Principal Congruence Links: Class Number Greater than 1

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1. Introduction

Let \( d \) be a square-free positive integer, let \( O_d \) denote the ring of integers in \( \mathbb{Q}(\sqrt{-d}) \), and let \( Q_d \) denote the Bianchi orbifold \( \mathbb{H}^3/\text{PSL}(2, O_d) \).

As is well known \( Q_d \) is a finite volume hyperbolic orbifold with \( h_d \) cusps, where \( h_d \) is the class number of \( \mathbb{Q}(\sqrt{-d}) \) (see [Maclachlan and Reid 03] Chapters 8 and 9 for example). A non-compact finite volume hyperbolic 3-manifold \( X \) is called arithmetic if \( X \) and \( Q_d \) are commensurable, that is to say they share a common finite sheeted cover (see [Maclachlan and Reid 03] Chapter 8 for more on this).

An important class of arithmetic 3-manifolds consists of the congruence manifolds. Recall that a subgroup \( \Gamma \subset \text{PSL}(2, O_d) \) is called a congruence subgroup if there exists an ideal \( I \subset O_d \) so that \( \Gamma \) contains the principal congruence group:

\[
\Gamma(I) = \ker[\text{PSL}(2, O_d) \to \text{PSL}(2, O_d/I)],
\]

where \( \text{PSL}(2, O_d/I) = \text{SL}(2, O_d/I)/[\pm Id] \). The largest ideal \( I \) for which \( \Gamma(I) < \Gamma \) is called the level of \( \Gamma \). For convenience, if \( n \in \mathbb{Z} \), we will denote the principal \( O_d \)-ideal \( < n > \) simply by \( n \). A manifold \( M = \mathbb{H}^3/\Gamma \) is called congruence (resp. principal congruence) if \( \Gamma \supset \Gamma(I) \) (resp. \( \Gamma = \Gamma(I) \)) for some ideal \( I \).

Since the 1980s there has been considerable interest in the question of which Bianchi orbifolds are commensurable to, or stronger still, covered by a link complement \( S^3 \setminus L \). This culminated in the solution of the Cuspidal Cohomology Problem for \( \text{PSL}(2, O_d) \) (see [Vogtmann 85]) which showed that \( Q_d \) can have a cover homeomorphic to a link complement in \( S^3 \) only if

\[
d \in \{1, 2, 3, 5, 6, 7, 11, 15, 19, 23, 31, 39, 47, 71\}.
\]

Furthermore, in [Baker 01] it was shown that for every \( d \) in this list, \( Q_d \) is indeed covered by such a link complement.

Given this finite list of \( d \), and the negative solution to the Congruence Subgroup Property for the Bianchi groups [Serre 70], one can refine these questions further, and ask for the levels for which there are congruence (resp. principal congruence) manifolds that are homeomorphic to a link complement in \( S^3 \). In a previous article [Baker and Reid 14], we started an enumeration of the finitely many levels for which a principal congruence manifold can be homeomorphic to a link complement in \( S^3 \) (see [Baker and Reid 14] or Section 4.1 for the proof of finiteness).

In particular, for \( h_d = 1 \), the main result of that article gave 9 new examples of such principal congruence link groups bringing the known total to 18. One consequence of the results in [Baker and Reid 14] is that when \( h_d = 1 \) (i.e., \( d = 1, 2, 3, 7, 11, 19 \)), \( Q_d \) is covered by a principal congruence link complement (see [Baker and Reid 14]).

In this article, we give a complete enumeration of all the principal congruence link complements in \( S^3 \), together with their levels in the case when the class number of \( \mathbb{Q}(\sqrt{-d}) \) is greater than 1. Our main result is the following:

**Theorem 1.1.** Suppose that \( h_d > 1 \). Then the following list of 16 pairs \((d, I)\) describes all principal congruence manifolds that are homeomorphic to a link complement in \( S^3 \):

\[
(d, I) = \{(1, \mathbb{Z}), (2, \mathbb{Z}), (3, \mathbb{Z}), (5, \mathbb{Z}), (6, \mathbb{Z}), (7, \mathbb{Z}), (11, \mathbb{Z}), (15, \mathbb{Z}), (19, \mathbb{Z}), (23, \mathbb{Z}), (31, \mathbb{Z}), (39, \mathbb{Z}), (47, \mathbb{Z}), (71, \mathbb{Z})\}.
\]
subgroups $\Gamma(I) < \text{PSL}(2, O_d)$ such that $H^3/\Gamma(I)$ is a link complement in $S^3$.

1. $d = 5$: $I = \langle 3, (1 \pm \sqrt{-5}) \rangle$.
2. $d = 15$: $I = \langle 2, (1 \pm \sqrt{-15})/2 \rangle$, $I = \langle 3, (3 \pm \sqrt{-15})/2 \rangle$.

We close the Introduction by outlining the plan of the article. In Section 2 we recall material from [Baker and Reid 14] as well as some other preliminary setup. In Section 3 we show that the principal congruence groups given in Theorem 1.1 are indeed link groups, and in Section 4, we eliminate the other (finitely many) possibilities. Finally, Section 5 contains the proof that there exist (non-principal) congruence link complements when $d = 6$ or 39.

2. Preliminaries and outline of proofs

In Sections 2.1–2.4 we gather facts and background to be used; in Section 2.5 we outline the proof of Theorem 1.1.

2.1. Indices of principal congruence subgroups

We begin by recalling the orders of the groups $\text{PSL}(2, R)$ where $R$ is a finite ring of the form $O_d/I$, with $I \subseteq O_d$ an ideal (see [Dixon 58]). For such an ideal $I$ we have a decomposition into powers of prime ideals. Assuming that $I = P_1^{i_1} \cdots P_m^{i_m}$, we have

$$|\text{PSL}(2, O_d/I)| = \begin{cases} 6, & \text{when } N(I) = 2, \\ N(I)^3 \prod_{P|I} \left(1 - \frac{1}{N(P)}\right)^2, & \text{when } I = \langle 2 \rangle, \\ \frac{N(I)^3}{2} \prod_{P|I} \left(1 - \frac{1}{N(P)}\right)^2, & \text{otherwise} \end{cases}$$

where $N(I) = |O_d/I|$ denotes the norm of the ideal $I$. Since $\text{PSL}(2, O_d)/\Gamma(I) \cong \text{PSL}(2, O_d/I)$, this gives a formula for the index of $\Gamma(I)$ in $\text{PSL}(2, O_d)$.

2.2. Presentations

The proof of Theorem 1.1 makes use of presentations for the Bianchi groups in the cases $d \in \{5, 6, 15, 23, 31, 39, 47, 71\}$. Note that the class numbers for the relevant quadratic imaginary number fields are 2 (when $d = 5, 6, 15$), 3 (when $d = 23, 31$), 4 (when $d = 39$), 5 (when $d = 47$), and 7 (when $d = 71$). The presentations for $d = 5, 6, 15$ are from [Swan 71] while the rest were done by Page using a suite of computer packages he recently developed (see [Page 15]) to study arithmetic Kleinian groups. We maintain the notation of [Swan 71] for $d = 5, 6, 15$ and for the others we use the notation of the presentations provided to us by Page.

$$\text{PSL}(2, O_d) = \langle a, b, c \mid a^2 = b^2 = (ta)^3 = (ab)^2 = (aub^{-1})^2 = acat^{-1}t^{-1} = ubu^{-1}cbt^{-1}ct^{-1}t^{-1} = 1, [t, u] = 1 \rangle,$$
PSL(2, O_b) = < a, t, u, b, c | a^2 = b^2 = (ta)^3 = (ab)^3 = (atu^b)^3 = t^{-1}ctub^{-1}c^{-1}b^{-1} = 1, \\
[t, u] = [a, c] = 1 >, \\
PSL(2, O_{13}) = < a, t, u, c | a^2 = (ta)^3 = uctu^{-1}c^{-1}u^{-1}t^{-1} = 1, [t, u] = [a, c] = 1 >, \\
PSL(2, O_{23}) = < g_1, g_2, g_3, g_4, g_5 | g_3^3 = (g_3g_2)^2 = g_3g_2g_3^{-1}g_3^{-1}g_2^{-1}g_3^{-1}g_1 = g_4^{-1}g_5g_4g_5^{-1}g_4g_5, \\
= 1, [g_1, g_2] = [g_4, g_5] = 1 >, \\
PSL(2, O_{33}) = < g_1, g_2, g_3, g_4, g_5, g_6, g_7 | g_3^3 = (g_3g_5)^2 = (g_5^{-1}g_3^{-1})^2 = (g_5^{-1}g_3^{-1})^3 = (g_7^{-1}g_5^{-1})^3 = g_5^{-1}g_6^{-1}g_4^{-1}g_5^4g_4g_5^{-1}g_6 = g_4^{-1}g_5g_4^{-1}g_7^{-1}g_8^{-1}g_7^{-1}g_6, \\
[g_2, g_1] = [g_3, g_7] = [g_4, g_6] = 1 >, \\
PSL(2, O_{47}) = < g_1, g_2, g_3, g_4, g_5, g_6, g_7 | g_3^3 = (g_3g_5)^2 = (g_5^{-1}g_3^{-1})^2 = (g_5^{-1}g_3^{-1})^3 = (g_7^{-1}g_5^{-1})^3 = g_5^{-1}g_6^{-1}g_4^{-1}g_5^4g_4g_5^{-1}g_6 = g_4^{-1}g_5g_4^{-1}g_7^{-1}g_8^{-1}g_7^{-1}g_6, \\
[g_2, g_1] = [g_3, g_7] = [g_4, g_6] = 1 >, \\
PSL(2, O_{71}) = < g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8, g_9 | g_3^3 = (g_3g_7)^2 = g_1^{-1}g_3^{-1}g_7^{-1} = g_3g_4g_5^{-1}, \\
= g_5^{-1}g_6^{-1}g_4^{-1}g_5^4g_4g_5^{-1}g_6 = g_4^{-1}g_5g_4^{-1}g_7^{-1}g_8^{-1}g_7^{-1}g_6, \\
[g_2, g_1] = [g_3, g_7] = [g_4, g_6] = 1 >, \\
PSL(2, O_{71}) = < g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8, g_9 | g_3^3 = (g_3g_7)^2 = g_1^{-1}g_3^{-1}g_7^{-1} = g_3g_4g_5^{-1}, \\
= g_5^{-1}g_6^{-1}g_4^{-1}g_5^4g_4g_5^{-1}g_6 = g_4^{-1}g_5g_4^{-1}g_7^{-1}g_8^{-1}g_7^{-1}g_6, \\
[g_2, g_1] = [g_3, g_7] = [g_4, g_6] = 1 >.

Throughout the article we use $\omega_d$ as follows:

$$\omega_d = \sqrt{-5}, \sqrt{-6}, \frac{1 + \sqrt{-15}}{2}, \frac{1 + \sqrt{-23}}{2}, \frac{1 + \sqrt{-31}}{2},$$

$$\frac{1 + \sqrt{-39}}{2}, \frac{1 + \sqrt{-47}}{2}, \frac{1 + \sqrt{-71}}{2}.$$

When $d = 5, 6, 15, t = \frac{1}{10}$ and $u = \left\{ \begin{array}{ll} \omega_5 & \text{if } 5 \text{ is even} \\
\omega_6 & \text{if } 6 \text{ is even} \\
\omega_{15} & \text{if } 15 \text{ is even} \end{array} \right.$

Let $\Gamma \leq PSL(2, O_d)$ be a finite index subgroup. For convenience in what follows, we assume that $d \neq 1, 3$. Then

- A cusp, $[c]$, of $\Gamma$ is a $\Gamma$-orbit of points in $\mathbb{P}^1(\mathbb{Q}(\sqrt{-d}))$.
- A peripheral subgroup of $\Gamma$ for $[c]$ is a maximal parabolic subgroup, $P_c \leq \Gamma$, fixing $x \in [c]$. Note that if $y \in [c]$, then $P_y$ and $P_c$ are conjugate; hence a peripheral subgroup for $[c]$ is determined up to conjugacy.

