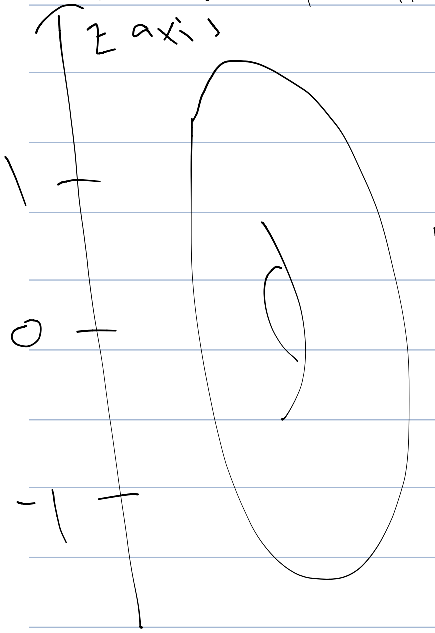
Morse Homology

Consider $T^2 \subseteq \mathbb{R}^3$.

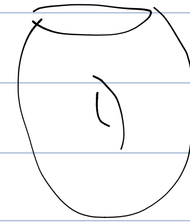


Let $f(x,y,z)=z$ and consider level sets

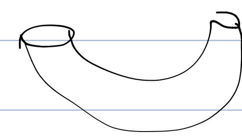
$$(T^2)^a = f^{-1}((-\infty,a))$$

Ex:

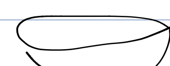
$(a=1)$



$(a=0)$



$(a=-1)$



Get decomp. into simpler pieces. But we need a particular kind of function. Let M^n be compact mfld, closed. A smooth fcn $f: M \to \mathbb{R}$ is Morse when f has only nondegenerate crit. pts. Recall that $p \in M$ critical if $df_p = 0$, and nondeg. means $\mathrm{Hess}_p(f)$ is nondegenerate:

$$H_f : TM \times TM \to \mathbb{R}$$
$$H_f(v,w) = \underbrace{v(\underbrace{df(w)}_{\text{smooth fcn, } \tilde{f}})}_{d\tilde{f}(v)}$$

or in local coord. (x_i), say,

$$H_f(p) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$$

Rmk: $p \in \mathrm{Crit}\, f$ for this to be well-defined.

Def The index of a critical point is defined as the dim.

of the maximal subspace on which $H_f(p)$ negative definite.

We also have normal form for such critical pts.

(Thm) (Morse Lemma) If $p \in \mathrm{Crit}(f)$ and p nondeg. then $\exists$ local coordinates so that (if $\mathrm{ind}(p) = k$)

$$f(x) = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2$$

Fact: This gives a handle decomposition of M. Moreover

- ($p \neq q$) If $p, q \in \mathrm{Crit}\, f$, w/ $f(p) \neq f(q)$.
- If $f(p) = a < b < f(q) = c$ then $\forall$ such b, M^b def. retracts to $M^{b'}$.
- If $\mathrm{ind}(p) = k$, then $M^c = M^a \cup (k\text{-handle})$.

Also, Morse fcns are easy to find.

Thm If $f: M \to \mathbb{R}$ smooth, $M \subseteq \mathbb{R}^N$ embedded mfld then for almost all $(a_i) \in \mathbb{R}^N$,

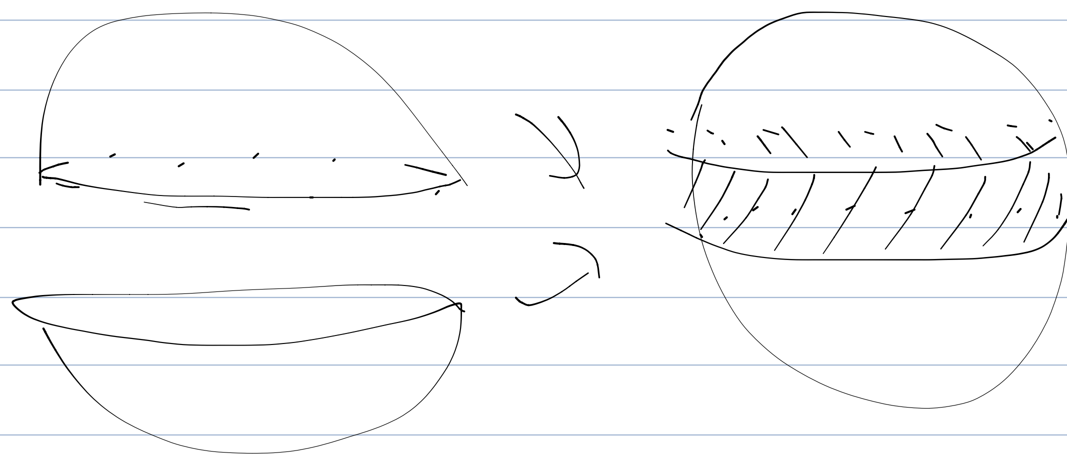
$$f_a(x) = f(x) - \sum a_i x_i$$

is Morse.

