Principal Congruence Link Complements

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1 Introduction

Let d be a square-free positive integer, let O_d denote the ring of integers in $\mathbf{Q}(\sqrt{-d})$, and let Q_d denote the Bianchi orbifold $\mathbf{H}^3/\mathrm{PSL}(2, \mathrm{O_d})$. A non-compact finite volume hyperbolic 3-manifold X is called *arithmetic* if X and Q_d are commensurable, that is to say they share a common finite sheeted cover (see [25] Chapter 8 for more on this). If N is a closed orientable 3-manifold and $L \subset N$ a link, then L is called *arithmetic* if $N \setminus L$ is an arithmetic hyperbolic 3-manifold.

In his list of problems in his Bulletin of the AMS article [32], Thurston states as Question 19:

Find topological and geometric properties of quotient spaces of arithmetic subgroups of $\mathrm{PSL}(2, \mathbf{C})$. *These manifolds often seem to have special beauty.*

For example, many of the key examples in the development of the theory of geometric structures on 3-manifolds (e.g. the figure-eight knot complement, the Whitehead link complement, the complement of the Borromean rings and the Magic manifold) are arithmetic.

The "beauty" referred to by Thurston is captured particularly well by *congruence* manifolds (which includes some of the above examples). Recall that a subgroup $\Gamma < \mathrm{PSL}(2, \mathrm{O_d})$ is called a *congruence subgroup* if there exists an ideal $I \subset O_d$ so that Γ contains the *principal congruence group*:

$$\Gamma(I) = \ker\{\mathrm{PSL}(2, \mathrm{O_d}) \to \mathrm{PSL}(2, \mathrm{O_d/I})\},$$

where $\mathrm{PSL}(2, \mathrm{O_d/I}) = \mathrm{SL}(2, \mathrm{O_d/I})/\{\pm \mathrm{Id}\}$

A manifold $M = \mathbf{H}^3/\Gamma$ is called *congruence* (resp. *principal congruence*) if $\Gamma > \Gamma(I)$ (resp. $\Gamma = \Gamma(I)$) for some ideal $I \subset O_d$. As above we will also refer to a link $L \subset N$ as congruence (resp. principal congruence) if the manifold $N \setminus L$ is so.

The largest ideal I for which $\Gamma(I) < \Gamma$ is called the *level* of Γ. For convenience, if $n \in \mathbf{Z}$, we will denote the principal O_d-ideal $< n >$ simply by n.

As we discuss in §2.2, for a fixed closed orientable 3-manifold N there are only finitely many principal congruence link complements in N. The aim of this paper is to make a start on enumerating all principal congruence link complements in S^3, together with their levels. Note that since links with at least 2 components are not generally determined by their complements (see [17]), one cannot just say "finitely many principal congruence links". Our main results are the following.

Theorem 1.1. *The following list of pairs* (d, I) *indicates the known Bianchi groups* $\mathrm{PSL}(2, \mathrm{O_d})$ *containing a principal congruence subgroup* $\Gamma(I)$ *such that* $\mathbf{H}^3/\Gamma(I)$ *is a link complement in* S^3. *Those annotated by * are new.*

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1. $d = 1$**:** $I \in \{2, < 2 \pm i >^*, < (1 \pm i)^3 >^*, 3^*\}$.

2. $d = 2$**:** $I \in \{2, < 1 \pm \sqrt{-2} >^*, < 2 \pm \sqrt{-2} >^*\}$.

3. $d = 3$**:** $I \in \{2, 3, < (5 \pm \sqrt{-3})/2 >, < 3 \pm \sqrt{-3} >\}$.

4. $d = 7$**:** $I \in \{< (1 \pm \sqrt{-7})/2 >, 2, < (3 \pm \sqrt{-7})/2 >^*, < 1 \pm \sqrt{-7} >^*\}$.

5. $d = 11$**:** $I \in \{< (1 \pm \sqrt{-11})/2 >^*, < (3 \pm \sqrt{-11})/2 >^*\}$.

6. $d = 15$**:** $I =< 2, (1 \pm \sqrt{-15})/2 >$.

7. $d = 19$**:** $I =< (1 \pm \sqrt{-19})/2 >$.

8. $d = 23$**:** $I =< 2, (1 \pm \sqrt{-23})/2 >$.

In the case when the level is a rational integer we can say more.

Theorem 1.2. *Let* $n \in \mathbf{Z}$. *Then* $\Gamma(n) < \mathrm{PSL}(2, \mathrm{O_d})$ *is a link group (in* S^3*) if and only if:*

$$(d, n) \in \{(1, 2), (2, 2), (3, 2), (7, 2), (1, 3), (3, 3)\}.$$

We close the Introduction by outlining the plan of the paper. In §2 we give some preliminary discussion for the methods we use to identify or rule out principal congruence groups from being link groups. In §3 we prove Theorem 1.1. This includes references to the previously known principal congruence link complements, as well as pictures of some of the links. In §4 we prove Theorem 1.2, and in §5 we discuss where these results leave the enumeration of all principal congruence groups that are link groups. Finally, in §6 we provide some discussion focused around the open question as to whether there are only finitely many congruence link complements in S^3.