A set of peripheral subgroups for $\Gamma$ is the choice of one peripheral subgroup for each cusp of $\Gamma$.

We will use the term cusp to mean $[c]$, a choice of point $x$ in $[c]$, as well as the end of $\mathbb{H}^3/\Gamma$ corresponding to $[c]$. Which one is meant should be clear from the context. Note that since $d \neq 1, 3$, each peripheral subgroup is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

### 2.4 Cusp representatives

As noted in the Introduction, $Q_d$ has $h_d$ cusps (or equivalently $PSL(2, O_d)$ has $h_d$ conjugacy classes of peripheral subgroups). In the setting of this article, $h_d > 1$ and so the orbifold $Q_d$ will have more than one cusp.

In the cases of $d = 5, 6, 15$, the peripheral subgroup of $PSL(2, O_d)$ that fixes $\infty$ is given by $< t, u >$ (in terms of the generators given in Section 2.2). Below we give a set of peripheral subgroups for each of the Bianchi groups in Section 2.2. We describe a choice of peripheral subgroup and cusp by $(x, P_x)$ (following the notation described in Section 2.3).

$$d = 5 : (\infty, < t, u >), \left( \frac{1 - \sqrt{-5}}{2}, < tb, tu^{-1}ct^{-1} > \right).$$

$$d = 6 : (\infty, < t, u >), \left( \frac{-\sqrt{-6}}{2}, < tb, cu > \right).$$

$$d = 15 : (\infty, < t, u >), \left( \frac{1 + \sqrt{-15}}{4}, < uca, c^{-1}au^{-1}c^{-1}u^{-1}ta > \right).$$
\[ d = 23 : (\infty, < g_1, g_2 >), \left( \frac{1 - \sqrt{-23}}{4}, < g_4, g_5 > \right), \]
\[ \left( -1 - \frac{\sqrt{-23}}{4}, < g_8 g_8 g_8, g_2^{-1} g_9 g_9 g_7 > \right). \]
\[ d = 31 : (\infty, < g_1, g_3 >), \left( \frac{1 - \sqrt{-31}}{4}, < g_4, g_5 > \right), \]
\[ \left( -1 - \frac{\sqrt{-31}}{4}, < g_1 g_5, g_3^{-1} g_2 g_8 g_4^{-1} g_2 g_5 > \right). \]
\[ d = 39 : (\infty, < g_1, g_2 >), \left( \frac{1 - \sqrt{-39}}{5}, < g_4, g_6 > \right), \]
\[ \left( 3 - \frac{\sqrt{-39}}{6}, < g_2 g_7, g_4 g_1^{-1} g_4 g_5 > \right). \]
\[ d = 47 : (\infty, < g_2, g_3 >), \left( \frac{1 - \sqrt{-47}}{4}, < g_5, g_7 > \right), \]
\[ \left( 3 - \frac{\sqrt{-47}}{4}, < g_2 g_7, g_4 g_1^{-1} g_4 g_5 > \right). \]
\[ d = 71 : (\infty, < g_7, g_1^{-1} g_3 >), \left( \frac{1 - \sqrt{-71}}{4}, < g_2, g_8 g_9 g_8 g_5^{-1} g_5^{-1} g_5 g_8 g_8 g_5^{-1} g_5 > \right), \]
\[ \left( 1 + \frac{\sqrt{-71}}{6}, < g_3, g_6 g_9^{-1} g_8 > \right). \]
\[ \left( -1 + \frac{\sqrt{-71}}{4}, < g_7 g_2, g_8 g_6 g_8 g_5^{-1} g_5 g_7^{-1} > \right). \]
\[ \left( 3 + \frac{\sqrt{-71}}{8}, < g_8 g_8^{-1} g_6^{-1} g_5^{-1} g_2 g_8^{-1} > \right). \]
\[ \left( 3 - \frac{\sqrt{-71}}{8}, < p_1, p_2 > \right) \]

where \( p_1 = g_8 g_8 g_8 g_8^{-1} g_5^{-1} g_3 g_3 g_9^{-1} g_3 g_8^{-1} g_4^{-1} \) and \( p_2 = g_8 g_8 g_8 g_8^{-1} g_5^{-1} g_3 g_3 g_9^{-1} g_3 g_8^{-1} g_4^{-1} g_4^{-1} \).

We remark that finding these peripheral subgroups and expressing them in terms of the given generators was a highly nontrivial exercise. One can check that the \( h_d \) cusps correspond to different elements of the ideal class group of \( \mathbb{Q}(\sqrt{-d}) \), hence they are inequivalent. Also, the generators of each of the above peripheral subgroups commute (using the relations in the corresponding Bianchi group, or by direct matrix calculation using the generators given in Appendix A), and these generators correspond to primitive parabolic matrices with the correct fixed point.

### 2.5. Outline of the proof of Theorem 1.1

We conclude this section with an outline of the proof of Theorem 1.1. The methods used are those in [Baker and Reid 14] adapted to the case \( h_d > 1 \).

We first note that there are only finitely many groups \( \Gamma(I) \) that can be link groups. Indeed, as mentioned above, the cuspidal cohomology of \( \text{PSL}(2, \mathbb{O}_d) \) reduces consideration to the eight Bianchi groups given in Section 2.2.

Furthermore, as explained in Section 4.1, if \( \Gamma(I) \) is a link group then \( N(I) < 39 \).

To establish the 16 principal congruence link groups in Theorem 1.1, and eliminate the remaining \( \Gamma(I) \), we use the following properties of hyperbolic link complements, which we state for the case of principal congruence groups.

1. If \( H^3/\Gamma(I) \cong S^3 \setminus L \), then \( \Gamma(I) \) is generated by parabolic elements.
2. \( H^3/\Gamma(I) \cong S^3 \setminus L \) if and only if \( \Gamma(I) \) can be trivialized by setting one parabolic from each cusp of \( \Gamma(I) \) equal to 1.

We briefly comment on these two items. The first is well known, following from the fact that a link group is generated by meridians, and such elements are represented by parabolic elements under the faithful discrete representation. For the second, if \( H^3/\Gamma(I) \cong S^3 \setminus L \), then for each component \( L_i \) of \( L \), there is a curve \( x_i \) (a meridian curve for \( L_i \)) so that Dehn filling each \( L_i \) along \( x_i \) results in \( S^3 \). Thus, trivializing the corresponding parabolic elements \( [x_i] \) in \( \Gamma(I) \) gives the trivial group. Conversely, given Perelman’s resolution of the Geometrization Conjecture, if \( \Gamma(I) \) can be trivialized by setting one parabolic from each cusp of \( \Gamma(I) \) equal to 1, then \( H^3/\Gamma(I) \) is homeomorphic to a link complement in \( S^3 \).

Given this, our method is as follows:

**Step 1:** Check whether \( \Gamma(I) \) is generated by parabolic elements.

Let \( \Gamma(I) \subset \text{PSL}(2, \mathbb{O}_d) \), and let \( P_i \) be the peripheral subgroup of \( \text{PSL}(2, \mathbb{O}_d) \) fixing the cusp \( c_i \) for \( i = 1, \ldots, h_d \) as given in Section 2.4. Now, comparing with the discussion in [Baker and Reid 14], \( P_i(I) = P_i \cap \Gamma(I) \) is the peripheral subgroup of \( \Gamma(I) \) fixing \( c_i \).

Let \( N_d(I) \) denote the normal closure in \( \text{PSL}(2, \mathbb{O}_d) \) of \( \{ P_1(I), \ldots, P_{h_d}(I) \} \). Note that \( N_d(I) < \Gamma(I) \) since \( \Gamma(I) \) is a normal subgroup.
of PSL(2, Od). It is clear that Γ(I) is generated by parabolic elements if and only if Nd(I) = Γ(I).

We use Magma [Bosma 97] to test whether Γ(I) = Nd(I). In Section 3 we show that the groups in Theorem 1.1 are generated by parabolics, while Section 4 is devoted to showing that the remaining Γ(I) are not and hence cannot be link groups.

Step 2: Find parabolic elements in Γ(I), one for each cusp, so that trivializing these elements trivializes the group.

First, we obtain a set of peripheral subgroups for Γ(I) as follows. Let {P1, . . . , Pn} be a set of peripheral subgroups for PSL(2, Od). Then, as in Step 1, set P(I) = P1 ∩ Γ(I). Note that S = {P1, . . . , Pn} is a partial set of peripheral subgroups for Γ(I). One obtains a full set of peripheral subgroups for Γ(I) by adding certain conjugates of the P1(I) to the partial set S as explained in Section 3.

Next, given a (full) set of peripheral subgroups for Γ(I), we choose one parabolic from each of these peripheral subgroups and use Magma to check that trivializing these elements trivializes Γ(I). This choice of parabolics involves trial and error. However, in all cases these parabolics are linear combinations of small powers of the generators of the P1(I) and their conjugates.

3. Proof of Theorem 1.1: Determining the principal congruence link groups

We now prove, as described in Section 2.5, that the groups given in Theorem 1.1 do correspond to link groups. We use the notation (d, I) to indicate the Bianchi group and level given in Theorem 1.1. The ideals involved are of norm 2 (5 groups), 3 (4 groups), 4 (4 groups), 5 (2 groups), and 6 (1 group). Note that if Γ(I) is determined to be a link group in S3, then Γ(I) is also. Therefore, in what follows we simply refer to one of the complex conjugate pair.

3.1. Normal closures

Doing Step 1 above, we check that Γ(I) is generated by parabolic elements. Since Nd(I) < Γ(I), it suffices to show that Nd(I) and Γ(I) have the same index in PSL(2, Od). The index of Γ(I) is given by the formula in Section 2.1 while the index of Nd(I) is calculated using Magma.

We now do this explicitly for the group (d, I) = (15, < 2, ω15>). Recall from Section 2.4 that the peripheral subgroups for PSL(2, O15) are given by P1 = < t, u >, P2 = < uca, (c−1au−1c−1u−1ta)>. In terms of the matrices for the presentation of PSL(2, O15), we have t = (1 10 1), u = (1 0 2ω15), uca = (1 + 2ω15 4 1 −2ω15), and (c−1au−1c−1u−1ta) = (−3 + 4ω15 8 −ω15 7 +ω15 1 −4ω15). Thus we obtain the peripheral subgroups P1(I) = P1 ∩ Γ(I) by reducing these matrices modulo I which gives P1(I) = < t2, u >, P2(I) = < uca, (c−1au−1c−1u−1ta)2 >.

Now from Section 2.1 we have [PSL(2, Od) : Γ(I)] = 6 and Magma gives [PSL(2, Od) : Nd(I)] = 6, hence Γ(I) = Nd(I) so that Γ(I) is generated by parabolics.

In our Magma routines below, G = PSL(2, Od), H = < P1(I), . . . , Pn(I) >, and N = Nd(I) = < H > (the normal closure of H).

