Lagrangian Floer Homology

Outline: ① Morse homology
② Floer's Remedy
③ Well-definedness

Given (M,ω) symplectic, $L_0, L_1 \subseteq M$ compact Lagrangians, choose J compatible almost complex structure s.o

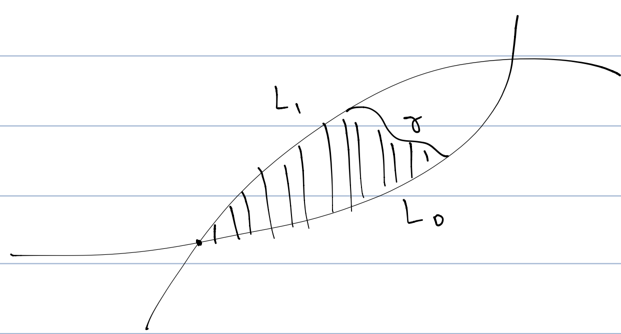
$$g(\cdot,\cdot) := \omega(\cdot, J(\cdot))$$

is a metric.

I.) Try to do Morse homology with symplectic action functional:

$$A : \Omega(L_0,L_1) \longrightarrow \mathbb{R} \text{ (generally multivalued)}$$

where $\Omega(L_0,L_1) = \{\gamma : [0,1] \to M \mid \gamma(i) \in L_i\}$. Idea is that $A(\gamma)$ measures area of disc swept out by "path of paths"



Facts: ① $T_\gamma \Omega = \{ V \in \Gamma(\gamma^* TM) \mid V(i) \in T_{\gamma(i)} L_i \}$
② $dA_\gamma(V) = \int_0^1 \omega(V, \dot\gamma)\, dx$, "action 1-form"
③ $\gamma \in \mathrm{Crit}(A) \iff \gamma \equiv x \in L_0 \cap L_1$

So $L_0 \cap L_1 = \mathrm{Crit}(\mathcal{A})$! Try to do Morse theory on $\mathcal{A}$ to find $L_0 \cap L_1$

At $\gamma \in \mathrm{Crit}(\mathcal{A})$,

$$\mathrm{Hess}_\gamma \mathcal{A}(V, W) = \int_0^1 \omega(V, \dot{W})\, dt = \int_0^1 g(V, -J\dot{W})$$

and as an endomorphism of $T_\gamma \Omega$,

$$\mathrm{Hess} = -J\, d/dt$$

Also, Hess_γ is nondegenerate $\Longleftrightarrow L_0 \pitchfork L_1$ at $x \equiv \gamma$.

⚠ But even so, no well-defined index; dim of negative eigenspace is infinite.

<u>Ex</u>: Consider e.g. solutions $\lambda \in \mathbb{C}$, $u : [0,1] \to \mathbb{C}$

$$\begin{cases} -i\dot{u} = i\lambda u \\ u(0) \in \mathbb{R} \\ u(1) \in i\mathbb{R} \end{cases}$$

Then $u(t) = u_0 e^{i\lambda t}$ for $u_0 \in \mathbb{R}$, $\lambda \in \frac{\pi}{2} + 2\pi k$ $(k \in \mathbb{Z})$ are all solns. :(

II.) Pretend we have reasonable gradient flowline

$$\gamma_s : \mathbb{R} \to \Omega(L_0, L_1)$$
$$\partial\gamma_s/\partial s = -J\, \partial\gamma_s/\partial t$$