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2 Preliminaries and discussion of proofs

In this section we gather some facts and background to be used, as well as discussing strategies involved in the proofs of Theorem 1.1 and 1.2.

2.1

We begin by recalling the orders of the groups $\mathrm{PSL}(2, \mathcal{R})$ where $\mathcal{R}$ is a finite ring of the form O_d/I, with $I \subset O_d$ an ideal (see [14]). For such an ideal I we have a decomposition into powers of prime ideals. Then,

$$|\mathrm{PSL}(2, \mathrm{O_d}/\mathrm{I})| = \begin{cases} N(I)^3 \prod_{\mathcal{P}|I}(1 - \frac{1}{N(\mathcal{P})^2}), & \text{when } \ 2 \in I \\ \frac{N(I)^3}{2} \prod_{\mathcal{P}|I}(1 - \frac{1}{N(\mathcal{P})^2}), & \text{otherwise} \end{cases}$$

where $N(I) = |O_d/I|$ denotes the norm of the ideal I.

2.2

The solution of the Cuspidal Cohomology Problem (see [33]) showed that there are only finitely many possible d (see below) so that Q_d has a cover homeomorphic to an arithmetic link complement in S^3. Moreover in [5] it was shown that for every such d there does exist a link complement. We summarize this in the following result:

Theorem 2.1. *Q_d is covered by an arithmetic link complement in S^3 if and only if*

$$d \in \{1, 2, 3, 5, 6, 7, 11, 15, 19, 23, 31, 39, 47, 71\}.$$

Although there is a unique arithmetic knot complement in S^3 (the figure-eight knot complement, [26]), it is easy to prove that there are infinitely many non-homeomorphic arithmetc link complements in S^3, even fixing links with 2 components (see for example [26]).

More generally, for a fixed closed orientable 3-manifold N, there are at most finitely many d for which an arithmetic link complement $N \setminus L$ covers some Q_d. To discuss the reason for this we recall the following description of *the degree 1 cuspidal cohomology*.

Suppose that $X = \mathbf{H}^3/\Gamma$ is an orientable, non-compact, finite volume hyperbolic 3-orbifold, and U_Γ the normal subgroup of Γ generated by the parabolic elements of Γ. Then the subspace of $H_1(X, \mathbf{Q})$ which defines the degree 1 cuspidal cohomology of X (or Γ) can be identified with:

$$V_X(\text{ or } V_\Gamma) = (\Gamma/U_\Gamma)^{\mathrm{ab}} \otimes_{\mathbf{Z}} \mathbf{Q}.$$

If now $X = N \setminus L$, is as above, then the dimension of V_X can be shown to be bounded above by $\dim \mathrm{H}_2(\mathrm{N}; \mathbf{Q})$, whilst on the other hand, it is known that (see [18]) as $d \to \infty$ the dimension of V_{Q_d} goes to infinity.

Thus we deduce the following corollary of this discussion.

Corollary 2.2. *Suppose that N is a fixed closed orientable 3-manifold. If $N \setminus L = \mathbf{H}^3/\Gamma(I)$ then the there are at most finitely many d such that $I \subset O_d$.*

We now prove the finiteness of principal congruence link complements in a fixed manifold.

Proposition 2.3. *Let N be a closed orientable 3-manifold. Then there are only finitely many principal congruence link complements in N.*

Proof: Corollary 2.2 shows that to establish finiteness for principal congruence manifolds, we need to prove finitely many possible levels, which in turn reduces to bounding the norm of the ideal.

We will discuss the case when N is not hyperbolic (which is straightforward). The case when N is hyperbolic is proved in [24] (see Corollary 4.2) and is a good deal more delicate. We shall give a proof that is the motivation for that in [24]. A different proof in the case of when Q_d has 1 cusp is given in [16] (see §4.1 for some discussion of this proof).

The proof is a consequence of the next two results, the first of which is proved in [1]. To state this result, recall that if M is a complete hyperbolic n-manifold of finite volume, by a *systole* of M we mean a shortest closed geodesic in M. By the *systole length* of M we mean the length of a systole. We denote this by sys(M).

Theorem 2.4. *Let N be a closed orientable 3-manifold which does not admit any Riemannian metric of negative curvature. Let L be a link in N whose complement admits a complete hyperbolic structure of finite volume. Then* $\mathrm{sys}(\mathrm{N} \setminus \mathrm{L}) \leq 7.35534...$

Lemma 2.5. *Let $\gamma \in \Gamma(I)$ be a hyperbolic element. Then* $\mathrm{tr}\ \gamma = \pm 2 \bmod \mathrm{I}^2$.