G<a,c,t,u>:=Group<a,c,t,u|a^2, (t*a)^3, u*c*u*a*t*u^-1*c^-1*u^-1*a*t^-1, (t,u), (a,c)>;
H:=sub<G|t^2,u, (c^-1*a*u^-1*c^-1*u^-1*t*a)^2, u*c*a>;
N:=NormalClosure(G,H);

print Index(G,N);

6

The remaining 15 groups are done in exactly the same way. The set of peripheral subgroups for each Γ(I) can be read off from H in its Magma routine.

3.2. Testing for link groups

Next, we implement Step 2: Find parabolic elements from Γ(I), one for each cusp of Γ(I), and use Magma to show that trivializing these elements trivializes Γ(I). We start by obtaining a set of peripheral subgroups for Γ(I) and then choose one parabolic from each subgroup. The method depends on the norm of the ideal I.

3.2.1. Cases (d,I), Nd(I) = 2

Here the Bianchi groups correspond to d = 15, 23, 31, 47, 71. Since Nd(I) = 2, we have PSL(2, Od/I) = S3. In order to find a set of peripheral subgroups for Γ(I), we use the following sequence of regular covers:

H3/Γ(I) → 3 H3/Γ1 → 2 Od

where Γ1 = < Γ(I), (1 0 1) >.

Since each peripheral subgroup P1 of PSL(2, Od) maps to a subgroup of order 2 in PSL(2, Od/I), it follows that each cusp of Od is covered by 3 cusps of H3/Γ(I) each with covering degree 2. Now Γ1/Γ(I) ≅ Z/3Z < PSL(2, Od/I), hence each cusp of Od is covered by 1 cusp of H3/Γ1 with covering degree 2. Thus a set of peripheral subgroups for Γ1 is given by {P1(I), . . . , Pn(I)} Conjugating this set by the elements {Id, (1 0 1), (1 −1)} gives a set of peripheral subgroups for Γ(I).
In what follows we give the Magma routines in the cases of $d = 15, 71$ and one of the prime ideals of norm 2 (since as remarked upon above the same will hold for the complex conjugate ideal). The remaining values of $d = 23, 31, 47$ and a prime ideal of norm 2 are handled similarly. Magma routines for these (and other calculations done in this section) are available from the authors upon request.

**The case (15, < 2, $\omega_{15}$ >)**

Let $I = < 2, \omega_{15} >$. We analyze this $\Gamma(I)$; the other four cases involving $N(I) = 2$ are dealt with in exactly the same way.

In Section 3.1, we computed the peripheral subgroups $P_1(I)$ and $P_2(I)$ and showed that $\Gamma(I)$ was generated by parabolics. This is given again in the following Magma routine.

Since $Q_{15}$ has 2 cusps, the manifold $\mathbb{H}^3/\Gamma(I)$ has 6 cusps. Recall that $P_1(I) = < t^2, u >$, $P_2(I) = < uca, (c^{-1}au^{-1}c^{-1}u^{-1}ta)^2 >$. Since $\left( \frac{-1}{0} \right) = ta$, conjugating $P_1(I)$ and $P_2(I)$ by the elements $[id, ta, (ta)^2]$ gives a set of six peripheral subgroups for $\Gamma(I)$. Now, we choose one element from each of these six peripheral subgroups:

$$\{ t^2, (ta)u(ta)^{-1}, (ta)^2u(ta)^{-2}, uca, (ta)uca(ta)^{-1}, (ta)^2(c^{-1}au^{-1}c^{-1}u^{-1}ta)^2(ta)^{-2} \}$$

In the Magma routine, $Q$ denotes the quotient of $\Gamma(I)$ by the normal closure of these six parabolics, and Magma calculates that $Q = < 1 >$ which shows that $\Gamma(I)$ is trivialized by setting these six elements equal to 1. Thus $\Gamma(< 2, \omega_{15} >)$ is indeed a 6-component link group.

```
G<a,c,t,u>:=Group<a,c,t,u|a^2,(t*a)^3,u*c*u*a*t*a*u^(-1)*c^(-1)*u^(-1)*a^*
    t^(-1), (t,u), (a,c)>;
H:=sub<G|t^2,u, (c^(-1)*a*u^(-1)*c^(-1)*
    u^(-1)*t*a)^2, u*c*a>;
N:=NormalClosure(G,H);
```

```
print Index(G,N);
6
\`
Q:=quo<N|t^2, (t*a)*u*(t*a)^(-1), (t*a)^2*u*(t*a)^(-2),
    u*c*a, (t*a)*u*c*a*(t*a)^(-1),
    (t*a)^2*(c^(-1)*a*u^(-1)*c^(-1)*
    u^(-1)*t*a)^2*(t*a)^(-2)>;
```

```
print Order(Q);
1
\`
```

