1) Determine the increasing and decreasing properties of \( f(x) = \frac{\sqrt{2}}{2} x + \cos x \) on the interval \([0, 2\pi]\).

2) Let \( f \) be the function defined by \( f(x) = x - 3x^3 \). Which of the following properties does \( f \) have?

   A) local maximum at \( x = 0 \)
   B) concave up on \((0, \infty)\)
   C) point of inflection at \( x = 0 \)
   D) increasing on the interval \((-\infty, 1)\) and decreasing otherwise

3) The following is the graph of the derivative of \( f \). Use it to determine the graph of \( f'' \). Does \( f \) have a local maximum? If so, where does it occur?

4) Determine the limits:

   A) \( \lim_{x \to 0} \frac{e^x - 1}{x^3 + 2x} \)  
   B) \( \lim_{x \to 0} \frac{3x - \ln x}{6x} \)  
   C) \( \lim_{x \to 0} \frac{(\cos 3x)^2}{x} \)  
   D) \( \lim_{x \to 1} \frac{1}{\ln x} - \frac{1}{x - 1} \)  
   E) \( \lim_{x \to \infty} xe^x \)
5) Sketch a graph of a function that satisfies the following:
   A) \( f \) is continuous on \((-\infty, \infty)\)
   B) \( f \) has a local max at \( x = 3 \) where \( f'(x) = 0 \)
   C) \( f \) has a local min at \( x = -2 \) where \( f'(x) \) is undefined
   D) \( \lim_{x \to \infty} f(x) = -7 \) and \( \lim_{x \to -\infty} f(x) = 4 \)

   When your graph is finished, state the intervals where \( f \) is increasing and the intervals where \( f \) is decreasing. For which values is \( f''(x) = 0 \) and for which values is \( f''(x) \) undefined? Where are the points of inflection?

6) A function \( f \) is continuous and twice-differentiable for all \( x \neq 2 \). It is known to have the properties
   (i) \( f(0) = 0 \)
   (ii) \( f'(4) = 0 \)
   (iii) \( f'' > 0 \) on \((-\infty, 2) \cup (6, +\infty)\)
   (iv) \( f'' < 0 \) on \((2, 6)\)

   If the lines \( x = 2 \) and \( y = -3 \) are asymptotes of the graph of \( f \), sketch a graph of \( f \) satisfying these conditions.

7) For each of the following graphs, discuss the (i) local extrema, (ii) increase/decrease (iii) concavity, (iv) points of inflection, (v) critical points

   A)
   ![Graph A]

   B)
   ![Graph B]

8) Find any vertical or horizontal asymptotes of the functions.
   A) \( f(x) = \frac{2x^3 + 5}{3x^2 - x} \) 
   B) \( g(x) = \frac{(x+3)^2}{9 - x^2} \)
9) Which of the following could be the graph of \( f(x) = \frac{x^3 + 1}{x^2 - 1} \)

10) A rectangular poster is to have a total area of 72 \( \text{in}^2 \) with 1-inch margins at the bottom and sides and a 3-inch margin at the top. What dimensions will give the largest printed area?

11) Find the coordinates of \( P \) that maximize the area of the rectangle shown in the figure below.

12*) An outdoor tour guide offers the following rates: $200 per person if 50 people (the minimum number necessary) go on the tour. For each additional person, up to a maximum of 80 people, everyone’s charge is reduced by $2. It costs $6000 (a fixed cost) plus $32 per person to conduct the tour. How many people does it take to maximize the guide’s profit?

13) A right circular cylinder (with a bottom and a top) is to be designed to hold 24 cubic inches of a soft drink and to use a minimum of material in its construction. Find the required dimensions for the container.

