1) Two cars start moving from the same point. One travels south at 60mph and the other travels west at 25 mph. At what rate is the distance between the cars increasing two hours later?

2) A balloon rises at the rate of 10 feet per second from a point of the ground 100 feet from an observer. Find the rate of change of the angle of elevation of the balloon from the observer when the balloon is 100 feet above the ground.

3) Use differentials to approximate \(3\sqrt{8.12}\)

4) Determine all critical values of the following function
\[ f(\theta) = \theta - \tan \theta \]

5) Find the absolute Maximum and Minimum of
\[ f(x) = (x^2 - 1)^3 \quad \text{on} \quad [-1,2] \]

6) Which of the following functions satisfy the hypotheses of the Mean Value Theorem?
A) \( f(x) = \frac{1}{x^2 - 2} \) on \([0,2]\)
B) \( f(x) = |x| \) on \([0,1]\)
C) \( f(x) = x^{2/3} \) on \([-1,1]\)

7) Determine all \( c \) values which satisfy the Mean Value Theorem for the function \( f(x) = x^3 - 3x + 5 \) on the interval \([-2,2]\)

8) For the function \( f(x) = 3x^5 - 5x^3 + 3 \)
A) Find intervals of Increase and Decrease
B) Find Local Maximum and Minimum
C) Find intervals of concavity & inflection points.
D) Sketch a rough graph

9) Find all values of \( x \) at which the graph of \( f(x) = \frac{x^2}{4} + \cos x \) changes concavity on \([-\pi, \pi]\)

10) Determine the limits
A) \( \lim_{x \to 0} \frac{\tan^{-1} x^2}{x^2} \)
B) \( \lim_{x \to \infty} (e^x + x)^{1/x} \)
C) \( \lim_{x \to 0^+} \sqrt{x} \ln x \)
D) \( \lim_{x \to 0^+} \left( \frac{1}{x} - \frac{1}{\sin x} \right) \)

11) A function \( f \) is continuous and twice-differentiable for all \( x \neq -3, x \neq 3 \). It is known to have the properties
(i) \( f(0) = 1 \)
(ii) \( f'(5) = 0 \)
(iii) \( f'' > 0 \) on \((0,3) \cup (3, +\infty)\)
(iv) \( f'' < 0 \) on \((-\infty, -3) \cup (-3, 0)\)
If the lines \( x = -3 \), \( x = 3 \) and \( y = 1 \) are asymptotes of the graph of \( f \), sketch a graph of \( f \) satisfying these conditions.

12) Find any vertical, horizontal, or slant asymptotes of the function. Determine the function’s behavior near any vertical asymptotes.
A) \( f(x) = \frac{5 - 2x^2}{x^2 - 4} \)
B) \( g(x) = \frac{3x - 3}{x^2 - 2x + 1} \)
1) Start Point

\[ x = \sqrt{h^2 + v^2} \]

So \( x = \sqrt{h^2 + v^2} \) and

\[ \frac{dx}{dt} = \frac{1}{2} (h^2 + v^2)^{-\frac{1}{2}} \left[ 2h \frac{dh}{dt} + 2v \frac{dv}{dt} \right]. \]

After two hours, \( h = 2(25) = 50 \) and \( v = 2(60) = 120 \) i.e. also \( \frac{dh}{dt} = 25 \) and \( \frac{dv}{dt} = 60 \).

So

\[ \frac{dx}{dt} = \frac{1}{2} \left( 50^2 + 120^2 \right)^{-\frac{1}{2}} \left[ 2 \cdot 50 \cdot 25 + 2 \cdot 120 \cdot 60 \right] \]

\[ = \frac{1}{2} \left( 130^2 \right)^{-\frac{1}{2}} \left[ 50^2 + 120^2 \right] = \frac{1}{2} \left( 130^2 \right)^{-\frac{1}{2}} \left( 130^2 \right) = \frac{1}{2} \cdot 130 = 65 \text{ mph} \]

2) Balloon

\[ x = \text{distance} \]

We have

\[ \tan(\theta) = \frac{h}{100}, \text{ or} \]

\[ \theta = \arctan \left( \frac{h}{100} \right). \]