pf: Consider $g = \left(\partial f/\partial x_1(x), \ldots, \partial f/\partial x_N(x)\right)$, so $df_a = g - a = 0$.
For $g: \mathbb{R}^n \to \mathbb{R}^n$, a is generically regular value $\Rightarrow$ dg an isom. $\blacksquare$

Thm (Reeb) M as above and $f: M \to \mathbb{R}$ Morse w/ exactly 2 critical pts (p, q) then $M^n \underset{\text{homeo}}{\cong} S^n$.

Sketch: WLOG p is max, q min. Then $\operatorname{ind} p = 0$, $\operatorname{ind} q = n$ so we glue (clutching fcn)



Morse-Smale Pairs & Gradient Flow

A Riemannian metric on M is symmetric, positive definite, bilinear pairing $\langle \cdot , \cdot \rangle : TM \otimes TM \to \mathbb{R}$.

The gradient of $f: M \to \mathbb{R}$ smooth is the vector field $\nabla f \in \mathfrak{X}(M)$ s.t. $df(v) = \langle \nabla f, v \rangle$.

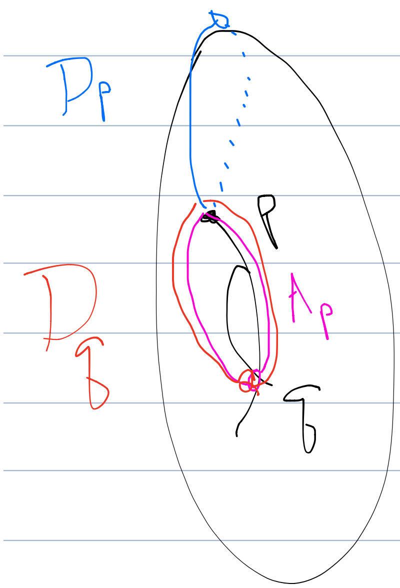
The flow along ∇f is the flow along ∇f, φ_t, so

$$\tfrac{d}{dt}\, \varphi_t(x) = \nabla f(\varphi_t(x)).$$

At $p \in \mathrm{Crit}(f)$, define the ascending and descending mflds

$$A_p = \{x \in M \mid \lim_{t\to\infty} \varphi_t(x) = p\}$$

$$D_p = \{x \in M \mid \lim_{t\to-\infty} \varphi_t(x) = p\}$$



Note: Each x lies in exactly one A_p. Moreover if $p \in \mathrm{Crit}\, f$, $p = A_p \cap D_p$.

Facts: ① A_p and D_p are mflds.
② $\dim A_p = \mathrm{ind}\, p$.

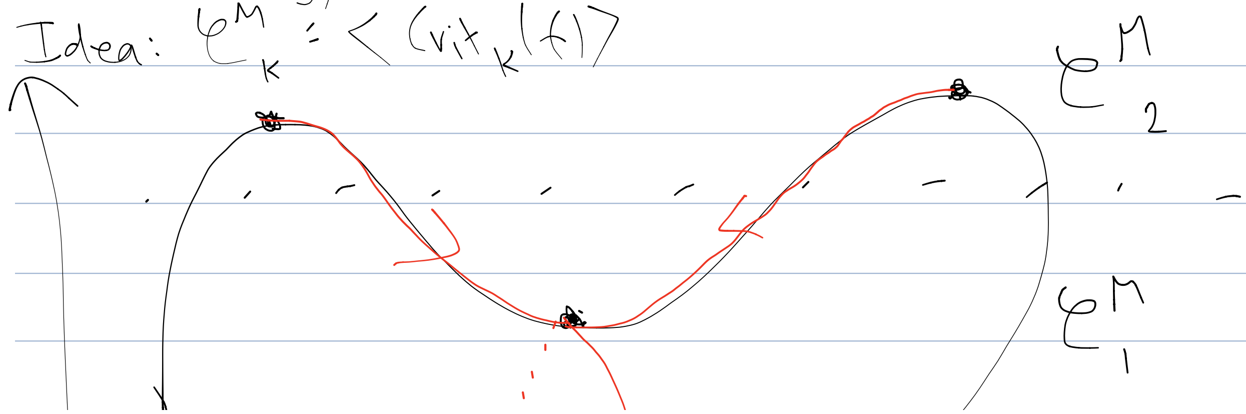
Def f is Morse-Smale when f is Morse and $\forall\, p, q \in \mathrm{Crit}\, f$, $A_p \pitchfork D_q$.