Proof: Since $\gamma \in \Gamma(I)$, it has the form

$$\gamma = \begin{pmatrix} \pm 1 + a & b \\ c & \pm 1 + d \end{pmatrix},$$

where $a, b, c, d \in I$. Thus tr $\gamma = \pm 2 + a + d$. In addition, since $\det \gamma = 1$, expanding we have

$$(\pm 1 + a)(\pm 1 + d) - bc = 1$$

from which it follows that

$$\pm(a + d) = -ad + bc$$

and the right hand side is easily seen to lie in I^2 as required. $\square$

To complete the proof of Proposition 2.3, Lemma 2.5 shows that if γ is a hyperbolic element corresponding to a systole of $\mathbf{H}^3/\Gamma(I)$ then tr $\gamma = \pm 2 + x$ for some $x \in I^2$. By assumption, N is not hyperbolic and so the systole bound of Theorem 2.4 applies to give a bound on $|x|$ (independent of N). This in turn bounds the norm of I and the finiteness is proved. $\square$

Remark: For the explicit systole bound given in Theorem 2.4, the bound for the norm of the ideal produced is 42.

2.3

We will make use of the following result. This is modelled on the case of $\mathrm{PSL}(2, \mathbf{Z})$ which is proved in [28] (see Lemma 1.42).

Recall that the set of fixed points of parabolic elements of $\mathrm{PSL}(2, \mathrm{O_d})$ coincides with $\mathbf{Q}(\sqrt{-d}) \cup \{\infty\}$, and that the number of orbits for the action of $\mathrm{PSL}(2, \mathrm{O_d})$ on the set of fixed points of parabolic elements is h_d (the class number of $\mathbf{Q}(\sqrt{-d})$), which is therefore the number of cusps of Q_d.

Theorem 2.6. *Suppose that $h_d = 1$, and let $c, c', d, d' \in O_d$ satisfy $(c, d) = (c', d') = 1$ (i.e the ideals $< c, d >=< c', d' >= O_d$) . Then d/c and d'/c' are equivalent modulo $\Gamma(I) < \mathrm{PSL}(2, \mathrm{O_d})$ if and only if*

$$\begin{bmatrix} d \\ c \end{bmatrix} = k \begin{bmatrix} d' \\ c' \end{bmatrix} \mod \mathrm{I},$$

where k is a unit of O_d.

Proof: Suppose that $I =< \pi >$, and $T \in \Gamma(I)$ so that $T(d/c) = d'/c'$. This determines an equation of the form $\pm(\frac{d+\pi_1}{c+\pi_2}) = \frac{d'}{c'}$, for $\pi_1, \pi_2 \in I$. This in turn implies that there exists $\lambda \in \mathbf{Q}(\sqrt{-d})^*$ with

$$\lambda \begin{bmatrix} d' \\ c' \end{bmatrix} = \begin{bmatrix} d + \pi_1 \\ c + \pi_2 \end{bmatrix}.$$

Writing $\lambda = m/n$ with $m, n \in O_d$ and $(m, n) = 1$ we deduce that

$$m \begin{bmatrix} d' \\ c' \end{bmatrix} = n \begin{bmatrix} d + \pi_1 \\ c + \pi_2 \end{bmatrix}.$$

Hence $< d' >$ and $< c' >$ are divisible by $< n >$ and so n is a unit by assumption. Now repeating this argument with $T^{-1}(m \begin{bmatrix} d' \\ c' \end{bmatrix}) = \begin{bmatrix} d \\ c \end{bmatrix}$ we also deduce that m is a unit and the required congruence condition is established.

Now assume that the congruence condition holds, we will construct an element $T \in \Gamma(I)$ with $T(d/c) = d'/c'$ as follows. Assume first that $d/c = 1/0$. Then $d' = k$ mod I, $c' = 0$ mod I and so $k^{-1}d' = 1$ mod I, and $k^{-1}c' = 0$ mod I. Using $k^{-1}d' = 1$ mod I, we deduce that $(1-k^{-1}d')/\pi \in O_d$, and so we can find $x, y \in O_d$ so that $d'x - c'y = (1-d')/\pi$. Now choose T to be the matrix

$$T = \begin{pmatrix} k^{-1}d' & ky\pi \\ k^{-1}c' & 1+kx\pi \end{pmatrix} \in \Gamma(I).$$

As can be checked, this has determinant 1.