**The case (71, < 2, $\omega_{71}$ >)**

Let $I = < 2, \omega_{71} >$. The following Magma routine shows that $\mathbb{H}^3/\Gamma(I)$ is a 21-component link complement in $S^3$. Note that $Q_{71}$ has 7 cusps and that $\left( \frac{-1}{0} \right) = g_8^{-1}$. The seven peripheral subgroups $R(I)$ can be read off from $H$ and $Q$ is $\Gamma(I)$ modulo the 21 parabolic elements to be trivialized.

```
G<g1,g2,g3,g4,g5,g6,g7,g8,
g9>: =Group<g1,g2,g3,g4,g5,g6,g7,g8,
g9|g8^3, (g8^(-1), g4), (g8*g7^(-1))^2,
g1^(-1)*g3*g7*g3^(-1)*g1*g7^(-1), g6*g3*
g6^(-1)*g7*g9^(-1)*g3^(-1)*g9*g7^(-1), g7^(-1)*
g6*g3*g6^(-1)*g5^(-1)*g2*g7*g5*g6^(-1)*
g6^(-1)*g2^(-1), g8*g7^(-1)*g1*g5*g6^(-1)*
g1*g5*g7*g8^(-1)*g5^(-1)*g1^(-1)*g3*g6^(-1)*
g5^(-1)*g1^(-1), g4^(-1)*g7^(-1)*g5^(-1)*g2^*
g1^(-1)*g3*g7*g9^(-1)*g4^(-1)*g7^(-1)*g5^(-1)*
g7^(-1)*g9^(-1)*g7^(-1)*g3^(-1)*g1,
g2*g6^(-1)*g1*g5*g7*g8^(-1)*g5^(-1)*g1^(-1)*g3^*
g6^(-1)*g7*g8^(-1)*g5^(-1)*g2^(-1)*g5*g7^*
g8^(-1)*g5^(-1)*g1^(-1)*g7*g8^(-1)*g1*g5*g6^*
g3^(-1)*g6^(-1)>;
```

```
H:=sub<G|g7^2, g7*g1^(-1)*g3, g2^2, g6^*
g1*g5*g7*g8^(-1)*g5^(-1)*g1^(-1)*g3*g6^(-1)*
g7*g8^(-1)*g5^(-1), g3^2, g6*g7*g7^*
g9^(-1), g7*g2, (g6*g3^(-1)*g5^(-1)*g5^(-1)*
g7^(-1))^2, (g7*g9^(-1))^2, g3^(-1)*g1^*g5^*
g8*g7^(-1)*g5^(-1)*g1^(-1), (g3*g9^(-1))^2,
g4^(-1)*g7^(-1)*g5^(-1)*g2^(-1)*g7*g1^(-1),
(g4^(-1)*g7^(-1)*g2^(-1)*g5^(-1)*g7^(-1)*g9^(-1)*
g8^(-1)*g4^(-1))^2, g6^(-1)*g1^(-1)*g5^*
g7^(-1)*g5^(-1)*g1^(-1)*g6^(-1)*g8^(-1)*
g4^(-1)>;
```

```
N:=NormalClosure(G,H);
print Index(G,N);
6
```

```
Q:=quo<N|g7*g1^(-1)*g3, g8*g7^2*g8^(-1),
g8^(-1)*g7^2*g8, g6*g1*g5*g7*g8^(-1)*
g5^(-1)*g1^(-1)*g3*g6^(-1)*g7*g8^(-1)*g5^(-1),
g8^(-1)*g5^(-1)*g1^(-1)*g3*g6^(-1)*
g3^(-1)*g1^(-1)*g7*g8^(-1)*g5^(-1)*g2^*
g6^(-1)*g7*g8^(-1)*g5^(-1)*g2^(-1)*g5^*
g6^(-1)*g7*g8^(-1)*g5^(-1)*g2^(-1)*g5^*
g7^(-1)*g5^(-1)*g2^(-1)*g5^*g6^(-1)*
g8^(-2), g7*g6^(-1)*
g7*g9^(-1), g8*(g3*g6^(-1)*g7*g9^(-1)) *
g8^(-1), g8^2*(g3^(-1)*g7^(-1)*g9^(-1)*
g8^(-2), g7*g2, g8*(g7*g2)*g8^(-1), g8^2*```
Here the Bianchi groups involved correspond to $I \in \{\bar{1}, \bar{2}, \bar{3}\}$. The remaining three cases are done in the same way. Since $Q_3$ is covered by 1 cusps of $H^3/\Gamma(I)$ each with covering degree 3. Also, each cusps of $Q_3$ is covered by 1 cusps of $H^3/\Gamma(I)$ with covering degree 3. Thus the set of peripheral subgroups of $H^3/\Gamma(I)$ is given by $\{\bar{P}_1(I), \ldots, \bar{P}_8(I)\}$. Conjugating this set by four elements of $\Gamma(I)$ that correspond to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ gives a set of peripheral subgroups for $\Gamma(I)$.

**The case (15, $< 3, 1 + \omega_{15}>$)**

Let $I = < 3, 1 + \omega_{15}>$. We do this case in detail; the remaining three cases are done in the same way. Since $Q_{21}$ has 2 cusps, $H^3/\Gamma(I)$ has 8 cusps. Reducing the matrix presentations of the $P_i$ for $Q_{21}$ given in Section 3.1 by the ideal $< 3, 1 + \omega_{15}>$, we obtain $P_1(I) = < t^3, tu >$, $P_3(I) = < (ua)^3, (c^{-1}au^{-1}c^{-1}u^{-1}ta) >$. Setting $\Gamma_1 = < \Gamma(I), a, h >$, for $h = ata^{-1}$, Magma checks that $\Gamma_1/\Gamma(I) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ so that conjugating $P_1(I)$ and $P_2(I)$ by $< id, a, h, ah >$ gives a set of eight peripheral subgroups for $\Gamma(I)$. The following set contains one element from each of these eight peripheral subgroups:

\[
\begin{align*}
&\{t^{-2}u, ata^{-1}, ht^{-2}uh^{-1}, (ha)tu(ha)^{-1}, \\
&\quad (c^{-1}au^{-1}c^{-1}u^{-1}ta)(uca)^3, a(c^{-1}au^{-1}c^{-1}u^{-1}ta)a^{-1}, \\
&\quad h(c^{-1}au^{-1}c^{-1}u^{-1}ta)h^{-1}, ha(c^{-1}au^{-1}c^{-1}u^{-1}ta)(ha)^{-1}\}
\end{align*}
\]

As above, Q denotes the quotient of $\Gamma(I)$ by the normal closure of these eight parabolics, and Magma calculates that $Q = < 1 >$; hence $\Gamma(I)$ is trivialized by setting these eight elements equal to 1. Thus $\Gamma(< 3, 1 + \omega_{15}>)$ is an 8-component link group.

**G**<a,c,t,u> := Group<a,c,t,u | (t,u),
(a,c),a^2,(t*a)^3, u*c*u*a*t*u^(-1)*c^(-1)*u^(-1)*t*a>; 
H := sub<G | N,a,h>;
print Order(Q); 1

\[
\begin{align*}
&\text{print Index(G,N); 12} \\
&\text{h := a*t*a*t^-1*a; A := sub<G | N,a,h>;} \\
&\text{print Index(A,N); 4} \\
&\text{Q := quo< N | t^-2*u, a*t*u*a, h*t^-2*u}\times\mathbb{Z}/2\mathbb{Z} < A_4. \\
\end{align*}
\]

Since each peripheral subgroup $P_i$ of $SL(2, O_d/I)$ maps to a subgroup of order 3 in $SL(2, O_d/I)$, it follows that each cusps of $Q_4$ is covered by 4 cusps of $H^3/\Gamma(I)$ each with covering degree 3. Also, each cusps of $Q_4$ is covered by 1 cusps of $H^3/\Gamma(I)$ with covering degree 3. Thus a set of peripheral subgroups for $H^3/\Gamma(I)$ is given by $\{P_1(I), \ldots, P_8(I)\}$. Conjugating this set by four elements of $\Gamma_1$ that correspond to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ gives a set of peripheral subgroups for $\Gamma_1(I)$.
The cases of \( d = 23, 47 \) and a prime ideal of norm 3 are handled similarly.

### 3.2.3. Cases \((d,i), N(I) = 4\)

Here the Bianchi groups involved correspond to \( d = 15, 23, 31, 47 \). In each case, \( I = P^2 \), where \( N(P) = 2 \). Hence \( \text{PSL}(2, O_k) : \Gamma(I) \) has 24. Each cusp of \( Q_d \) is covered by 6 cusps of \( H^3/\Gamma(I) \), each with covering degree 4. Since \( \Gamma(P)/\Gamma(I) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) and \( \Gamma(P) = < \Gamma(I), (12)^3, (01) > \), we exploit the intermediate covers

\[
H^3/\Gamma(I) \xrightarrow{2} H^3/\Gamma_1 \xrightarrow{2} H^3/\Gamma(P),
\]

where each covering has degree 2 and \( \Gamma_1 = < \Gamma(I), (12) >. \)

In the cases \( d = 15, 23 \), we show that \( H^3/\Gamma_1 \) is a link complement in \( \mathbb{R}P^3 \), from which it follows that \( H^3/\Gamma(I) \) is a link complement in \( S^3 \). For the cases \( d = 31, 47 \) we show that \( H^3/\Gamma_1(I) \) is a link complement in \( S^3 \).

#### The case \((15, < 4, \omega_{15} >)\)

Here \( P = < 2, \omega_{15} > \) so that \( I = < 4, \omega_{15} > = P^2 \). Note also that \( \{12\} = at^2a^{-1} \).

As shown in Section 3.2.1, \( H^3/\Gamma(P) \) has 6 cusps, each cusp of \( Q_{15} \) being covered by 3 cusps. Note that \( H^3/\Gamma_1 \) has 12 cusps and the intermediate cover \( H^3/\Gamma_1 \) has 8 cusps.

In the following Magma routine, \( P_1(I) = < \Gamma, u > \) and \( P_2(I) = < \Gamma, c^{-1}a^{-1}c^{-1}u^{-1}ta > \) are read off from \( H, A = \Gamma_1 \), and \( Q \) equals \( \Gamma_1 \) modulo 8 parabolics, one from each peripheral subgroup of \( \Gamma_1 \). Since Magma gives \( Q \cong \mathbb{Z}/2\mathbb{Z} \), it follows that \( H^3/\Gamma_1 \) is an 8-component link in \( \mathbb{R}P^3 \). Finally, since the eight peripheral elements trivialized are also in \( \Gamma(I) \), we have that \( H^3/\Gamma(I) \) is a 12-component link complement in \( S^3 \).

\[
G_a,t,u,c := \text{Group}(a,t,u,c | a^2, (t*a)^3,u*c*u*a*t*u-1*c^1*u^1-1 \}
\]

\[
a^1-1*t^1, (t,u),(a,c) >;
\]

\[
H := \text{sub}(G,t^4,u, (c^1-1*a*u^1-1*c^1-1 \}
\]

\[ u^1-1*t^1*a)^4, u*c*a); \]

\[
N := \text{NormalClosure}(G,H);
\]

\[
\text{print Index}(G,N);
\]

\[
24
\]

\[
A := \text{sub}(G,N, a*t^2*a); \]

\[
\text{print Index}(G,A);
\]

\[
12
\]

\[
Q := \text{quo}(A,t^4,u*a*(t*a)^1-1, (t*a)^1-1 \}
\]

\[ t^4*u*t*a, u*c*a, (t*a)*u*c*a*(t*a)^1-1, \]

\[ (t*a)^1-1*u*c*a*(t*a), t^2*(t*a)*t^4*u*a \}
\]

\[
(t*a)^1-1*t^2, t^2*u*c*a*t^2 \> \};
\]

\[
\text{print Order}(Q);
\]

\[
2
\]

The following Magma routine shows that \( \Gamma^3/\Gamma(I) \) is a 30-component link complement in \( S^3 \). Note that \( Q_{15} \) has 5 cusps, and that \( (12) = g_1, (12)^3 = g_2, (12)^4 = h := g_1^{-1}g_2^{-1} \). The peripheral subgroups \( P(I) \) can be read off from \( H \), and one obtains a set of peripheral subgroups for \( \Gamma(I) \) by conjugating the \( P(I) \) first by the elements \( [Id, g_1, g_1^3] \) and then by those of \( [Id, g_1^3, h, g_2h] \). Here some care must be taken to remove 30 redundant groups, leaving a set of 30 peripheral subgroups for \( \Gamma(I) \).

\( Q \) is the quotient of \( \Gamma(I) \) by the normal closure of 30 parabolic elements, 1 from each of the peripheral subgroups. Magma calculates that \( Q = < 1 > \), hence \( \Gamma^3/\Gamma(I) \) is a 30-component link complement in \( S^3 \).

\[
G := \text{Group}(g_1, g_2, g_3, g_4, g_5, g_6, g_7, \{g_1 \}
\]

\[ 3, (g_3, g_2), (g_2^{-1}*g_1)^2, (g_5, g_7), g_2^{-1}*g_1*g_6 \}
\]

\[ g_1^{-1}*g_2*g_6^{-1}1, g_2*g_1^{-1}*g_4^{-1}g_5 \}
\]

\[ g_3^{-1}*g_6^{-1}1*g_4^2*g_3*g_5^{-1}1, g_7^{-1}1 \}
\]

\[ g_2^{-1}g_5^{-1}1*g_4^1*g_4^{-1}g_1^{-1}g_2^{-1}g_7^1 \}
\]

\[ g_1^{-1}g_4^{-1}1^1g_5^3, g_3^1g_5^{-1}1^1g_4^1g_4^{-1}1 \}
\]

\[ g_2^1g_5^{-1}1^1g_2^{-1}1^1g_1^{-1}g_4^1 \} >;
\]

\[
H := \text{sub}(G,g_2^4, g_7^2, g_2^{-1}g_3, g_5^4, g_5^1g_7, g_2^1g_7, \}
\]

\[ g_4^1g_1^{-1}g_4^{-1}1^1g_5^1, (g_5^1g_2^{-1}g_4^{-1} \}
\]

\[ g_4^{-1}1^2, (g_6^1g_2^{-1}g_4^{-1}1^1g_5^1) \}
\]

\[ g_2^{-1}1^1g_4^{-1}1^1, (g_3^1g_5^{-1}1^1g_4^1g_1^{-1}g_2^{-1} \}
\]

\[ g_6^{-1}1^1g_1^{-1}g_4^1) > \);
If $\Gamma_1 = \langle \Gamma, a, b >$ and $\Gamma_2 = \langle \Gamma, r >$, we obtain a set of peripheral subgroups for $H^3/\Gamma(I)$ by conjugating $\{P_1(I), \ldots, P_d(I)\}$ first by the elements $[Id, r, r^2]$, and then further conjugating by $[Id, a, b, ab]$. 

**The case (15, < 5, 2 + \omega_{15} >)**

The following Magma routine shows that $H^3/\Gamma(I)$ is a 24-component link complement in $S^3$. Note that $H^3/\Gamma(I)$ has 24 cusps. Reducing the matrix presentations for $P_1$ and $P_2$ modulo $I$ gives $P_1(I) = < i^2, t^u >$ and $P_2 = < (ua)^5, (c^2u^{-1}1c^{-1}u^{-1}ta) >$. The group $\Gamma_1$ (resp. $\Gamma_2$) is given by $A$ (resp. $B$). A set of peripheral subgroups for $\Gamma(I)$ is calculated as described above. Magma then calculates that quotient of $\Gamma(I)$ by the normal closure of the 24 peripheral elements given below trivializes the group.

We use the following sequence of covers of $Q_d$ to find a set of peripheral subgroups for $\Gamma(I)$:

\[ H^3/\Gamma(I) \to H^3/\Gamma_1 \to H^3/\Gamma_2 \to Q_d \]

where $\Gamma_1/\Gamma(I) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\Gamma_2/\Gamma(I) \cong A_4$. It follows that $H^3/\Gamma_1$ has $h_d$ cusps, $H^3/\Gamma_1$ has $3h_d$ cusps, and $H^3/\Gamma(I)$ has $12h_d$ cusps.

The cases $d = 23, 31$ and an ideal of norm 4 which is the square of a prime ideal of norm 2 are handled similarly. Note that the principal ideal $< 2 >$ also has norm 4. However we show in Section 4 that $\Gamma(2)$ is not a link group if $h_d > 1$.
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3.2.5. Case (d,l), \( N(I) = 6 \)

Here the Bianchi group corresponds to \( d = 15 \) and \( I = < 6, -2 + \omega_{15} > \). Since \( \Gamma(I) = A_4 \), and \( \Gamma(P) = < \Gamma(I), (ta) \rangle \), we obtain a set of peripheral subgroups for \( \Gamma(I) \) by first conjugating \( \{ P_1, P_2 \} \) by the elements \( \{ Id, ta, (ta)^2 \} \) and then further conjugating by the elements \( \{ Id, r, s \} \). In the following Magma routine, \( A = \Gamma(P), (\frac{1}{6}) = ta, r = t^2at^{-2}a, \) and \( s = at^{-2}a^2 \).

