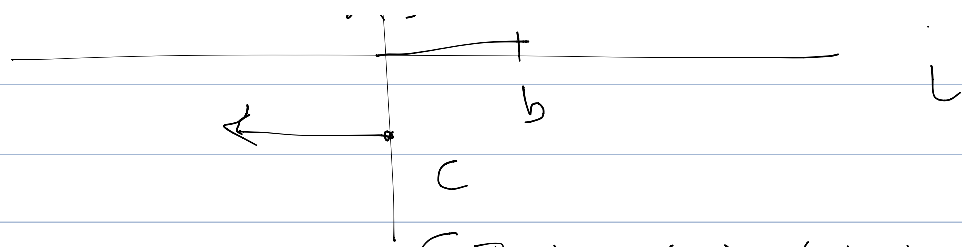
<u>Differential</u>: $$\partial x=\sum_{y\in T_\alpha\cap T_\beta}\sum_{\substack{\phi\in\pi_2(x,y)\\ \mu(\phi)=1\\ n_{w_i}(\phi)=n_{z_i}(\phi)=0}}\#\hat{\mathcal{M}}(\phi)\,y$$

Homology of this is $\widehat{HFK}(\mathcal{H})$, with $\widehat{HFK}_i(K,s)$ summands. Note that ∂x drops i by 1 but <u>preserves</u> s. So in this case

Alexander grading

Maslov grading





So then

$$\widehat{HFK}(K,s) = \begin{cases} \mathbb{Z} & \text{when } (s,i) = (-1,0), (0,1), (1,2) \\ 0 & \text{else} \end{cases}$$

§3: Properties

① The graded Euler characteristic of $\widehat{HFK}$ is $\Delta_K(q)$, Alexander polynomial:

$$\sum_{i,s \in \mathbb{Z}} (-1)^i q^s \left(\operatorname{rk}_{\mathbb{Z}}(\widehat{HFK}_i(K,s))\right) = \Delta_K(q)$$

② $\widehat{HFK}_i(K,s) = \widehat{HFK}^{-i}(\overline{K},s)$

mirror

defined via univ. coeff.

③ $\widehat{HFK}_i(K,s) = \widehat{HFK}_{i-2s}(K,-s)$

Similarly: Kunneth formula for $K_1 \# K_2$, $\widehat{HFK}$ detects mutants, $\exists$ LES for skein relations

__Thm__ (OS) $g(K) = \max\{s \geq 0 : \widehat{HFK}_*(K,s) \neq 0\}$
$\Rightarrow \widehat{HFK}$ detects the unknot.

__Thm__ (OS, Ghiggini, Ni, Juhasz) For $K \subseteq S^3$, K fibered iff

$\widehat{HFK}_*(K, g(K)) \cong \mathbb{Z}$.

<u>Fact</u>: Both 3_1, 4_1 (trefoil & fig. 8) are the only genus 1 fibered knots, so above $\Rightarrow$ $\widehat{HFK}$ detects $3_1, 4_1$.