So

\[ \frac{d\theta}{dt} = \frac{1}{1 + \left( \frac{h}{100} \right)^2} \cdot \frac{1}{100} \cdot \frac{dh}{dt} \]

and when \( h = 100 \), since \( \frac{dh}{dt} = 10 \)

\[ \frac{d\theta}{dt} = \frac{1}{1 + 1} \cdot \frac{1}{100} \cdot 10 = \frac{10}{200} = 0.05 \text{ radians/seconds}. \]
3. Note that this is from 3.10, which many classes did not cover.

Nevertheless, here’s the solution.

Let \( f(x) = \sqrt[3]{x} \). Then we can approximate \( f(x) \) near 8 by

\[
f''(x) \approx f(8) + f''(8)(x-8).
\]

So

\[
f(8.12) \approx \sqrt[3]{8} + \frac{1}{3} (8)^{-\frac{2}{3}} (8.12 - 8), \quad \text{since } f'(x) = \frac{1}{3} x^{-\frac{2}{3}}
\]

\[
= 2 + \frac{1}{3} \cdot \frac{1}{4} (0.12) = 2.01.
\]

4. \( f'(\theta) = \theta - \tan \theta \)

So \( f'(\theta) = 1 - \sec^2 \theta = \frac{-1}{\cos^2 \theta} \)

\( f \) has critical points where \( f' = 0 \) or \( f' \) is undefined.

\( f' \) is undefined where \( \cos^2(\theta) = 0 \) —

that is, at \( \theta = \frac{\pi}{2} + 2\pi n \) or \( \theta = \frac{3\pi}{2} + 2\pi n, \, n \in \mathbb{Z} \)

\( f' = 0 \) wherever \( 1 = \frac{1}{\cos^2 \theta} \iff 1 = \cos^2 \theta \iff \cos(\theta) = \pm 1 \)

that is, at

\( \quad \theta = 2\pi n \quad \text{or} \quad \theta = \pi + 2\pi n, \, n \in \mathbb{Z} \)

But we were asked for critical values

Note that when \( f'(\theta) = 0 \) (i.e., \( \cos \theta = 0 \)), \( f(\theta) \) is undefined, too,

so our only critical points are \( \theta = 2\pi n \) or \( \theta = \pi + 2\pi n \).

\[
f(\pi n) = \pi n - \tan(2\pi n) = 2\pi n
\]

\[
f(\pi + 2\pi n) = 2\pi n + \pi - \tan(2\pi n + \pi) = 2\pi n + \pi
\]

Our critical values are of the form \( \pi n \) or \( \pi + 2\pi n, \, n \in \mathbb{Z} \).

Alternatively, of the form \( n\pi \), \( n \in \mathbb{Z} \).
5. \( f(x) = (x^2-1)^3 \) on \([-1, 3]\).

Note that \( f'(x) = 3(x^2-1)^2 \cdot 2x \), so \( f \) has critical points when \( x^2 = 1 \) = 0 (i.e. \( x = \pm 1 \)) or \( 2x = 0 \).

So to find our absolute min, max I will just evaluate \( f \) at critical points and boundary points.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>( 8^3 )</td>
</tr>
</tbody>
</table>

absolute mins:

absolute max:

6.

4. \( f(x) = \frac{1}{x^2+2} \) is not continuous at \( \sqrt{2} \in [0, 2] \), and hence does not satisfy the hypotheses of the Mean Value Theorem.

5. \( f(x) = |x| \) is continuous on \([0, 1]\) and differentiable on \((0, 1)\), hence does satisfy .... of the MVThm.

6. \( f(x) = x^{2/3} \) is not differentiable at \( 0 \in (-1, 1) \), hence does not satisfy .... of MVThm.
\[ F(x) = x^3 - 3x + 5, \quad \text{on } [-2, 2] \]

The average change of \( F \) on \([-2, 2]\) is

\[
\frac{F(2) - F(-2)}{2 - (-2)} = \frac{7 - 3}{4} = 1.
\]

So we're looking for \( c \in (-2, 2) \) with \( F'(c) = 1 \).