\[
\begin{align*}
P &= < 2, \omega_{15} >. & \\
\text{Note that } N(I) &= 6 \\
\text{and so using Section 2.1 we deduce that } & \text{PSL}(2, O_{15}/I) : \Gamma(I) = 72. \text{Indeed, in this case we have } \text{PSL}(2, O_{15}/I) \cong \text{PSL}(2, \mathbb{Z}/2\mathbb{Z}) \times \text{PSL}(2, \mathbb{Z}/3\mathbb{Z}) \cong S_3 \times A_4. \text{Now } P_1 \text{ and } P_2 \text{ both have order 6 in } \text{PSL}(2, O_{15}/I), \text{so that each cusp of } Q_{15} \text{ is covered by 12 cusps of } \mathbf{H}^3/\Gamma(I), \text{each with covering degree 6. We will show that } \mathbf{H}^3/\Gamma(I) \text{ is a 24-component link group.}
\end{align*}
\]

We use the following sequence of covers of \( Q_{15} \):

\[
\mathbf{H}^3/\Gamma(I) \overset{12}{\longrightarrow} \mathbf{H}^3/\Gamma(P) \overset{6}{\longrightarrow} Q_{15}
\]

where \( \mathcal{P} = < 2, \omega_{15} >. \text{ Since } \Gamma(P)/\Gamma(I) \cong A_4, \text{ and } \Gamma(P) = < \Gamma(I), (ta) \rangle, r, s >, \text{ we obtain a set of peripheral subgroups for } \Gamma(I) \text{ by first conjugating } \{ P_1, P_2 \} \text{ by the elements } \{ Id, ta, (ta)^2 \} \text{ and then further conjugating by the elements } \{ Id, r, s \}. \text{ In the following Magma routine, } A = \Gamma(P), (\frac{1}{6}) = ta, r = t^2at^{-2}a, \text{ and } s = at^{-2}a^2.

\[
\begin{align*}
G &= \langle a, c, t, u \rangle : \langle a, c, t, u \rangle (ta), (a, c), (a, t)^2, (t^a)^3, u^c * u * a * t * u^{-1} * \\
\text{H} &= \langle 2 * a, t^2 * a, t^{-2} * a \rangle; \text{print Index}(G, N); 72
\end{align*}
\]
\begin{verbatim}s := (a*t^2 - 2*a*t^2); A := sub<G[N, t*a, r, s]; print Index(A, N);\end{verbatim}

4. **Proof of Theorem 1.1: Excluding the remaining levels**

In this section we eliminate (for the eight values of $d$ in Section 2.2) the groups $\Gamma(I)$ that do not appear in $\text{Th}(\Gamma(I))$ using several techniques. In Section 4.1 we establish a bound on the norm of $I$ for which $\Gamma(I)$ can be a link complement. In Section 4.2 we use results on cuspidal cohomology to quickly eliminate certain $\Gamma(I)$. Finally, in Section 4.3, we follow the procedure described in Step 1 of Section 2.5 using Magma to show that the remaining $\Gamma(I)$ are not generated by parabolics hence are not link groups.

We remind the reader that for an ideal $I$ for which $I \neq \mathbb{T}$, it suffices to exclude $\Gamma(I)$ as this will automatically exclude $\Gamma(\mathbb{T})$.

### 4.1. Systole bounds

First, we remind the reader that there are only finitely many levels that can give principal congruence link complements in $S^3$ (see [Baker and Reid 14] Proposition 2.3 for example). We recall a proof of that here, and exhibit an explicit bound for the norm of the ideal.

From [Adams and Reid 00] we know that if $\mathbb{H}^3/\Gamma(I)$ is homeomorphic to a link complement in $S^3$, its systole (i.e., the length of a shortest closed geodesic) is at most 7.35534... Note that the argument in [Adams and Reid 00] used the $2\pi$-Theorem of Gromov and Thurston, but using the 6-Theorem of Agol [Agol 00] and Lackenby [Lackenby 00] the argument of [Adams and Reid 00] can be redone to improve this systole bound to 7.17164... (see also [Palapathi 13]).

**Lemma 4.1.** Suppose that $I \subset O_d$ is an ideal such that $\mathbb{H}^3/\Gamma(I)$ is homeomorphic to a link complement in $S^3$. Then $N(I) < 39$.

**Proof.** If $\gamma \in \Gamma(I)$ is a hyperbolic element, its complex length is $l(\gamma) = \ell_0(\gamma) + i\theta(\gamma)$, where $\ell_0(\gamma)$ is the translation length of $\gamma$ and $\theta(\gamma)$ is the angle incurred in translating along the axis of $\gamma$ by distance $\ell_0(\gamma)$. Now, as is well known koshox($l(\gamma)/2) = \pm \text{tr}(\gamma)/2$, and so we get the following inequality for $\ell_0(\gamma): |\text{tr}(\gamma)/2| \leq \cos(\ell_0(\gamma)/2)$. With the systole bound given above, it follows that

$$|\text{tr}(\gamma)/2| \leq \cos(7.1717/2) \leq 18.1 \text{ and so } |\text{tr}(\gamma)| < 37.$$ 

From Lemma 2.5 of [Baker and Reid 14] we have that if $\gamma \in \Gamma(I)$ is a hyperbolic element, then $\text{tr} \gamma = \pm 2 \mod I^2$. Hence $|\text{tr}(\gamma)| \pm 2 \in I^2$, and this together with the bound on $|\text{tr}(\gamma)|$ quickly gives $N(I) < 39$. 

**Remark 4.2.** The proof of Lemma 4.1 actually shows more: if $\Gamma(I)$ is a link group, then there exists $x \in I$ such that $|x|^2 < 39$. Since $h_d > 1$, there are ideals $I$ of norm less than 39 for which no such element exists and hence $\Gamma(I)$ is not a link group. In particular, this eliminates $\Gamma(I)$ for the levels $I = (-23, 8 + \omega_3)$, $I = (-29, 9 + \omega_6)$, and $I = (-13, -1 + 2\omega_{39})$. 

---

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4.2. Cuspidal cohomology

Given Lemma 4.1, and the Remark 4.2, we further reduce the number of groups $\Gamma(I)$ of $N(I) < 39$ that can be link groups. The case of rational integer level can be dealt with by the following result of the first author [Baker 82].

Theorem 4.3. If $h_d > 1$, and $\Gamma(n) < \text{PSL}(2, O_d)$, then $H^3/\Gamma(n)$ is not homeomorphic to a link complement in $S^3$.

Indeed, for certain $d$, it will be useful in what follows to note a stronger version of Theorem 4.3 when the primes have small norm. To state this, we recall the definition of degree 1 cuspidal cohomology in a form that is useful to us. Suppose that $X = H^3/\Gamma$ is an orientable, non-compact, finite volume hyperbolic 3-orbifold, and $U_\Gamma$ the normal subgroup of $\Gamma$ generated by the parabolic elements of $\Gamma$. Then the subspace of $H_1(X, Q)$ which defines the degree 1 cuspidal cohomology of $X$ (or $\Gamma$) can be identified with:

$$V_X \text{ (or } V_\Gamma) = (\Gamma/U_\Gamma)^{ab} \otimes \mathbb{Q}.$$ 

Theorem 4.4. If $h_d > 1$, then $\Gamma(n) < \text{PSL}(2, O_d)$ has nontrivial degree 1 cuspidal cohomology in the following cases (using the notation introduced earlier to indicate the level and $d$):

(23, 3), (23, 5), (31, 2), (47, 2), (47, 3), (71, 2), (71, 3).

Proof. To prove this we use Zimmert sets as in the proof of Proposition 4.6 of [Baker and Reid 14] (following [Grunewald and Schwermer 81]). As in [Baker and Reid 14] for the cases stated in the theorem, the Zimmert sets all have at least two elements, and this allows one to conclude the existence of nontrivial cuspidal cohomology. □

Note that as a corollary to this we have the following.

Corollary 4.5. Let $d \in \{23, 31, 47, 71\}$, $I \subset O_d$ an ideal, and $p = 2, 3, 5$. Suppose that $(d, p)$ is as in Theorem 4.4 and $I$ is divisible by $< p >$. Then $\Gamma(I)$ has nontrivial degree 1 cuspidal cohomology. In particular $H^3/\Gamma(I)$ is not homeomorphic to a link complement in $S^3$.

4.3. Ruling out small norm levels

We now use Magma to deal with $\Gamma(I)$ for those ideals $I$ with $N(I) < 39$ that are not eliminated by the results in Section 4.2.

Recall from Section 2.5 that $H^3/\Gamma(I)$ a link complement implies that $\Gamma(I)$ is generated by parabolics. Using the presentations for the Bianchi groups in Section 2.2, the peripheral subgroups $P_i$ of $\text{PSL}(2, O_d)$ in Section 2.4, and the matrix representatives in Appendix A, we identify the peripheral subgroups $P_i(I) = P_i \cap \Gamma(I)$ for $i = 1, \ldots, h_d$.

As in Section 2.5, $N_d(I)$ denotes the normal closure in $\text{PSL}(2, O_d)$ of $< P_1(I), \ldots, P_{h_d}(I) >$.

Consider the quotient group $B_d(I) = \text{PSL}(2, O_d)/N_d(I)$. If $\Gamma(I)$ is a link group then $B_d(I)$ is a finite group with order equal to $|\text{PSL}(2, O_d/I)|$. Hence if $B_d(I)$ is infinite or has order greater than $|\text{PSL}(2, O_d/I)|$, then $\Gamma(I)$ cannot be a link group. Since the groups $N_d(I)$ are often not of finite index in $\text{PSL}(2, O_d)$, we calculate $B_d(I)$ in the Magma routines below as

$$B_d(I) = \langle \text{PSL}(2, O_d)|P_1(I) = \cdots = P_{h_d}(I) = 1 \rangle$$

that is, by adding the peripheral subgroups $P_i(I)$ to the relations of $\text{PSL}(2, O_d)$. We distinguish two cases:

Case 1: $B_d(I)$ is a finite group but has order larger than $|\text{PSL}(O_d/I)|$.

Case 2: $B_d(I)$ has a finite index subgroup with infinite abelianization.

Remark 4.6. Note that if $B_d(I)$ is infinite or larger than $|\text{PSL}(O_d/I)|$ for an ideal $I \subset I$, then so is $B_d(J)$ and hence $\Gamma(I)$ is not a link group.

We single out the case of $d = 6$ and discuss it in some detail in Section 4.3.1. The remaining values of $d$ are discussed in the following subsections. All Magma routines not included are available on request.

Notation. For the reader’s convenience, we recall that Magma uses the following notation to encode the abelianization of a group $G$. We illustrate this by way of example. If the abelianization of a group $G$ is

$$\mathbb{Z}^4 \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/38\mathbb{Z}$$

then Magma encodes this as $[2, 2, 38, 0, 0, 0, 0]$.

4.3.1. $d = 6$

We begin with some comments about the behavior of rational primes of norm $< 39$ in the extension $Q(\sqrt{-6})/Q$. The primes 2 and 3 are the only ramified primes; 5, 7, 11, 29, and 31 split; and 13, 17, 19, 23, and 37 are inert and so have norm exceeding 39. Hence, we exclude these and any ideal that they divide from further consideration. Recall also that $I = < 29, 9 + \omega_6 >$ is excluded by Remark 4.2.

Table 1 gives the ideals in $O_6$ whose norms are $< 39$, which were ruled out using Magma. We give the order of $B_d(I)$ as $\infty$ if it contains a finite index subgroup with infinite abelianization, and as $\geq 1$ if its order is $\geq 10^6$. We also include peripheral subgroups $P_1(I)$ and $P_2(I)$. 
Table 1. Levels and peripheral subgroups for \( d = 6 \).

<table>
<thead>
<tr>
<th>Ideal</th>
<th>Norm</th>
<th>Peripheral subgroups</th>
<th>( \text{Order}(B_d(I)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(&lt; 2, \omega_6 &gt;)</td>
<td>2</td>
<td>(&lt; t^2, u &gt;, &lt; (tb)^2, cu &gt;)</td>
<td>24</td>
</tr>
<tr>
<td>(&lt; 3, \omega_6 &gt;)</td>
<td>3</td>
<td>(&lt; t^3, u &gt;, &lt; (tb)^3, cu &gt;)</td>
<td>( \infty )</td>
</tr>
<tr>
<td>(&lt; 2 &gt;)</td>
<td>4</td>
<td>(&lt; t^2, u^2 &gt;, &lt; (tb)^2, (cu)^2 &gt;)</td>
<td>( \infty )</td>
</tr>
<tr>
<td>(&lt; 5, 2 + \omega_6 &gt;)</td>
<td>5</td>
<td>(&lt; t^5, t^2u &gt;, &lt; (tb)^5, (tb)^2cu &gt;)</td>
<td>( \gg 1 )</td>
</tr>
<tr>
<td>(&lt; 7, 1 + \omega_6 &gt;)</td>
<td>7</td>
<td>(&lt; t^7, tu &gt;, &lt; (tb)^7, (tb)^2cu &gt;)</td>
<td>( \gg 1 )</td>
</tr>
<tr>
<td>(&lt; 11, 4 + \omega_6 &gt;)</td>
<td>11</td>
<td>(&lt; t^{11}, t^2u &gt;, &lt; (tb)^{11}, (tb)^2cu &gt;)</td>
<td>( \gg 1 )</td>
</tr>
</tbody>
</table>

The ideals of norm 8, 9, 12, 14, 15, 16, 18, 20, 21, 24, 27, 28, 30, 32, 33, 35, and 36 are sub-ideals of those in Table 1 and are eliminated by Remark 4.6.

**Magma routine for \( I = < 2, \omega_6 >\)**

Note that from Section 2.1, the order of \( \text{PSL}(2, O_6/I) \) is 6.