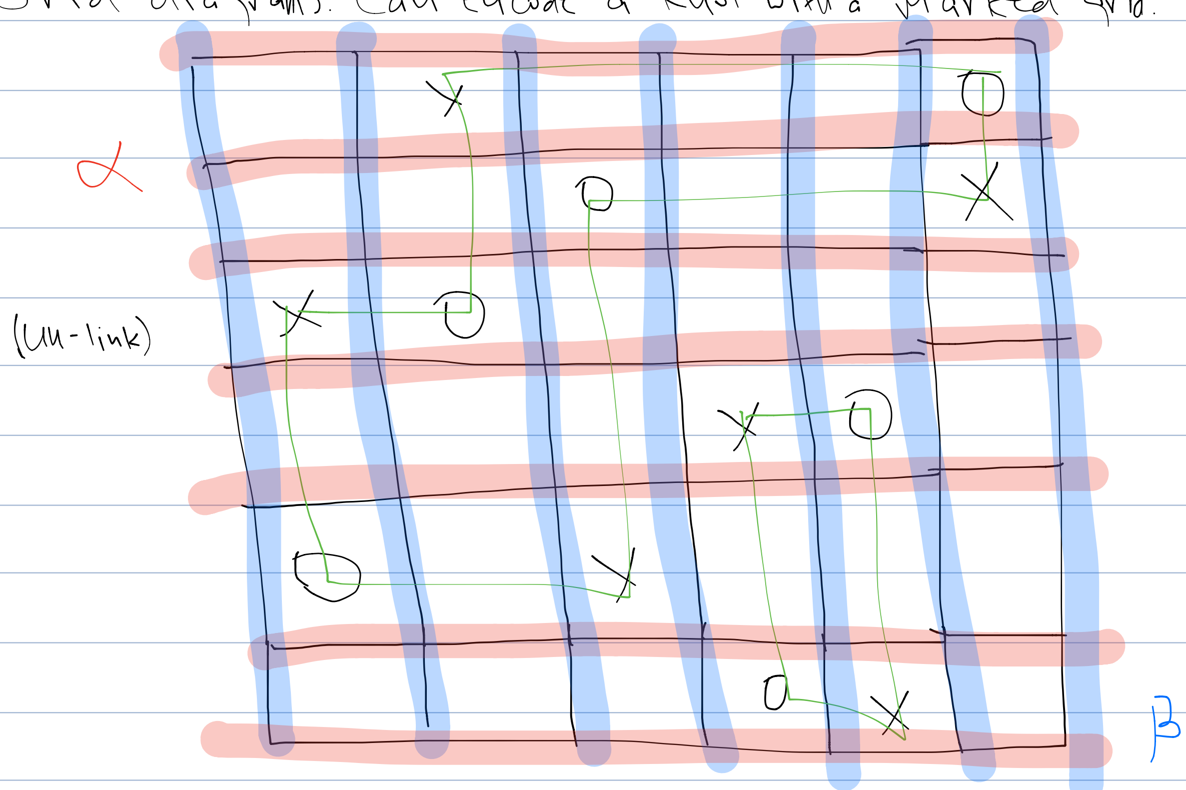
<u>Fact</u> (OS) $\exists$ a knot invariant $\tau(K)$ s.t. if $K_1 \sim K_2$ (concordant) $\tau(K_1) = \tau(K_2)$ and $|\tau(K)| \leq |g_4(K)| \leq u(K)$

4 ball genus

unknotting number

<u>§4</u>: Computation

Grid diagrams: can encode a knot with a marked grid.



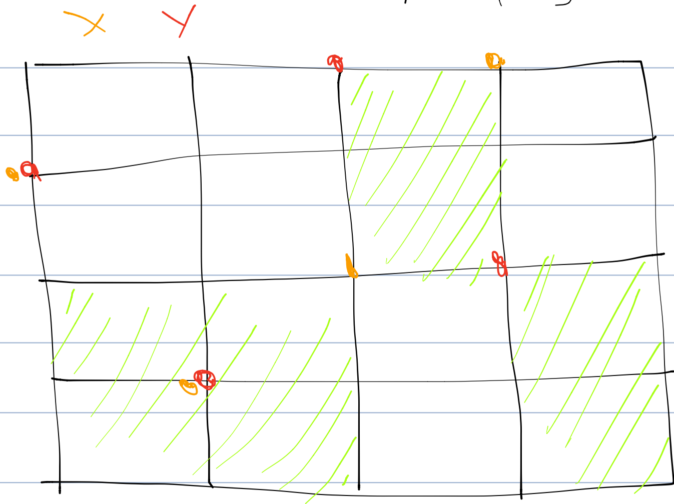
This determines a pointed Heegaard diagram of S^3, with K.

Where are J-curves w/ ∂ on $\underline{\alpha}, \underline{\beta}$?

We have $|T_\alpha \cap T_\beta| = n!$; take x, y in this set.

<u>Def</u> A rectangle between x, y is a rectangle in grid s.t. its

bottom left corner is on x, top right is on y.



A rectangle is *empty* if no other pts of X in r.

Let $Rect(x,y) = \{$empty rectangles from x to y$\}$.

Fact: $r \in Rect(x,y)$ can be assigned $\varepsilon(r) \in \{\pm 1\}$.

Thm (OSM) Empty rectangles are in 1-1 correspondence w/ J-curves.

For $g\,\widehat{CFK}(G)$,

$$\partial x = \sum_{y \in T_\alpha \cap T_\beta} \sum_{r \in Rect(x,y)} \varepsilon(r) \cdot y$$

$$O_i(r) = X_i(r) = 0$$

times r crosses i^{th} O

sim. for X's