For the general case, let $x, y \in O_d$ satisfy $dx+cy = 1$ and let $U = \begin{pmatrix} d & -y \\ c & x \end{pmatrix}$. Then $U(1/0) = d/c$ and so the assumption of the congruence condition gives

$$U^{-1}\begin{bmatrix} kd' \\ kc' \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ mod I},$$

for unit k. From above, one can find an element $T \in \Gamma(I)$ so that

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = U^{-1}\begin{bmatrix} kd' \\ kc' \end{bmatrix}.$$

Then $UTU^{-1} \in \Gamma(I)$ is the required element. $\square$

Also note that the number of cusps of $\mathbf{H}^3/\Gamma(I)$ can be computed for any given I. In the case of $h_d = 1$, this follows directly using the order of the finite group $\mathrm{PSL}(2, \mathrm{O_d}/\mathrm{I})$ given in §2.1 and the image of the peripheral subgroup P_∞ (i.e. the subgroup of $\mathrm{PSL}(2, \mathrm{O_d})$ consisting of those elements fixing ∞) in $\mathrm{PSL}(2, \mathrm{O_d}/\mathrm{I})$. When $d \neq 1, 3$, $P_\infty \cong \mathbf{Z} \oplus \mathbf{Z}$, and when $d = 1, 3$, P_∞ is an extension of $\mathbf{Z} \oplus \mathbf{Z}$ by a group of order 2 or 3. Thus when $d \neq 1, 3$, the image of P_∞ in $\mathrm{PSL}(2, \mathrm{O_d}/\mathrm{I})$ is either a cyclic or a rank 2 finite abelian group, and when $d = 1, 3$, the image is an extension of a cyclic or a rank 2 finite abelian group.

2.4

The proof of Theorems 1.1 and 1.2 will make use of the following presentations from [29] for the Bianchi groups in the cases for $d \in \{1, 2, 3, 7, 11, 19\}$. Note that in these cases $h_d = 1$, and so Q_d has one cusp.

$$\mathrm{PSL(2, O_1)} = <\mathrm{a}, \ell, \mathrm{t}, \mathrm{u} \mid \ell^2 = (\mathrm{t}\ell)^2 = (\mathrm{u}\ell)^2 = (\mathrm{a}\ell)^2 = \mathrm{a}^2 = (\mathrm{ta})^3 = (\mathrm{ua}\ell)^3 = 1, [\mathrm{t}, \mathrm{u}] = 1>,$$

$$\mathrm{PSL(2, O_2)} = <\mathrm{a}, \mathrm{t}, \mathrm{u} \mid \mathrm{a}^2 = (\mathrm{ta})^3 = (\mathrm{au}^{-1}\mathrm{au})^2 = 1, [\mathrm{t}, \mathrm{u}] = 1>,$$

$$\mathrm{PSL(2, O_3)} = <\mathrm{a}, \ell, \mathrm{t}, \mathrm{u} \mid \ell^3 = \mathrm{a}^2 = (\mathrm{a}\ell)^2 = (\mathrm{ta})^3 = (\mathrm{ua}\ell)^3 = 1, \ell^{-1}\mathrm{t}\ell = \mathrm{t}^{-1}\mathrm{u}^{-1}, \ell^{-1}\mathrm{u}\ell = \mathrm{t}, [\mathrm{t}, \mathrm{u}] = 1>,$$

$$\mathrm{PSL(2, O_7)} = <\mathrm{a}, \mathrm{t}, \mathrm{u} \mid \mathrm{a}^2 = (\mathrm{ta})^3 = (\mathrm{atu}^{-1}\mathrm{au})^2 = 1, [\mathrm{t}, \mathrm{u}] = 1>,$$

$$\mathrm{PSL(2, O_{11})} = <\mathrm{a}, \mathrm{t}, \mathrm{u} \mid \mathrm{a}^2 = (\mathrm{ta})^3 = (\mathrm{atu}^{-1}\mathrm{au})^3 = 1, [\mathrm{t}, \mathrm{u}] = 1>,$$

$$\mathrm{PSL(2, O_{19})} = <\mathrm{a}, \mathrm{b}, \mathrm{t}, \mathrm{u} \mid \mathrm{a}^2 = (\mathrm{ta})^3 = \mathrm{b}^3 = (\mathrm{bt}^{-1})^3 = (\mathrm{ab})^2 = (\mathrm{at}^{-1}\mathrm{ubu}^{-1})^2 = 1, [\mathrm{t}, \mathrm{u}] = 1>.$$

In addition in all cases $a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $u = \begin{pmatrix} 1 & \omega_d \\ 0 & 1 \end{pmatrix}$ (with the obvious abuse of notation between SL and PSL) and where

$$\omega_d = i, \sqrt{-2}, \frac{-1+\sqrt{-3}}{2}, \frac{1+\sqrt{-7}}{2}, \frac{1+\sqrt{-11}}{2}, \frac{1+\sqrt{-19}}{2}.$$

2.5

To establish the new examples of principal congruence link complements in Theorem 1.1, we employ two different strategies. First, in two cases, explicit links in S^3 are identified in the literature, and we prove them to be principal congruence links. Second, and more usually, we are unable to describe an explicit link, but use experiment and computation to prove that certain principal congruence subgroups determine link groups in S^3. The outline for this is as follows, and is based on three facts about link complements and their groups that we now recall.