But \( F'(x) = 3x^2 - 3 \).

So \( F'(x) = 3x^2 - 3 = 1 \) \( \Rightarrow 3x^2 = 4 \) \( \Rightarrow x^2 = \frac{4}{3} \) \( \Rightarrow x = \pm \frac{2}{\sqrt{3}} = \pm \frac{2\sqrt{3}}{3} \).

8. \( F(x) = 3x^5 - 5x^3 + 3 \).

\( F'(x) = 15x^4 - 15x^2 = 15x^2(x^2 - 1) \).

So \( F \) has critical points at \( x = 0, \pm 1 \).

\( F''(x) = 60x^3 - 30x = 30x(2x^2 - 1) \)

So \( F'' = 0 \) when \( x = 0 \) or \( \pm \frac{1}{\sqrt{2}} = \pm \frac{\sqrt{2}}{2} \).

So \( F \) has inflection points at \( 0, \pm \frac{\sqrt{2}}{2} \).
9.  \( F(x) = \frac{x^2}{4} + \cos(x) \) on \([-\pi, \pi]\)

\[ F'(x) = \frac{x}{2} - \sin(x) \]

\[ F''(x) = \frac{1}{2} - \cos(x). \]

So \( F'' = 0 \) when \( \cos(x) = \frac{1}{2} \), i.e. when \( x = \pm \frac{\pi}{3} \).

So \( F \) changes concavity exactly at \( \pm \frac{\pi}{3} \).

\[
\begin{array}{ccc}
\cup & \cap & \cup \\
-\pi & \pi/3 & \pi/3 \\
\cup & 0 & \cup \\
\end{array}
\]

\( P'' > 0 \quad P'' < 0 \quad P'' > 0 \).

10. \( \lim_{x \to 0} \sqrt{\arctan(x^2)} \)

\[ = \lim_{x \to 0} \frac{x^2}{x^2} \cdot \frac{2x}{2x} \]

\[ = \lim_{x \to 0} \frac{1}{x^4} = 1. \]

11. \( \lim_{x \to \infty} (e^x + x)^{\frac{1}{x}} = \lim_{x \to \infty} \frac{\ln(e^x + x)}{x} \)

\[ = e^{\lim_{x \to \infty} \frac{\ln(e^x + x)}{x}} \]

\[ = e^{\lim_{x \to \infty} \frac{e^x + 1}{e^x + x}} \]

\[ = e^{e^{\lim_{x \to \infty} \frac{1}{e^x + 1}}} \]

\[ = e^{\frac{1}{e^{\lim_{x \to \infty} \sqrt{\ln(e^x + x)}}}} \]

\[ = e^{\frac{1}{e^{e}}} = e^{\frac{1}{e}} = e. \]
\[ \lim_{x \to 0^+} \sqrt{x} \ln(x) = \lim_{x \to 0^+} \frac{\ln(x)}{x^{-\frac{1}{2}}} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2} x^{-\frac{3}{2}}} \]
\[ = \lim_{x \to 0^+} -2 \cdot \frac{1}{x} \cdot x^{\frac{3}{2}} = \lim_{x \to 0^+} -2 \cdot x^{\frac{1}{2}} = 0 \]

\[ \lim_{x \to 0^+} \left( \frac{1}{x} - \frac{1}{\sin x} \right) = \lim_{x \to 0^+} \left( \frac{\sin(x) - x}{x \sin x} \right) = \lim_{x \to 0^+} \left( \frac{\cos x - 1}{\sin x + x \cos x} \right) \]
\[ = \lim_{x \to 0^+} \left( \frac{-\sin x}{\cos x + x \cos x - x \sin x} \right) = \frac{0}{2} = 0 \]
\[ f(x) = \frac{5 - 2x^2}{x^2 - 4} \]

Undefined at \( x = \pm 2 \).