```magma
B<a,t,u,b,c> := Group<a,t,u,b,c | a^2, b^2, (t*a)^3, (a*t*b)^3, (a*t*u*b*u^-1)^3, t^-1*c*t*u*b*u^-1*c^-1*b^-1, (t,u), (a,c), t^2, u, (t*b)^2, c*u>;

print Order(B); 24
```

**Magma routine for \( I = < 3, \omega_6 >\)**

Note that from Section 2.1, the order of \( \text{PSL}(2, O_6/I) \) is 12.

```magma
B<a,t,u,b,c> := Group<a,t,u,b,c | a^2, b^2, (t*a)^3, (a*t*b)^3, (a*t*u*b*u^-1)^3, t^-1*c*t*u*b*u^-1*c^-1*b^-1, (t,u), (a,c), t^3, u, (t*b)^3, c*u>;

L:=LowIndexNormalSubgroups(G,48);
print #L; 8
```

**Magma routine for \( I = < 2 >\)**

We first point out that this case does not follow from Remark 4.6 using the ideal \(< 2, \omega_6 >\), since from Section 2.1, the order of \( \text{PSL}(2, O_6/I) \) is \( 48 > 24 \). Also note that it does follow from Theorem 4.3 that \( \Gamma(2) \) is not a link group. However, to deal with other levels divisible by 2 we need to use the stronger statement that \( B_6(I) \) is infinite.

```magma
B<a,t,u,b,c> := Group<a,t,u,b,c | a^2, b^2, (t*a)^3, (a*t*b)^3, (a*t*u*b*u^-1)^3, t^-1*c*t*u*b*u^-1*c^-1*b^-1, (t,u), (a,c), t^2, u^2, (t*b)^2, (c*u)^2>;

L:=LowIndexNormalSubgroups(B,48);
print #L; 98
```

**Magma routine for \( I = < 5, 2 + \omega_6 >\)**

Note that from Section 2.1, the order of \( \text{PSL}(2, O_6/I) \) is 60.

```magma
B<a,t,u,b,c> := Group<a,t,u,b,c | a^2, b^2, (t*a)^3, (a*t*b)^3, (a*t*u*b*u^-1)^3, t^-1*c*t*u*b*u^-1*c^-1*b^-1, (t,u), (a,c), t^5, u^2, (t*b)^5, (t*b)^2*(c*u)>;

print Order(B); 1966080
```

**Magma routine for \( I = < 11, 4 + \omega_6 >\)**

Note that from Section 2.1, the order of \( \text{PSL}(2, O_6/I) \) is 660.

```magma
B<a,t,u,b,c> := Group<a,t,u,b,c | a^2, b^2, (t*a)^3, (a*t*b)^3, (a*t*u*b*u^-1)^3, t^-1*c*t*u*b*u^-1*c^-1*b^-1, (t,u), (a,c), t^11, u^4, (t*b)^11, (t*b)^4*(c*u)^-1>;

L:=LowIndexNormalSubgroups(B,660);
print #L; 2
```

**Magma routine for \( I = 0 \)**