Let $L = L_1 \cup \ldots \cup L_n \subset S^3$ be a link, $X(L)$ denote the exterior of L, and $\Gamma = \pi_1(S^3 \setminus L)$ be the link group. Then:

1. Γ^{ab} *is torsion-free of rank equal to the number of components of L; i.e.* $\Gamma^{\mathrm{ab}} \cong \mathbf{Z}^n$.

2. Γ *is generated by parabolic elements.*

3. *For each component L_i, there is a curve $x_i \subset \partial X(L)$ so that Dehn filling $S^3 \backslash L$ along the totality of these curves gives S^3. Following Perelman's resolution of the Geometrization Conjecture, this can be rephrased as saying that the group obtained by setting $x_i = 1$ for each i is the trivial group.*

Given this, our method is:

Step 1: *Show that* $\Gamma(I)$ *is generated by parabolic elements.*

We briefly discuss how this is done. Let $P = P_\infty(I)$ be the peripheral subgroup $\mathrm{P}_\infty \cap \Gamma(I)$, and let $< P >$ denote the normal closure in $\mathrm{PSL}(2, \mathrm{O_d})$. Since $\Gamma(I)$ is a normal subgroup of $\mathrm{PSL}(2, \mathrm{O_d})$, then $< P >< \Gamma(I)$. Thus if $< P >= \Gamma(I)$ then $\Gamma(I)$ is generated by parabolic elements. Note that the converse also holds in this case (i.e. $h_d = 1$). For if $\Gamma(I)$ is generated by parabolic elements, then since $\Gamma(I)$ is a normal subgroup and Q_d has 1 cusp, all such generators are $\mathrm{PSL}(2, \mathrm{O_d})$-conjugate into P.

From §2.1 we know the order of $\mathrm{PSL}(2, \mathrm{O_d/I})$, and we can use Magma [8] to test whether $\Gamma(I) =< P >$. In some cases Magma is unable to decide whether the index is finite, and so is of no help here. In addition Magma can also return an index greater than $[\mathrm{PSL}(2, \mathrm{O_d}) : \Gamma(\mathrm{I})]$. In this case we can prove that $\Gamma(I)$ is not a link group. To see this we argue as follows.

If $\Gamma(I)$ is a link group then it is normally generated by its peripheral subgroups. On the other hand, since Q_d has 1 cusp, all the peripheral subgroups of $\Gamma(I)$ are $\mathrm{PSL}(2, \mathrm{O_d})$-conjugate to P. If $< P >$ is a proper subgroup of finite index in $\Gamma(I)$, then all the peripheral subgroups of $\Gamma(I)$ lie in a proper subgroup of $\Gamma(I)$. In particular the peripheral subgroups cannot normally generate $\Gamma(I)$.

Step 2: *Find parabolic elements in* $\Gamma(I)$ *so that as above, trivializing these elements, trivializes the group.*

This step is largely done by trial and error, however, the motivation behind our experimentation can be usefully described. Given Step 1, if $\mathbf{H}^3/\Gamma$ has n cusps, we attempt to find n parabolic fixed points that are $\Gamma(I)$-inequivalent, and for which the corresponding parabolic elements of $< P >$ provide curves that can be Dehn filled as in Step 3 above. The inequivalence can be shown using Theorem 2.6. Our search for inequivalent fixed points is informed by what happens for $\mathrm{PSL}(2, \mathbf{Z})$, and additionally we sometimes use an intermediate group $\Gamma(I) < \Gamma < \mathrm{PSL}(2, \mathrm{O_d})$ that is easier to work with.

One upshot of attempting to implement the above strategy is the following question, for which a positive answer would greatly simplify our work.

Question: *Let $M = \mathbf{H}^3/\Gamma$ be a finite volume orientable hyperbolic 3-manifold for which Γ is generated by parabolic elements. Is M homeomorphic to a link complement in S^3?*

3 Proof of Theorem 1.1

We now give the details of how to implement the strategies described in §2.5. Before that we list the known principal congruence link complements with references. We use the notation (d, I) to indicate the Bianchi group and level given in Theorem 1.1. Note that if $\Gamma(<\alpha>)$ is determined to be a link group in S^3, then $\Gamma(<\overline{\alpha}>)$ also is. Therefore, in what follows we simply refer to one of the complex conjugate pair.

3.1

The cases $(1,2)$,$(2,2)$, $(3,2)$ and $(7,2)$ were described in the Ph.D thesis of the first author [3]. The links in question are shown in Figure 1 below.