14*) A Norman window has the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is 30 ft, find the length of the base that will allow the window to admit the most light.
\[ -100 \text{ (50)} + 268 \text{ (50)} - (120) \text{ (50)} = 48 \text{ (50)} \]

\[ 268 \text{ (50)} - 120 \text{ (50)} + 268 \text{ (17)} \]

\[ -2(67) \text{ (50)} - 2(67) \text{ (17)} \]

\[ (268 - 120 - 134) \text{ (50)} + (268 - 2(67)) \text{ (17)} \]

\[ \frac{254}{267 - 134} \]

\[ 14 \cdot 50 \]

\[ 700 \]

\[ 133.17 \]

\[ 1330 + \frac{931}{931} \]

\[ \frac{2262}{2962} \]

\[ 80 \left(-160 + 268 - 75\right) \]

\[ 108 - 75 \]

\[ 80 \cdot 33 \]

\[ 2400 \]

\[ 240 \]
1. \( f(x) = \frac{\sqrt{2}}{2} x + \cos(x) \) is increasing when \( f'(x) > 0 \) and decreasing when \( f'(x) < 0 \).

   \[ f'(x) = \frac{\sqrt{2}}{2} - \sin x. \]

So if I first find critical points of \( f \) on \([0, 2\pi]\),

I set \( f'(x) = 0 \) and get

\[ \sin(x) = \frac{\sqrt{2}}{2}, \]

which has two solutions on \([0, 2\pi]\) -

\[ x = \frac{\pi}{4} \text{ and } x = \frac{3\pi}{4}. \]

So if I test points in intervals \((0, \frac{\pi}{4}), (\frac{\pi}{4}, \frac{3\pi}{4}), \text{ and } (\frac{3\pi}{4}, 2\pi)\)

I get that \( f'(x) \) is positive (and \( f \) is increasing) on \((0, \frac{\pi}{4}), (\frac{3\pi}{4}, 2\pi)\)

\[ f'(x) \text{ is negative (and } f \text{ is decreasing) on } (\frac{\pi}{4}, \frac{3\pi}{4}) \]
\( F(x) = x - 3x^{1/3} \).

So \( F'(x) = 1 - x^{-2/3}, \quad F''(x) = \frac{2}{3} x^{-5/3}. \)

Since \( F'(0) \) is undefined, \( F \) has a critical point at 0. However, for \( x \) close to zero (but negative), \( F'(x) \) is negative and for \( x \) close to zero (but positive) \( F'(x) \) is negative. So 0 is not a local max.

\( F''(x) \) is defined and positive for \( x \in (0, \infty) \), so \( F \) is concave up on \((0, \infty)\).

\( F''(x) \) is negative for \( x < 0 \), positive for \( x > 0 \) so 0 is a point of inflection.

\( F'(x) \) is positive (i.e. \( F \) is increasing) whenever

\[ 1 - x^{-2/3} > 0 \quad \text{i.e.} \quad 1 > x^{-2/3} \]

\[ 1 > \frac{1}{x^{2/3}} \]

\[ x^{2/3} > 1 \]

\[ x^2 > 1 \]

\[ x > 1 \quad \text{or} \quad x < -1 \]

\( F \) is increasing on \((-\infty, -1), (1, \infty)\)

and decreasing only on \(( -1, 1)\).
3) \( p'' \) is positive when \( p' \) is increasing (on \((-\infty, -5), (-1, 5)) \)
   negative \( " \) decreasing (on \((-5, -1), (5, \infty)) \).

So \( p'' \) looks something like.

\[ \text{Diagram showing } p'' \text{ behavior.} \]

\[ \text{So } p'' \text{ looks something like.} \]

\[ \text{So } p'' \text{ looks something like.} \]

\(\begin{align*}
\text{B) } & p' \text{ is positive on } (2, 7), \text{ equal to zero at 7,} \\
& \text{ and negative on } (7, 8). \text{ (p is decreasing)} \\
& \text{So 7 is a local max.}
\end{align*}\]

\(\begin{align*}
\text{H) } & \lim_{x \to 0} \frac{e^x - 1}{x^4 + 2x} = \lim_{x \to 0} \frac{e^x}{6x^4 + 2} = \frac{e^0}{6(0)^4 + 2} = \frac{1}{2}.
\end{align*}\)
\[ \lim_{x \to 0^+} \frac{3x - \ln(x)}{6x} = 0, \text{ positive} \]

\[ \lim_{x \to 0} \cos(\frac{5}{x})^{\frac{5}{x}} = \lim_{x \to 0} e^{\ln(\cos(\frac{5}{x}))^{\frac{5}{x}}} = \lim_{x \to 0} e^{\frac{5}{x} \ln(\cos(3x))} = e^{\lim_{x \to 0} \frac{5 \ln(\cos 3x)}{x}} = e^0 = 1 \]