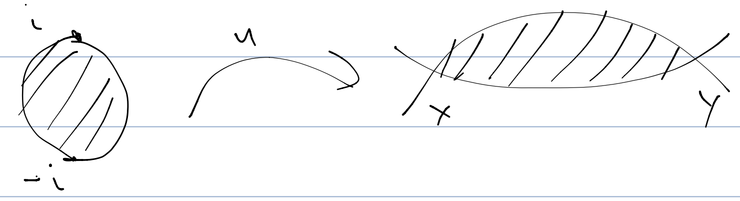
between $\gamma_\pm \in \mathrm{Crit}(A)$ where $\gamma_\pm \equiv x_\pm \in L_0 \cap L_1$.
Then we rewrite this ODE in $\Omega(L_0, L_1)$ to a PDE in M:
find a map $u(s,t) = \gamma_s(t)$ that solves

$$\begin{cases} \partial u/\partial s + J\, \partial u/\partial t = 0 \quad (\Leftrightarrow du \circ j = J \circ du) \\ \lim_{s \to \pm\infty} u(s,t) = x_\pm \\ u(s,i) \in L_i \end{cases}$$

Floer's remedy is to count these maps $\{u\}$ to find flowlines as if we were doing Morse theory.

Recall that for $x, y \in L_0 \cap L_1$,

$$\pi_2(x,y) = \{\text{htpy classes of whitney discs from } x \text{ to } y\}$$



<u>Defn</u>: For $[\varphi] \in \pi_2(x,y)$, define the <u>moduli space</u> of holomorphic discs

$$\mathcal{M}_J[\varphi] = \left\{ u : \mathbb{D}^2 \to M \;\middle|\; \begin{array}{l} du \circ j = J \circ du \\ [u] = [\varphi] \in \pi_2(x,y) \end{array} \right\}$$

<u>Thm</u> For generic $J \in \mathcal{J}(M, \omega)$, $\mathcal{M}_J[\varphi]$ is a finite dim'l mfld

$\forall [\phi] \in \pi_2(x,y)$ and $x,y \in L_0 \cap L_1$.

<u>Defn</u> The "<u>Maslov index</u>" is $\dim_{\mathbb{R}}(\mathcal{M}_J[\phi])$, when $[\phi] \neq 0 \in \pi_2(x,y)$.

<u>Facts</u> ① $\exists\ \mathbb{R}$-action of translation on $\mathbb{R}\times[0,1] \cong D^2\setminus\{\pm i\}$.