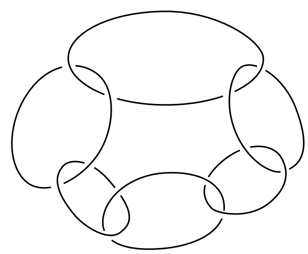

Figure 1a

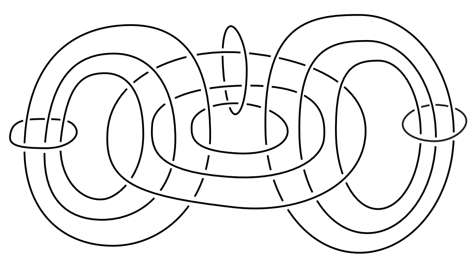

Figure 1b

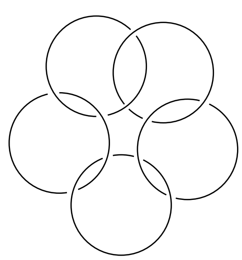

Figure 1c

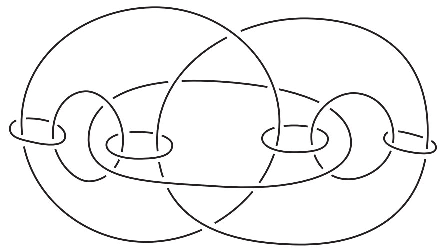

Figure 1d

For $(3,(5+\sqrt{-3})/2)$, this was shown by Thurston and described in [16]. Thurston's link is shown in Figure 2 below.

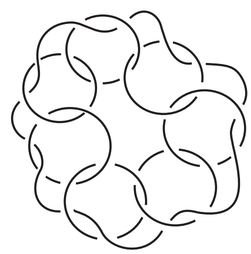

Figure 2

Note that, as ideals $< (5+\sqrt{-3})/2 >=< 2-\sqrt{-3} >$, and so $\mathbf{H}^3/\Gamma(< 2 \pm \sqrt{-3} >)$ is also homeomorphic to a link complement in S^3.

The cases $(3,3)$ and $(3,3+\sqrt{-3})$ are described in Chapter 1 of [16]. These are 12 and 20 component link complements respectively. We refer the reader to [16] for a description of the links.

The case of $(7,(1+\sqrt{-7})/2)$ is described in [20] (the link is shown below in Figure 3, and the complement is known as the Magic Manifold).

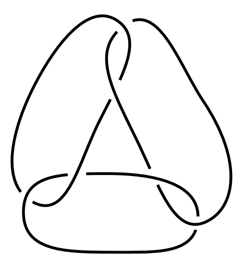

Figure 3

The case of $(19,(1+\sqrt{-19})/2)$ was shown in [5], and those of $(15, < 2, (1+\sqrt{-15})/2 >)$ and $(23, < 2, (1+\sqrt{-23})/2 >)$ in [6]. In the first case, no link was described, but in the other two cases, the following links in Figure 4 were identified.

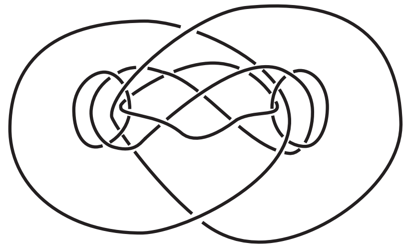

Figure 4a

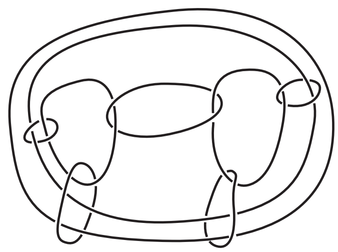

Figure 4b

3.2

We now deal with the cases $(2, 1+\sqrt{-2})$ and $(11, (1+\sqrt{-11})/2)$. Consider the links that are shown below in Figures 5a and 5b.

$L_1 =$

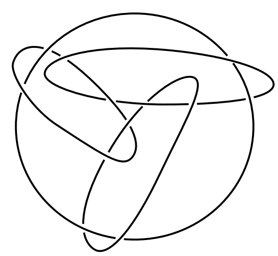

Figure 5a

$L_2 =$

Figure 5b

These links were previously described in Thurston's Notes [31], and [21] respectively, and in both cases, commensurability of the link complement with the Bianchi orbifolds Q_2 and Q_{11} respectively was established in [31] and [21]. Since $\mathbf{Q}(\sqrt{-2})$ and $\mathbf{Q}(\sqrt{-11})$ have class number 1, it follows from [26] Theorem 3, that these link complements cover Q_d. Now either using the calculation of volume in [31] or using SnapPy [10], volume considerations apply to show that the degree of the cover of Q_d in both cases is 12.

In the remainder of this subsection, I denotes either of the ideals $< 1+\sqrt{-2} >$ or $< (1+\sqrt{-11})/2 >$. Now $N(< 1+\sqrt{-2} >) = N(< (1+\sqrt{-11})/2 >) = 3$, so that in both cases $\Gamma(I)$ is a normal subgroup of the Bianchi group of index 12, and the principal congruence manifolds each have 4 cusps (the image of P_∞ is cyclic of order 3).