\[ \text{(Since } \lim_{x \to 0} \frac{5 \ln(\cos 3x)}{x} = \lim_{x \to 0} \frac{5 \cdot (-\sin(3x))}{3 \cos(3x)} \cdot \frac{1}{3} = 0 \text{)} \]

\[ \lim_{x \to 1} \left( \frac{1}{\ln(x)} - \frac{1}{x-1} \right) = \lim_{x \to 1} \left( \frac{x-1 - \ln(x)}{\ln(x)(x-1)} \right) \]
\[ = \lim_{x \to 1} \frac{x - \frac{1}{x}}{\ln(x) + \frac{1}{x}(x-1)} = \lim_{x \to 1} \frac{x - 1}{x \ln(x) + x - 1} \]
\[ = \lim_{x \to 1} \frac{1}{\ln(x) + x + \frac{1}{x} + 1} = 1 \to 0 \]
\[ 0+1+1 = \frac{1}{2} \]

\[ \lim_{x \to -\infty} x e^x = \lim_{x \to -\infty} \frac{x}{e^{-x}} = \lim_{x \to -\infty} \frac{1}{-e^{-x}} = \lim_{x \to -\infty} -e^x = 0 \]
$f(x)$ is increasing on $(-2, 3)$

decreasing on $(-\infty, -2), (3, \infty)$.

$f''(x) = 0$ at $x=5$ (the point of inflection).

$f''(x)$ is undefined at $x=-2$. 
7) Local Extrema
   \( x = -2 \) a local max
   \( x = 1 \) a local max

8) Increase / Decrease
   Increasing \((-2, 1)\)
   Decreasing \((-\infty, -3) \cup (1, 3) \cup (-3, -2)\)

9) Concavity
   Concave Up \((-\infty, -3) \cup (-2, -1)\)
   Concave Down \((-3, -2) \cup (-1, 3)\)

10) Points of Inflection
    \( x = -3, -2, -1 \)

11) Critical points
    All local maxes, mins and
    \( x = -3 \quad (f' = 0) \)
    \( x = -1 \quad (f' \text{ undefined}) \)
    (All local maxes, mins and
    \( x = 0 \quad (f' = 0) \)
9. \[ f(x) = \frac{x^3+1}{x^2-1} \]

\[
\lim_{x \to \infty} f(x) = \infty, \text{ via L'Hopital, so our answer is not } D.
\]

\[
\lim_{x \to 1^+} f(x) = \infty, \text{ so our answer is not } A.
\]

\[
\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} \frac{x^3+1}{x^2-1} = \lim_{x \to -1^-} \frac{3x^2}{2x} = \lim_{x \to -1^-} \frac{3}{2} x = -\frac{3}{2},
\]

so our answer is not B.

So our answer is C.

10. \[ \text{Total Area = (Printed area) + Margins} \]

\[ 72 = (\text{Printed area}) + (3w+w+(l-4)) + (l-4) \]

So we want to maximize

\[ PA = 72 - 4w - 2l + 8 \]

But we're requiring \( \text{Total area} = w \cdot l = 72 \). So \( l = \frac{72}{w} \), and

\[ PA = 80 - 4w - \frac{144}{w} \]

\[ (PA)' = -4 + \frac{144}{w^2} \]

Setting \((PA)' = 0\), we get \( w^2 = \frac{144}{4} = 36 \), or \( w = 6 \), \( l = 12 \).

Also, boundary values at \( w = 0 \) and \( l = 0 \) both give \( A = 0 \).
\( F(x) = \frac{2x^3 + 5}{3x^2 - x} \) is undefined when \( 3x^2 - x = 0 \)

i.e. when \( x(3x-1) = 0 \), \( x = 0 \) or \( x = \frac{1}{3} \)

Since \( \lim_{x \to 0} F(x) = \infty \), \( F \) has a vertical asymptote at 0.

\( \lim_{x \to \frac{1}{3}} F(x) = -\infty \), \( F \) has a vertical asymptote at \( \frac{1}{3} \).