```magma
print Index(B,L[2] 'Group); 660
```

```magma
print AbelianQuotientInvariants(L[8]'Group); [ 4, 120, 120, 120, 120, 120, 120, 120 ]
```
Magma routine for \( I = < 31, 5 + \omega_6 > \)

Note that from Section 2.1, the order of \( \text{PSL}(2, O_6/I) \) is 14880.

\[
G = \langle a, t, u, b, c \rangle := \text{Group} < a, t, u, b, c | a^2, b^2, (t, u), (t^a)^3, (a, c)^{t^{-1}c*t^u}, b^u = 1, (a*t*b)^3, (a*t*b*u*b^u - 1)^3, t^31, t^5*u, (t*b)^31, (t*b)^5*(c*u)^{-1} >;
\]

\[
L := \text{LowIndexNormalSubgroups}(G, 14880);
\]

print Index(G, L[2] \langle\text{Group}\rangle);

14880

\[
M := \text{sub}\langle G | (L[2] \langle\text{Group}\rangle), t >;
\]

print Index(M, (L[2] \langle\text{Group}\rangle));

31

\[
\text{AbelianQuotientInvariants}(M);
\]

\[
\begin{cases}
3, 3, 3, 3, 3, 3, 3, 3, 68433864, 1847714328
\end{cases}
\]

In the following subsections, we have followed what was done for the case of \( d = 6 \) above and applied the results and comments of Sections 4.1 and 4.2 to the remaining values of \( d \). Specifically we have used Appendix B (which contains lists of prime ideals of norm less than 39 in each case), together with Lemma 4.1, Remark 4.6, Theorem 4.3, Theorem 4.4, and Corollary 4.5 to eliminate certain levels. Tables 2–6 below summarize the Magma calculations that treat all of the remaining cases to be eliminated. We do not list those ideals \( J \subset I \) that are eliminated directly by the size of \( B_d(I) \) (as per Remark 4.6).

4.3.3. \( d = 15, 23, 31, 39 \)

Referring to Tables 3–6, in Table 3, set \( x = c^{-1}au^{-1}c^{-1}u^{-1}ta \) to be the parabolic element commuting with \( uca \).

In Table 4, set \( p_1 = g_1g_2g_3 \) and \( p_2 = g_4^{-1}g_5g_6 \) to be the parabolic elements generating the peripheral subgroup of \( \text{PSL}(2, O_{31}) \) fixing \( -1 + \sqrt{31} / 4 \).

In Table 5, set \( p = g_1^{-1}g_2g_3^{-1}g_4 \) to be the parabolic element commuting with \( g_5g_6 \). These elements generate the peripheral subgroup of \( \text{PSL}(2, O_{31}) \) fixing \( -1 + \sqrt{31} / 4 \).

Table 6 needs no further explanation.

4.3.4. \( d = 47 \) and \( d = 71 \)

For the remaining two Bianchi groups, we simply comment on a few cases which together with Theorems 4.3, 4.4, and Corollary 4.5 eliminate all other possible levels.

When \( d = 47 \), \( B_7(I) \) is infinite (resp. \( \geq 1 \)) for \( I = < 6, \omega_{47} > \) (resp. for \( I = < 7, 1 + \omega_{47} >, < 8, 4 + \omega_{47} > \) and \( < 9, 2 + \omega_{47} > \)).

When \( d = 71 \), \( B_{71}(I) \) is infinite for \( I = < 3, \omega_{71} > \) (resp. \( \geq 24 \times (33)^3 \) for \( I = < 4, -2 + \omega_{71} > \), (resp. \( \geq 1 \)) for \( I = < 5, 1 + \omega_{71} > \)).

5. Congruence links when \( d = 6 \) and \( d = 39 \)

In this section we exhibit congruence link groups for \( d = 6 \) and \( d = 39 \) which proves Theorem 1.3. We begin with some notation.

For an ideal \( I \subset O_d \), let \( \Gamma_0(I) \) (resp. \( \Gamma_1(I) \)) denote the congruence subgroup obtained as the preimage of the subgroup \( \{ (a, x) : a, x \in O_d/I|/\{\pm Id\} \} \) (resp. \( \{ (a, x) : x \in O_d/I|/\{\pm Id\} \}) \) under the reduction homomorphism \( \text{PSL}(2, O_d) \rightarrow \text{PSL}(2, O_d/I) \). Note that by definition, as long as \( N(I) \geq 5 \), the group \( \Gamma_1(I) \) is torsion-free.
## Table 2. Levels and peripheral subgroups for $d = 5$.

<table>
<thead>
<tr>
<th>Ideal</th>
<th>Norm</th>
<th>Peripheral subgroups</th>
<th>Order($B_d(I)$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&lt; 2 &gt;$</td>
<td>4</td>
<td>$&lt; t^4, u^2, &lt; (tb)^5 &gt;, (tu^{-1}ct^{-1})^2 &gt;$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$&lt; \omega_5 &gt;$</td>
<td>5</td>
<td>$&lt; t^5, u, &lt; (tb)^5, tu^{-1}ct^{-1} &gt;$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$&lt; 6, 1 + \omega_5 &gt;$</td>
<td>6</td>
<td>$&lt; t^6u, tu, &lt; (tb)^5, (tu^{-1}ct)^{-1} &gt;$</td>
<td>144</td>
</tr>
<tr>
<td>$&lt; 7, 3 + \omega_5 &gt;$</td>
<td>7</td>
<td>$&lt; t^7, t^{-1}u, &lt; (tb)^5, (tu^{-1}ct^{-1})^2 &gt;$</td>
<td>$\geq 168 \times 3^2$</td>
</tr>
<tr>
<td>$&lt; 3, 1 + \omega_5 &gt;$</td>
<td>9</td>
<td>$&lt; t^3, t^{-1}u, &lt; (tb)^5, (tu^{-1}ct^{-1})^{-1} &gt;$</td>
<td>$\geq 324 \times 2^2$</td>
</tr>
<tr>
<td>$&lt; 3 &gt;$</td>
<td>9</td>
<td>$&lt; t^3, u^3, &lt; (tb)^5, (tu^{-1}ct^{-1})^3 &gt;$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$&lt; 29, -13 + \omega_5 &gt;$</td>
<td>29</td>
<td>$&lt; t^{29}, t^{-1}u^3, &lt; (tb)^{29}, (tb)^{-13}(tu^{-1}ct^{-1}) &gt;$</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

## Table 3. Levels and peripheral subgroups for $d = 15$.

<table>
<thead>
<tr>
<th>Ideal</th>
<th>Norm</th>
<th>Peripheral subgroups</th>
<th>Order($B_d(I)$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&lt; 2, \omega_3 &gt;$</td>
<td>3</td>
<td>$&lt; t^8, t^{-4}u &gt;, &lt; (uca)x^4, (uca)^2 &gt;$</td>
<td>$192 \times 3^3$</td>
</tr>
<tr>
<td>$&lt; 3 &gt;$</td>
<td>9</td>
<td>$&lt; t^3, u^3, &lt; (uca)^3, x^3 &gt;$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$&lt; 10, 2 + \omega_3 &gt;$</td>
<td>10</td>
<td>$&lt; t^{10}, t^2u &gt;, &lt; (uca)^5, x^2 &gt;$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$&lt; 12, 4 + \omega_3 &gt;$</td>
<td>12</td>
<td>$&lt; t^{12}, t^4u &gt;, &lt; (uca)^2, x^4 &gt;$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$&lt; 15, -8 + \omega_3 &gt;$</td>
<td>15</td>
<td>$&lt; t^{15}, t^2u &gt;, &lt; (uca)^5x &gt;$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$&lt; 17, 5 + \omega_3 &gt;$</td>
<td>17</td>
<td>$&lt; t^{17}, t^2u &gt;, &lt; (uca)^{17}, (uca)^{-9}x &gt;$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$&lt; 19, 10 + \omega_3 &gt;$</td>
<td>19</td>
<td>$&lt; t^{19}, t^9u &gt;, &lt; (uca)^{19}, (uca)^{3}x &gt;$</td>
<td>$\gg 1$</td>
</tr>
<tr>
<td>$&lt; 31, -14 + \omega_3 &gt;$</td>
<td>31</td>
<td>$&lt; t^{31}, t^{-14}u &gt;, &lt; (uca)^{31}, (uca)^{13}x &gt;$</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

## Table 4. Levels and peripheral subgroups for $d = 23$.

<table>
<thead>
<tr>
<th>Ideal</th>
<th>Norm</th>
<th>Peripheral subgroups</th>
<th>Order($B_d(I)$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&lt; 2 &gt;$</td>
<td>4</td>
<td>$&lt; g_1^3, g_2^3 &gt;, &lt; g_3^2 &gt;, &lt; p_1^2, p_2^2 &gt;$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$&lt; 6, \omega_3 &gt;$</td>
<td>6</td>
<td>$&lt; g_1^3, g_2^3, g_3 &gt;, &lt; g_4^3, g_5 g_6 &gt;, &lt; p_1^3, p_2^3 &gt;$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$&lt; 2, \omega_3 &gt;$</td>
<td>8</td>
<td>$&lt; g_1^3, g_3 &gt;, &lt; g_4^3, g_5 ^2g_6 &gt;, &lt; (g_1g_3)^2, (g_4g_5)^2p &gt;$</td>
<td>$\geq 192 \times 3^6$</td>
</tr>
<tr>
<td>$&lt; 3, \omega_3 &gt;$</td>
<td>9</td>
<td>$&lt; g_1^3, g_3 &gt;, &lt; g_4^3, g_5 ^2g_6 &gt;, &lt; p_1^3, p_2^3, p_1^2p_2 &gt;$</td>
<td>$\geq 324 \times 2^8$</td>
</tr>
<tr>
<td>$&lt; 13, 4 + \omega_3 &gt;$</td>
<td>13</td>
<td>$&lt; g_1^3, g_2^3, g_4 &gt;, &lt; g_5 ^2g_6 &gt;, &lt; p_1^3, p_2^3, p_1^2, p_2 &gt;$</td>
<td>$\gg 1$</td>
</tr>
<tr>
<td>$&lt; 23, \omega_3 &gt;$</td>
<td>23</td>
<td>$&lt; g_1^3, g_2 ^3g_4 &gt;, &lt; g_3 ^2g_4 ^2g_5 &gt;, &lt; p_1^3, p_2^3, p_1^2, p_2 &gt;$</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

## Table 5. Levels and peripheral subgroups for $d = 31$.

<table>
<thead>
<tr>
<th>Ideal</th>
<th>Norm</th>
<th>Peripheral subgroups</th>
<th>Order($B_d(I)$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&lt; 7, 2 + \omega_3 &gt;$</td>
<td>7</td>
<td>$&lt; g_1^7, g_1 ^3g_3 &gt;, &lt; g_4 ^2g_5 ^2g_6 &gt;, &lt; g_7 &gt;, &lt; (g_1g_3)^2p &gt;$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$&lt; 2, \omega_3 &gt;$</td>
<td>8</td>
<td>$&lt; g_1 ^3, g_3 &gt;, &lt; g_4 ^3, g_5 ^2g_6 &gt;, &lt; (g_1g_3)^2, (g_4g_5)^2p &gt;$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$&lt; 3 &gt;$</td>
<td>9</td>
<td>$&lt; g_1 ^3, g_3 &gt;, &lt; g_4 ^3, g_5 ^2g_6 &gt;, &lt; (g_1g_3)^2, p_1^3 &gt;$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$&lt; 10, 1 + \omega_3 &gt;$</td>
<td>10</td>
<td>$&lt; g_1^{10}, g_1 ^3g_3 &gt;, &lt; g_4 ^2g_5 ^2g_6 &gt;, &lt; (g_1g_3)^{10}, (g_4g_5)^2p &gt;$</td>
<td>$\gg 1$</td>
</tr>
<tr>
<td>$&lt; 19, 5 + \omega_3 &gt;$</td>
<td>19</td>
<td>$&lt; g_1^9, g_1 ^3g_3 &gt;, &lt; g_4 ^3, g_5 ^2g_6 &gt;, &lt; (g_1g_3)^9, (g_4g_5)^2p &gt;$</td>
<td>$\gg 1$</td>
</tr>
<tr>
<td>$&lt; -1 + 2\omega_3 &gt;$</td>
<td>31</td>
<td>$&lt; g_1^3, g_1 ^3g_3 &gt;, &lt; g_4 ^3, g_5 ^2g_6 &gt;, &lt; (g_1g_3)^3, (g_4g_5)^2p^{-1} &gt;$</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

## Table 6. Levels and peripheral subgroups for $d = 39$.

<table>
<thead>
<tr>
<th>Ideal</th>
<th>Norm</th>
<th>Peripheral subgroups</th>
<th>Order($B_d(I)$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&lt; 2, \omega_3 &gt;$</td>
<td>2</td>
<td>$&lt; g_1 ^3, g_2 ^2 &gt;, &lt; g_3 ^2g_4 &gt;, &lt; (g_5 ^2g_6 ^2, g_7 ) &gt;, &lt; (g_5 ^2g_6 ^2g_7 ) &gt;$</td>
<td>18</td>
</tr>
<tr>
<td>$&lt; 3, 1 + \omega_3 &gt;$</td>
<td>3</td>
<td>$&lt; g_1 ^3, g_2 ^3, g_3 &gt;, &lt; g_4 ^2g_5 ^2g_6 &gt;, &lt; (g_1g_2)^3g_3g_6 &gt;, &lt; (g_1g_2)^3g_5g_6 &gt;, &lt; (g_1g_2)^3g_5g_6 &gt;$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$&lt; 2, \omega_3 &gt;$</td>
<td>4</td>
<td>$&lt; g_1 ^3, g_2 ^3, g_3 &gt;, &lt; g_4 ^2g_5 ^2g_6 &gt;, &lt; (g_1g_2)^3g_3g_6 &gt;, &lt; (g_1g_2)^3g_5g_6 &gt;, &lt; (g_1g_2)^3g_5g_6 &gt;$</td>
<td>72</td>
</tr>
<tr>
<td>$&lt; 2 &gt;$</td>
<td>4</td>
<td>$&lt; g_1 ^3, g_2 ^3, g_3 &gt;, &lt; g_4 ^2g_5 ^2g_6 &gt;, &lt; (g_1g_2)^3g_3g_6 &gt;, &lt; (g_1g_2)^3g_5g_6 &gt;$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$&lt; 6, \omega_3 &gt;$</td>
<td>5</td>
<td>$&lt; g_1 ^3, g_2 ^3, g_3 &gt;, &lt; g_4 ^2g_5 ^2g_6 &gt;, &lt; (g_1g_2)^3g_3g_6 &gt;, &lt; (g_1g_2)^3g_5g_6 &gt;, &lt; (g_1g_2)^3g_5g_6 &gt;$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$&lt; 11, 3 + \omega_3 &gt;$</td>
<td>11</td>
<td>$&lt; g_1 ^3, g_2 ^3, g_3 &gt;, &lt; g_4 ^2g_5 ^2g_6 &gt;, &lt; (g_1g_2)^3g_3g_6 &gt;, &lt; (g_1g_2)^3g_5g_6 &gt;, &lt; (g_1g_2)^3g_5g_6 &gt;$</td>
<td>$\gg 1$</td>
</tr>
</tbody>
</table>