Case 1, d=2: Table 5 of [20] lists the $\mathrm{PSL}(2,\mathrm{O}_2)$-conjugacy classes of subgroups Γ of $\mathrm{PSL}(2,\mathrm{O}_2)$ of index ≤ 12 with Γ^{ab} being torsion-free. Thus if Γ denotes the link group associated to the link L_1 above, then Γ must appear in Table 5 of [20]. There are 3 groups listed in Table 5 of [20] whose first homology is $\mathbf{Z}^4$. In the notation of [20] these are $\Gamma_{-2}(12,9)$, $\Gamma_{-2}(12,10)$, and $\Gamma_{-2}(12,11)$. As noted in [20], the first two of these are conjugate in $\mathrm{PSL}(2,\mathbf{C})$, and the other group is not torsion-free. Hence the group Γ must be conjugate to one from the first pair.

Also upon consulting Table 5 of [20], one sees that the group $\Gamma_{-2}(12,10)$ is generated by (in the notation of §2.4)

$$\{g_1 = tu, g_2 = t^3, g_3 = atua, g_4 = t^{-1}at^{-1}u^{-1}at\}$$

Now $g_1 = \begin{pmatrix} 1 & 1+\sqrt{-2} \\ 0 & 1 \end{pmatrix}$, g_3, g_4 are conjugates of $g_1^{\pm 1}$ (recall that a has order 2) and so also lie in $\Gamma(I)$. It is also clear that $g_2 \in \Gamma(I)$ (since $N(I) = 3$). Hence, by index considerations, $\Gamma(I)$ coincides with $\Gamma_{-2}(12,10)$, and we deduce that $\Gamma = \Gamma(I)$.

Case 2, d=11: Since [20] does not deal with $d = 11$, we need to argue differently. We begin with an observation. As noted above the link group Γ associated to L_2 is a subgroup of index 12 in $\mathrm{PSL}(2,\mathrm{O}_{11})$. For future use, we note that $\mathrm{Vol}(\mathrm{S}^3 \setminus \mathrm{L}_2) = 16.591299695\ldots$

Claim: Γ *is a normal subgroup of* $\mathrm{PSL}(2,\mathrm{O}_{11})$ *with quotient group* A_4.

Proof of Claim: Using SnapPy the symmetry group of the link complement is order 24 (it is the rotational octahedral group). SnapPy also computes that the link complement has no orientation-reversing isometries. Thus $S^3 \setminus L_2$ is a regular cover of an orbifold Q commensurable with Q_{11}. Using the volume of $S^3 \setminus L_2$ given above we deduce that $\mathrm{Vol(Q)} = 0.691304154192559\ldots$.

Now using the structure of minimal orbifolds in the commensurability class of Q_{11} (see [25] Chapter 11), it can be checked that although there are infinitely many minimal elements in the commensurability class of Q_{11}, there is a unique minimal volume orientable orbifold in the commensurabilty class of Q_{11}, namely $\mathbf{H}^3/\mathrm{PGL}(2,\mathrm{O}_{11})$. It also follows from [25] Chapter 11, that the volume of this orbifold is that of the orbifold Q above. Hence $Q = \mathbf{H}^3/\mathrm{PGL}(2,\mathrm{O}_{11})$ and so it follows that $S^3 \setminus L_2$ is a regular cover of $\mathbf{H}^3/\mathrm{PGL}(2,\mathrm{O}_{11})$, and hence Q_{11} as required. The covering group of $S^3 \setminus L_2 \to Q_{11}$ is then A_4 (since the full group is the octahedral group).

Given this claim we now run the following routine in Magma that takes a group G and outputs all normal subgroups of index 12 with A_4 quotient and for which abelianization is $\mathbf{Z}^4$. In the case at hand, $G = \mathrm{PSL}(2,\mathrm{O}_{11})$ and the presentation used is that given in §2.4.

```
G<a,t,u>:=Group<a,t,u|a^2,(t*a)^3,(a*t*u^-1*a*u)^3,(t,u)>;
L := LowIndexNormalSubgroups(G,12);
for i in [1..#L] do N := L[i]`Group;
```

```
if Index(G,N) eq 12 then Q := CosetImage(G,N);
  if IsIsomorphic(Q,Alt(4)) then if AQInvariants(N) eq [0,0,0,0] then
    N := Rewrite(G,N); print ""; print N; end if; end if;
  end if; end for;
```

Magma outputs only three subgroups, N_1, N_2 and N_3. Hence Γ corresponds to one of these. The generators produced by Magma are listed below:

$$N_1 =< u, aua, tauat^{-1}, t^{-1}auat >, \; N_2 =< ut^{-1}, aut^{-1}a, t^3, tauata >,$$
$$\text{and } N_3 =< tu, at^2u^{-1}a, uauat^{-1}a, auatu^{-1}at^{-1} >.$$

Note that $u \in \Gamma(< (1+\sqrt{-11})/2 >)$ and so since all generators of N_1 are conjugates of u (recall a has order 2) we see that $N_1 < \Gamma(< (1+\sqrt{-11})/2 >)$ and so we must have $N_1 = \Gamma(< (1+\sqrt{-11})/2 >)$.