\( F \) has no horizontal asymptotes, since

\[ \lim_{x \to \infty} F(x) = \infty \quad \text{(use L'Hospitil)} \]

\[ \lim_{x \to -\infty} F(x) = -\infty \]

\[ \text{Similarly, } G(x) = \frac{(x+3)^2}{9-x^2} \text{ is undefined when } 9-x^2 = 0 \]

i.e. \( x = \pm 3 \).

\[ \lim_{x \to 3} G(x) = \infty, \text{ so } G \text{ has a vertical asymptote at } x = 3 \]

However

\[ \lim_{x \to 3} G(x) = \lim_{x \to -3} \frac{(x+3)^2}{9-x^2} = \lim_{x \to -3} \frac{2(x+3)}{-2x} = 0, \]

so \( G \) does not have a vertical asymptote at \( x = -3 \).

We can use L'Hospital to find

\[ \lim_{x \to \infty} g(x) = -1 = \lim_{x \to -\infty} g(x), \]

so \( g \) has a horizontal asymptote at \( y = -1 \).
Maximize Area.

\[
\text{Area} = l \cdot w
\]

Calling \( P = (x, y) \), \( l = x \) and \( w = y \).

\[
\text{Max } A = xy.
\]

Requiring \( P \) to lie on the line \( y = \frac{3}{4}x + 3 \), gives

\[
\text{Max } A = x\left(\frac{3}{4}x + 3\right).
\]

Then \( A' = \frac{3}{2}x + 3 \), so \( A' = 0 \iff x = 2 \), \( y = \frac{3}{2} + 3 = \frac{3}{2} \)

\[
A = 3.
\]

(Note boundary values are \( x = 0 \) or \( y = 0 \); both give \( A = 0 \).)

Max Profit = Revenue - Cost

Cost to take \( x \) people: \( 6000 + 32x \).

When we have 50 people, $200 each.

When we get one more person, the fee decreases by $2. This implies the fee is a linear function of the number of people.

\[
\text{So price/person} = -2(\text{# of people}) + 300.
\]

So Revenue = \( (\text{price/person})(\text{# of people}) = (-2x + 300)x \).
So we want to maximize

\[
\text{Profit} = (-2x + 300) x - (6000 + 32x) \\
P = -2x^2 + 268x - 6000, \text{ and we're requiring } 50 \leq x \leq 80.
\]

So \( P' = -4x + 268 \).

To find critical points, we set \( P' = 0 \) and get \( x = \frac{268}{4} = 67 \).

To see what our absolute max is, we evaluate \( P(50), P(67), P(80) \).

\[
P(50) = -2(50)(50) + 268(50) - 6000 \\
= 2400
\]

\[
P(67) = -2(67)(67) + 268(67) - 6000 \quad \text{\{Absolute Max with 67 people\}}
\]

\[
P(80) = -2(80)(80) + 268(80) - 6000(80) \\
= 2640
\]

13

\[
\text{Min}(SA)
\]

Minimize \( SA = 2(\pi r^2) + 2\pi rh \)

Requiring \( V = \pi r^2 h = 24 \)

or \( h = \frac{24}{\pi r^2} \)
So \( \text{Min } SA = 2\pi r^2 + 2\pi r \left( \frac{24}{\pi r^2} \right) = 2\pi r^2 + \frac{48}{r} \).

\[ SA' = 4\pi r - \frac{48}{r^2} \]

\[ SA' = 0 \quad \text{when} \quad 4\pi r - \frac{48}{r^2} = 0 \]

\[ 4\pi r = \frac{48}{r^2} \quad \Rightarrow \quad \pi r^3 = 12 \]

\[ r = \sqrt[3]{\frac{12}{\pi}} \]

14. \( \text{Perimeter } = \pi r + 2h + b \)

\[ = \pi r + 2h + 2r = 30 \]

Maximise \( A = \frac{1}{2} \pi r^2 + (2r)h \).

\[ = \frac{1}{2} \pi r^2 + 2r \left( \frac{30 - \pi r - 2r}{2} \right) \]

\[ = \frac{1}{2} \pi r^2 - \pi r^2 - 2r^2 + 30r \]

So \( A' = \pi r - 2\pi r - 4r + 30 \), and

\[ A' = 0 \quad \text{when} \quad r = \frac{30}{\pi + 4}, \quad b = 2 \left( \frac{30}{\pi + 4} \right) = \frac{60}{\pi + 4}. \]