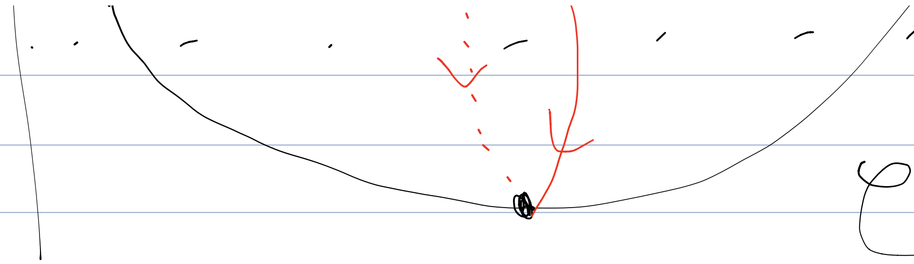
Fact: Morse-Smale fcns are generic. (λ-Lemma).

So above example not Morse-Smale; A_p and D_q not transverse.

Morse Homology

Idea: $\mathcal{C}^M_k = \langle \mathrm{Crit}_k(f) \rangle$





$\mathcal{C}_0^M$

Then for $p \in Crit(f)$, $\delta(p) = \sum_{q \in \mathcal{C}_{k-1}^M} q(\#\text{ oriented flow lines from } p \text{ to } q)$

<u>Def</u> $\mathcal{M}(p,q)' := A_p \cap D_q$ and reduced is $\mathcal{M}(p,q) := \mathcal{M}'/\mathbb{R}$.

We are interested in
① $ind(q) = ind(p) - 1$
② $\dim(A_p) = k$
③ $\dim(D_q) = n-(k-1)$
④ $\dim(\mathcal{M}(p,q)') = 1$
$\Downarrow$
$\dim(\mathcal{M}(p,q)) = 0$ ✓

So we can reformulate ∂ map as

$$\partial(p) = \sum_{q \in \mathcal{C}_{k-1}^M} q(\#\mathcal{M}(p,q))$$

<u>Ex</u>: (Sphere above) $\mathcal{C}_2^M = \langle p, r \rangle$, $\partial(p) = q$, $\partial(r) = q$

$\mathcal{C}_1^M = \langle q \rangle$, $\partial(q) = 0$ since signs cancel

$\mathcal{C}_0^M = \langle s \rangle$, $\partial(s) = 0$

$\rightarrow$ get $H_*(S^2)$.

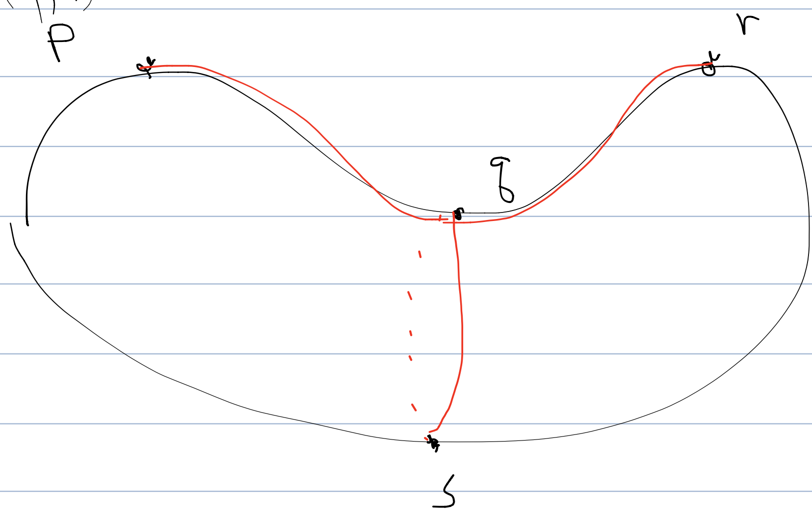
Broken Flowlines

How to see $\partial^2 = 0$? Can equate w singular homology, but this isn't very concrete.

$$\partial^2(p) = \partial\left(\sum q \,\#\mathcal{M}(p,q)\right) = \sum r \,\#\mathcal{M}(p,q)\,\#\mathcal{M}(q,r)$$

Fact: Count here is boundary of compact 1-mfld.

Recall previous example, S^2, consider $\mathcal{M}(p,s)$ and its compactification $\overline{\mathcal{M}}(p,s)$.



$$\overline{\mathcal{M}}(p,s) = \bigcup_{\{q_i \text{ distinct}\}} \mathcal{M}(p,q_1) \times \mathcal{M}(q_1,q_2) \times \cdots \times \mathcal{M}(q_i,s)$$

and $\partial\overline{\mathcal{M}}(p,s) = \bigcup_{\{q_i \ldots\}}$ (broken flowlines) ... etc.