\[
\lim_{{x \to 2^+}} f(x) = \lim_{{x \to 2^+}} \frac{5 - 2x^2}{x^2 - 4} = -\infty, \quad \text{so } x = 2 \text{ is an asymptote.}
\]

\[
\lim_{{x \to 2^-}} f(x) = \lim_{{x \to 2^-}} \frac{5 - 2x^2}{x^2 - 4} = -\infty, \quad \text{similarly, so } x = -2 \text{ is an asymptote.}
\]

Finally, L'Hôpital gives us that

\[
\lim_{{x \to \infty}} f(x) = \lim_{{x \to -\infty}} f(x) = -2, \quad \text{so } y = -2 \text{ is a horizontal asymptote.}
\]

\[ g(x) = \frac{3x - 3}{x^2 - 2x + 1} \]

Undefined at \( x = -1 \).

\[
\lim_{{x \to -1^+}} g(x) = \lim_{{x \to -1^+}} \frac{3x - 3}{x^2 - 2x + 1} = -\infty, \quad \text{so } x = -1 \text{ is an asymptote.}
\]

\[
\lim_{{x \to -1^-}} g(x) = \lim_{{x \to -1^-}} \frac{3x - 3}{x^2 - 2x + 1} = \frac{3(-1) - 3}{(-1)^2 - 2(-1) + 1}, \quad \text{is positive}
\]

L'Hôpital quickly gives that \( \lim_{{x \to \infty}} g(x) = \lim_{{x \to -\infty}} g(x) = 0, \)

so \( y = 0 \) is our horizontal asymptote.

Neither function has a slant asymptote.
Max \ Area = xy = x(12-x^2) \\
= 12x - x^3 \\
A' = 12-3x^2 \ , \text{so our critical points are at} \\
x = 2 \quad (x=-2 \ does \ not \ make \ sense \ in \ this \ context).

Since A' > 0 for x < 2 and A' < 0 for x > 2, \ x = 2 \ is \ our \ absolute \ maximum, \ with \ a \ Area \ of \ 12(2) - 2^3 = 16.

Length = 2x + y = 800 \\
Maximise \\
Area = xy = x(800 - 2x) \\
= 800x - 2x^2 \\
Note \ A' = 800 - 4x, \ so \ A \ has \ a \ critical \ point \ at \ x = 200.

Since A' > 0 for x < 200, \ A' < 0 for x > 200, \ x = 200 \\
is \ our \ maximising \ point, \ with \ A = 200(800 - 2(200)) \\
= 200(400) = 80000 \ ft^2.

Profits = (# sold) \ (price/unit) \\
call this \ p. \\
We \ want \ to \ write \ (# \ sold) = n \ as \ a \ function \ of \ p. \\
This \ set \ up \ assumes \ that \ n \ is \ a \ linear \ function \ of \ p— \\
that \ is, \ there \ are \ some \ m, b \ such \ that \\
\ n = mp + b.
Since when \( p \uparrow \) by 3, \( n \downarrow \) by 1, we must have

\[ m = -\frac{1}{3}. \]

So \( n = -\frac{1}{3} p + b \). We have data point \( (p=30, n=20) \)

so \( 20 = -\frac{1}{3} (30) + b \), or \( b = 30 \).

So \( \max \ p \cdot n = p \left( -\frac{1}{3} p + 30 \right) = -\frac{1}{3} p^2 + 30p \).

Our derivative is \( -\frac{2}{3} p + 30 \); setting it equal to zero we get

\[ \frac{2}{3} p = 30 \implies p = 45. \]

15. \( f''(t) = 8 \cos(2t) \quad f(0) = 1 \quad f'(\frac{\pi}{2}) = 1 \)

\[ f'(t) = 4 \sin(2t) + c \]

\[ 1 = f'(\frac{\pi}{2}) = 4 \sin(\pi) + c \quad \implies c = 1, \quad f'(t) = 4 \sin(2t) + 1. \]

So \( f(t) = -2 \cos(2t) + t + k. \)

\[ 1 = f(0) = -2 \cos(0) + 0 + k = -2 + k, \quad \text{so} \quad k = 3. \]

So \( f(x) = -2 \cos(2x) + x + 3 \)

\[ f(\pi) = -2 \cos(2\pi) + \pi + 3 = -2 + \pi + 3 = \pi + 1. \]
17. \[ g'(x) = 6e^{2x} - \sec^2 x + x, \quad g(0) = 1 \]
\[
g(x) = 3e^{2x} - \tan(x) + \frac{1}{2}x^2 + C
\]
1 = g(0) = 3e^0 - \tan(0) + \frac{1}{2}(0)^2 + C = 3 + C, \text{ so } C = -2
and \[ g(x) = 3e^{2x} - \tan(x) + \frac{1}{2}x^2 - 2. \]

18.