$\Rightarrow \mu[\phi] \geq 1$ (when $\exists$ nonconst. repesentative)#

② $\mu([\varphi]\cdot[\psi]) = \mu[\varphi]+\mu[\psi]$ again when #

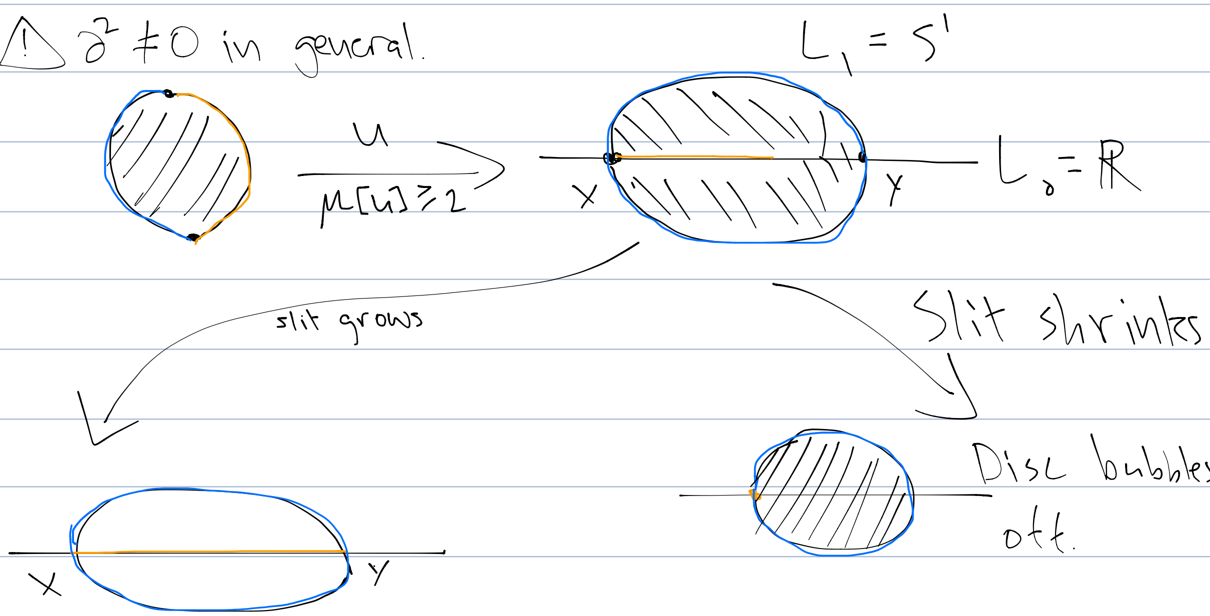
concatenation

Define the <u>Floer chain complex</u> $CF(L_0,L_1)$

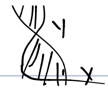
$$CF(L_0,L_1) := \mathbb{Z}\langle L_0\cap L_1\rangle$$

$$\partial x = \sum_{y\in L_0\cap L_1}\ \sum_{\substack{[\phi]\in\pi_2(x,y)\\ \mu[\phi]=1}} \left(\#\mathcal{M}_J[\phi]/\mathbb{R}\right)$$

⚠ $\partial^2 \neq 0$ in general.



Good degeneration, looks like 



III.) Assume ① $L_0, L_1 \subset M$ compact transverse Lagrangians,

② M cpt or "convex at infinity"

③ $\langle \omega, A \rangle = 0 \ \forall A \in \pi_2(M)$ and $\pi_2(M, L_i)$

④ $\forall x \in L_0 \cap L_1$,

Energy$(\ker \mu : \pi_2(x,x) \to \mathbb{Z})$ is bounded above.

Rmk! $E(u) = \int_D |d\varphi|^2_{\omega,J} \, dvol_D$. Note that $\forall x, y \in L_0 \cap L_1$, $\exists C > 0$ such that $E(u) < C \ \forall u \in \mathcal{M}[\phi]$, $[\phi] \in \pi_2(x,y)$.

Thm (Floer) Under above hypotheses,

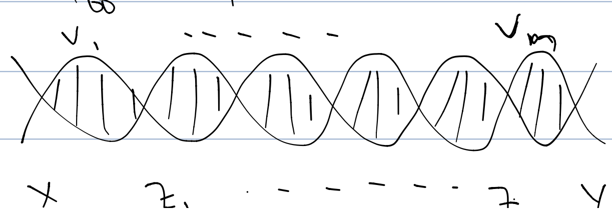
$$\mathcal{M}(x,y) := \coprod_{\substack{[\phi] \in \pi_2(x,y) \\ \mu[\phi] = 1}} \left(\mathcal{M}_J[\phi] / \mathbb{R} \right)$$

is smooth 0-mfld.

Prop $\forall x, y \in L_0 \cap L_1$, $\mathcal{M}(x,y)$ compact.

pf: Let (u_n) be sequence of maps in $\mathcal{M}(x,y)$. Then ④ $\Rightarrow$ uniform energy bounds, so Gromov compactness applies. Passing to some subsequence, get $u_n \xrightarrow{C^\infty_{loc}} \bar{u}_\infty$, stable map.

Then ③ $\Rightarrow \bar{u}_\infty = v_1 \cdot v_2 \cdots v_n$



$1 = M[\bar{u}_\infty] = \sum M[v_i]$

Want to say that v_i are all const. except for one. For generic J, negative index $\Rightarrow$ empty moduli space, so no negative index trajectories. Similarly, for generic J, can show index 0 trajectories must have zero area, so constant. Then all Gromov limits are , so compact. $\square$

Prop $\partial^2 = 0$ under above hypotheses.

Def $HF(L_0, L_1) = H_*(CF(L_0, L_1), \partial)$.

Fact $HF(L_0, L_1)$ doesn't depend on choice of J.