In the case when \( I = \mathcal{P} \) is a prime ideal of \( O_d \) lying over a split prime \( p \in \mathbb{Z} \) (as will be the case of interest below), then \([\text{PSL}(2, O_d) : \Gamma_0(\mathcal{P})] = p + 1\), the quotient group \( \Gamma_0(\mathcal{P})/\Gamma(\mathcal{P}) \) is cyclic of order \((p - 1)/2\), and \( \Gamma_1(\mathcal{P})/\Gamma(\mathcal{P}) \cong \mathbb{Z}/p\mathbb{Z} \) with \( \Gamma_1(\mathcal{P}) = < \Gamma(\mathcal{P}), (\omega^2)^\chi >. \)

Finally, note that by definition, a peripheral subgroup of \( \Gamma_0(I) \) is necessarily contained in \( \Gamma_1(I) \), so that a set of peripheral subgroups for \( \Gamma_0(I) \) is a partial set of peripheral subgroups for \( \Gamma_1(I) \).

**Proposition 5.1.** \( \Gamma_1(\< 7, 1 + \omega_b >) \) and \( \Gamma_1(\< 5, \omega_{39} >) \) are congruence link subgroups of \( \text{PSL}(2, O_b) \) and \( \text{PSL}(2, O_{39}) \) respectively.

**Proof.** The proof follows the method in Section 3. We do \( \Gamma_1(I) \) for \( I = \< 7, 1 + \omega_b > \) in detail, and then comment on the case \( I = < 5, \omega_{39} > \). Both Magma routines are given below.

Let \( I = \< 7, 1 + \omega_b > \). From the remarks preceding the statement of Proposition 5.1, we have \([\text{PSL}(2, O_b) : \Gamma_0(I)] = 8 \) and \([\Gamma_0(I) : \Gamma_1(I)] = 3 \). Now \( Q_b \) has 2 cusps, and in the covering \( \mathbb{H}^3/\Gamma_0(I) \rightarrow Q_b \) each one of these 2 cusps is covered by two cusps, one of which has covering degree 1 and the other has degree 7.

Hence we deduce that \( \mathbb{H}^3/\Gamma_1(I) \) has 12 cusps, and a set of peripheral subgroups for \( \Gamma_1(I) \) is obtained by conjugating those of \( \Gamma_0(I) \) by the elements \( \{Id, x, x^{-1}\} \) where \( \Gamma_0(I) = < \Gamma_1(I), x > \). A particular choice of \( x \) is \((a^{-1} - a^{-1})/t^2(t^{-1}a^{-1})\).

Using the matrix representations for the peripheral subgroups \( P_1 \) and \( P_2 \) of \( \text{PSL}(2, O_b) \) given in Section 2.4, we find the following set of peripheral subgroups for \( \Gamma_0(I) \):

\[
(\infty, < t, u >), (0, a \neq t^2, tu > a^2),
\]

\[
\left( -\frac{\sqrt{6}}{2}, < (tb)^7, (tb)(cu)^{-1} > \right),
\]

\[
\left( -\frac{6 - \sqrt{6}}{14}, (at^2a^{-1}) < tb, cu > (at^2a^{-1})^{-1} \right)
\]

Now we obtain a set of peripheral subgroups for \( \Gamma_1(I) \) by conjugating the above set for \( \Gamma_0(I) \) by the elements \( \{Id, x, x^{-3}\} \).

In the following Magma routine, \( G = \text{PSL}(2, O_b) \), \( K \) is the subgroup generated by the four peripheral subgroups of \( \Gamma_0(I) \), \( L[1] = \Gamma_0(I) \), and \( M[1] = \Gamma_1(I) \). As before, \( Q \) is the quotient of \( \Gamma_1(I) \) by the normal closure the 12 parabolic elements to be trivialized. Since Magma calculates \( Q = < 1 > \), it follows that \( \Gamma_1(I) \) is indeed a 12-component link group.

Note that we use the \texttt{LowIndexSubgroups} routine in Magma to obtain \( \Gamma_0(I) \) from \( \text{PSL}(2, O_b) \) and \( \Gamma_1(I) \) from \( \Gamma_0(I) \). Indeed, since \( N_d(I) \) is not of finite index in \( \Gamma(I) \) (see Section 4.3.1), we cannot use Magma to obtain a presentation for \( \Gamma(I) \) as in Section 3, and hence we cannot present \( \Gamma_1(I) = < \Gamma(I), (\omega^2)^\chi > \) in this way.

Now let \( I = < 5, \omega_{39} > \). Note that \( Q_{39} \) has 4 cusps and in the cover \( \mathbb{H}^3/\Gamma_0(I) \rightarrow Q_{39} \) each one of these 4 cusps is covered by 2 cusps, one of which has covering degree 1 and the other has degree 5. Thus we deduce that \( \mathbb{H}^3/\Gamma_1(I) \) has 16 cusps, and a set of peripheral subgroups for \( \Gamma_1(I) \) is obtained by conjugating one for \( \Gamma_0(I) \) by the elements \( \{Id, x\} \) where \( \Gamma_0(I) = < \Gamma_1(I), x > \).

In the following Magma routine, \( G = \text{PSL}(2, O_{39}) \), \( K \) gives a set of peripheral subgroups for \( \Gamma_0(I) \), \( L[1] = \Gamma_0(I) \), and \( M[1] = \Gamma_1(I) \). Magma confirms that \( \Gamma_0(I) = < \Gamma(I), x > \) for \( x = (g_3g_4g_5g_6g_1)(g_3g_4g_5g_6g_1)^{-1} \). Note that Magma returns two possibilities for \( \Gamma_0(I) \): \( L[1] \) and \( L[2] \); however
$L[2]$ is eliminated by homology considerations. Finally, $Q = <1>$, so that $\Gamma_1(I)$ is a 16-component link group.

\[
G g_1, g_2, g_3, g_4, g_5, g_6, g_7 : = \text{Group}\langle g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_3^3, (g_4, g_6), (g_3^g_5)^2, (g_2, g_1), (g_1^{g_3-1})^{g_1}, (g_3-1, g_7^{-1}), (g_5^{-1} g_1)^{g_3}, g_5^{-1} g_1 g_6^{-1} g_4^{-1} g_5^{-1} g_4 g_1^{-1} g_6, g_4^{-1} g_5 g_4 g_2^{-1} g_7, g_5^{-1} g_7^{-1} g_2, (g_7 g_5^{-1} g_7^{-1} g_1)^{g_3}, g_6 g_1^{-1} g_5 g_6^{-1} g_4^{-1} g_5 g_4 g_1^{-1} g_4^{-1} g_5 g_4 g_1^{-1} > \\
K : = \text{sub}\langle G | g_1, g_2, (g_3^g_1) * g_1^g_5 * (g_3^g_1), (g_3^g_1) * g_2^g_1 (g_3^g_1), g_5^{-1} g_6, g_4 g_1^{-1} g_6, (g_3^g_1) * (g_4^g_1) * (g_6^g_4) * (g_5^{-1} g_6) * (g_3^g_1), g_4^{-1} g_6^{-1}, (g_3^g_1) * g_2^g_1 * g_4^g_1 * (g_5^g_1), (g_3^g_1) * (g_1^g_2 * g_4^g_1) * (g_5^g_1), g_5 g_4 (g_2^{-1} g_7), (g_3^g_1) * g_1^g_6 (g_5^g_1) * g_1^{-1} (g_3^g_1), (g_3^g_1) * g_1^g_6 (g_5^g_1) * g_1^{-1} (g_3^g_1) ;
\]

\[
L : = \text{LowIndexSubgroups}(G, <6, 6>) ; \\
\text{Subgroup} := K ; \\
\text{print} \ #L ; \\
2 \\
\|
\text{print} \ \text{AbelianQuotientInvariants}(L[1]) ; \\
[2, 2, 0, 0, 0, 0, 0, 0, 0, 0] \\
\|
\text{print} \ \text{AbelianQuotientInvariants}(L[2]) ; \\
[2, 0, 0, 0, 0, 0, 0, 0, 0, 0] \\
\|
\text{M} := \text{LowIndexSubgroups}(L[1], <2, 2>) ; \\
\text{Subgroup} := K ; \\
\text{print} \ #M ; \\
1 \\
\|
\text{x} := g_3^g_1 g_1^g_2 g_3^g_1 g_1^g_3^g_1^{-1} g_4^g_1 g_1^g_2^g_1 g_3^g_1 ; \\
\text{A} := \text{sub}\langle G | M[1], x > ; \\
\text{A} \ \text{eq} \ L[1] ; \\
\text{true} \\
\|
\text{Q} := \text{quo}\langle M | g_1, x^g_2 x^g_1, (g_3^g_1) g_2^g_1 g_3^g_1 x^g_1, g_4 g_1^{-1} g_6, x^g_4 x^g_1^{-1} g_6 x^g_1, (g_3^g_1) * (g_4^g_1) * (g_5^g_1) * (g_3^g_1), x^g_1 (g_3^g_1) * (g_4^g_1) * (g_6^g_4) * (g_5^g_1) * (g_6^g_4) * (g_3^g_1) * x^g_1, (g_4^g_1 g_4^g_6) * x^g_1 g_4^g_6^{-1} x^g_2 (g_5^{-1} g_6) * (g_3^g_1) * x^g_1, x^g_1 g_4^g_6^{-1} ; \\
\text{Q} := \text{ReduceGenerators}(Q) ; \\
\text{print} \ \text{Order}(Q) ; \\
1 \
\|

\text{Appendix A}

In this appendix we gather together the matrix generators for the groups $\text{PSL}(2, O_d)$ as given in Section 2.2 and used throughout. For $d = 5, 6, 15$, $t = \left(\begin{array}{c}1
\\omega_d \\omega_d \\omega_d \end{array}\right)$, and $a = \left(\begin{array}{c}1
0 \\omega_d \omega_d \omega_d \omega_d \end{array}\right)$. Also, recall that:

\[
\omega_d = \sqrt{-5}, \sqrt{-6}, \frac{1 + \sqrt{-15}}{2}, \frac{1 + \sqrt{-23}}{2}, \frac{1 + \sqrt{-31}}{2}, \frac{1 + \sqrt{-39}}{2}, \frac{1 + \sqrt{-47}}{2}, \frac{1 + \sqrt{-71}}{2}. \\
\]

\[
d = 5 : a, b = \left(\begin{array}{c}0
-w_5 \frac{2}{5} \omega_5 \end{array}\right), \\
c = \left(\begin{array}{cc}
-w_5 - 4 & -2w_5 \\
2w_5 & w_5 - 4
\end{array}\right), t, u. \\
d = 6 : a, b = \left(\begin{array}{cc}
-1 - w_6 & 2 - w_6 \\
2 & 1 + w_6
\end{array}\right), \\
c = \left(\begin{array}{c}
5 \\
2w_6
\end{array}\right), t, u. \\
d = 15 : a, c = \left(\begin{array}{cc}
4 & 1 - 2w_{15} \\
2w_{15} - 1 & 4
\end{array}\right), t, u. \\
d = 23 : g_1 = \left(\begin{array}{cc}1 & 1 - w_{23} \\
0 & 1
\end{array}\right), g_2 = \left(\begin{array}{c}1 \\
0 1
\end{array}\right), \\
g_3 = \left(\begin{array}{cc}0 & 1 \\
1 & 1
\end{array}\right), g_4 = \left(\begin{array}{cc}3 + w_{23} & -4 + w_{23} \\
-2 + w_{23} & -1 - w_{23}
\end{array}\right), \\
g_5 = \left(\begin{array}{cc}5 - w_{23} & 1 + 2w_{23} \\
2 + w_{23} & -3 + w_{23}
\end{array}\right). \\
d = 31 : g_1 = \left(\begin{array}{c}1 \\
0 1
\end{array}\right), g_2 = \left(\begin{array}{c}0 \\
-1 1
\end{array}\right). \\
\]
\[ g_5 = \begin{pmatrix} 3 - 2\omega_{31} & 7 + \omega_{31} \\ 4 & -1 + 2\omega_{31} \end{pmatrix}. \]

\[ d = 39: g_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ g_2 = \begin{pmatrix} 1 & \omega_{39} \\ 0 & 1 \end{pmatrix}, \]

\[ g_3 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \ g_4 = \begin{pmatrix} -3 - \omega_{39} & 7 - 2\omega_{39} \\ 2 - \omega_{39} & 5 + \omega_{39} \end{pmatrix}. \]

\[ g_5 = \begin{pmatrix} 3 - \omega_{39} & 2 + \omega_{39} \\ 3 & -1 + \omega_{39} \end{pmatrix}, \]

\[ g_6 = \begin{pmatrix} 7 - \omega_{39} & 2 + 3\omega_{39} \\ 2 + \omega_{39} & -5 + \omega_{39} \end{pmatrix}, \]

\[ g_7 = \begin{pmatrix} 6 - \omega_{39} & -1 + 2\omega_{39} \\ 1 - 2\omega_{39} & 5 + \omega_{39} \end{pmatrix}. \]

\[ g_8 = \begin{pmatrix} -1 + 2\omega_{71} & 12 \\ -6 & -1 + 2\omega_{71} \end{pmatrix}, \]

\[ g_9 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \ g_8 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \]

\[ g_9 = \begin{pmatrix} 1 + \omega_{71} & -7 \\ 3 & -2 + \omega_{71} \end{pmatrix}. \]

**Appendix B**

In this appendix we list the prime ideals of norm \(< 39\) in \(O_d\) (as dictated by Lemma 4.1). We also include information on the splitting type of the rational prime \(p\) (i.e., whether \(p\) is ramified, inert, or split). The norm of any \(O_d\)-
prime \(P\) dividing \(p\) is then easily computed: it is \(p\) when \(p\) is ramified or split, and \(p^2\) when \(p\) is inert.

\[ d = 5 \]

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<tr>
<td>3</td>
<td>ramified</td>
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<tr>
<td>5</td>
<td>split</td>
</tr>
<tr>
<td>7</td>
<td>split</td>
</tr>
<tr>
<td>23</td>
<td>split</td>
</tr>
<tr>
<td>29</td>
<td>split</td>
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</tr>
<tr>
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<td>ramified</td>
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<tr>
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</tr>
<tr>
<td>7</td>
<td>split</td>
</tr>
<tr>
<td>11</td>
<td>split</td>
</tr>
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<tr>
<td>29</td>
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\[ d = 23 \]

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<tr>
<td>31</td>
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References