19. \[ y = \cos x \]
\[
\int_0^{\pi/2} \cos(x) \, dx \approx \cos(0) \cdot \frac{\pi}{3} + \cos\left(\frac{\pi}{2}\right) \cdot \frac{\pi}{3} + \cos\left(\frac{\pi}{3}\right) \cdot \frac{\pi}{3}
\]
\[
= \frac{\pi}{3} \left(1 + \frac{\sqrt{3}}{2} + \frac{1}{2}\right) = \frac{\pi}{3} \left(3 + \sqrt{3}\right) = \frac{\pi (3 + \sqrt{3})}{6}
\]
\[
\int_0^{\pi/2} \cos(x) \, dx = \sin x \bigg|_0^{\pi/2} = \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1 - 0 = 1.
\]
$f(x) = 3x^2$ on $[1, 6]$.

\[
\int 3x^2 \, dx = \lim_{n \to \infty} \left( \frac{1}{n} \sum_{i=1}^{n} \frac{6i}{n} \left( 3 \left( 1 + \frac{6i}{n} \right)^2 \right) \right)
= \lim_{n \to \infty} \left( \frac{1}{n} \sum_{i=1}^{n} \frac{18i^2}{n^2} \right)
\]

21.

A Very false. Could take almost any $f, g$ and have this fail. (e.g. $f(x) = g(x) = x^2$)

B True.

Since $f(x) \geq g(x)$, $f(x) - g(x) \geq 0$, so

\[
\int_{a}^{b} f(x) - g(x) \, dx \geq 0,
\]

C Sometimes true, sometimes false. (e.g. $[a, b] = [0, 1]$)

$\begin{align*}
F(x) &= 3x \\
g(x) &= 2x
\end{align*}$

\begin{align*}
F(x) &= 8x \\
g(x) &= 2x
\end{align*}

D Always true.

\[
\int_{a}^{b} 3f(x) \, dx = \int_{a}^{b} 2f(x) \, dx = \int_{a}^{b} 2g(x) \, dx
\]

since $f > 0$ \hspace{2cm} since $f \geq g$. 

\[ F(x) = \int_0^x \frac{1}{1 + t^2} \, dt. \]

So
\[ F'(x) = \frac{1}{1 + (3x)^2} \cdot 3 = \frac{3}{1 + 9x^2}, \quad F'(\frac{1}{3}) = \frac{3}{1 + 9} = \frac{3}{10}. \]

\[ \int_{-1}^{\frac{1}{3}} -\sin(\pi x) + 3x^2 \, dx = \left[ \frac{\cos(\pi x)}{\pi} + x^3 \right]_{-1}^{\frac{1}{3}} \]
\[ = \left( \frac{\cos(\frac{2\pi}{3})}{\pi} + \left(\frac{1}{3}\right)^3 \right) - \left( \frac{\cos(\pi)}{\pi} + 1 \right) = \left( \frac{1}{\pi} + \frac{8}{27} \right) - \left(1 - \frac{1}{\pi} \right) = \frac{2}{\pi} + \frac{7}{27}. \]

\[ \int_0^1 \frac{e^{2x} + 2}{e^x} \, dx = \int_0^1 \frac{e^{2x}}{e^x} + \frac{2}{e^x} \, dx \]
\[ = \left[ e^x + 2e^{-x} \right]_0^1 \]
\[ = (e - 2e^{-1}) - (e^0 - 2e^0) = e - \frac{2}{e} - (1 - 2) = e - \frac{2}{e} + 1. \]