Similarly, for N_2, it is easy to see that the first three generators in this case all lie in $\Gamma(< (1-\sqrt{-11})/2 >)$. The fourth generator is

$$\begin{pmatrix} (3-\sqrt{-11})/2 & -(1-\sqrt{-11})/2 \\ (1-\sqrt{-11})/2 & (1+\sqrt{-11})/2 \end{pmatrix},$$

which can be checked to be congruent to the identity modulo $< (1-\sqrt{-11})/2 >$. Hence $N_2 = \Gamma(< (1-\sqrt{-11})/2 >)$.

Now the group Γ must appear in the list enumerated by Magma; i.e. it must be one of N_1, N_2 or N_3. We wish to show that Γ is either of N_1 and N_2; i.e. we are done if we can show that $\Gamma \neq N_3$. To do this we look at the first homology groups of double covers of $S^3 \setminus L_2$ and compare with the first homology groups of index 2 subgroups of the group to be eliminated. To that end, using SnapPy, we can check double covers of $S^3 \setminus L_2$ and we see that there is a double cover with first homology group $\mathbf{Z}^4 \oplus \mathbf{Z}/3\mathbf{Z}$. On the other hand, as can be checked with Magma, there is no subgroup of index two in N_3 having this first homology group. This completes the proof.

3.3

We now discuss the remaining cases of Theorem 1.1. In particular, $d \in \{1, 2, 7, 11\}$.

d=1:

Proposition 3.1. *1.* $\Gamma(< 2+i >)$ *is a six component link group.*

2. $\Gamma(< (1+i)^3 >)$ *is a twelve component link group.*

3. $\Gamma(3)$ *is a twenty component link group.*

Proof: We describe the cases of $2+i$ and 3 in some detail. The case of $(1+i)^3$ is similar to that of $2+i$. The relevant information is included at the end of the proof.

First, since $N(< 2+i >) = 5$, $\Gamma(< 2+i >)$ is a normal subgroup of $\mathrm{PSL}(2, \mathrm{O}_1)$ of index 60. Since P_∞ has order 10 in this case, we deduce that $\mathbf{H}^3/\Gamma(< 2+i >)$ has 6 cusps. Recalling Step 1 from §2.5, the subgroup P in this case can be seen to be $< t^2u, t^5 >$. We now use Magma as discussed above (see below for the Magma routine used in this proof) to see that $[\mathrm{PSL}(2, \mathrm{O}_1) :< \mathrm{P} >] = 60$, and so $\Gamma(< 2+i >) =< P >$. Note also that the Magma routine shows that $< P >^{\mathrm{ab}} \cong \mathbf{Z}^6$.

We now find six parabolic elements that are not conjugate in $\Gamma(< 2+i >)$, that generate $\Gamma(< 2+i >)$ and have the property that trivializing these elements trivializes the group.

Lemma 3.2. *Let* $S = \{\infty, 0, \pm 1, \pm 2\}$*. Then each element of* S *is a fixed point of some parabolic element of* $\Gamma(< 2+i >)$ *and moreover they are all mutually inequivalent under the action of* $\Gamma(< 2+i >)$*.*

Proof: That each element of S is a fixed point is clear as $\Gamma(<2+i>)$ has finite index index in $\mathrm{PSL}(2,\mathrm{O}_1)$. An explicit collection of parabolic elements of $\Gamma(<2+i>)$ that fix the elements of S is:

$$S' = \{t^2u, at^2ua, t^{-1}at^2uat, tat^2uat^{-1}, t^{-2}at^2uat^2, t^2at^{-3}uat^{-2}\}.$$

It is also easy to see that ∞ is not equivalent to 0, since any element $T \in PSL(2,\mathbf{C})$ with $T\infty = 0$ has $(1,1)$-entry 0 which is impossible for an element of $\Gamma(<2+i>)$. Similarly ∞ is not equivalent to ± 1 or ± 2. For if $T\infty = \pm 1$, then $a = \pm c$, but $c = 0 \bmod\ <2+\mathrm{i}>$ and $a = 1 \bmod\ <2+\mathrm{i}>$, a contradiction. A similar argument holds for ± 2 (as $<2+i>$ has norm 5).

A similar argument also works to rule out the equivalence of 0 and ± 1 and the equivalence of 0 and ± 2.

Finally an application of Theorem 2.6 rules out the other equivalences. For example, suppose that 1 and 2 are equivalent with $T(1) = 2$ for some $T \in \Gamma(<2+i>)$. This determines an equation:

$$1 + (2+i)\alpha = 2(1 + (2+i)\beta),$$

which implies $1 \in <2+i>$, a contradiction. $\square$

Magma routine for $\Gamma(<2+i>)$

```
G<a,l,t,u>:=Group<a,l,t,u|l^2,a^2,(t*l)^2,(u*l)^2,(a*l)^2,(t*a)^3,(u*a*l)^3,
(t,u)>;
h:=sub<g|t^2*u,t^5>;
> n:=NormalClosure(G,h);
> print Index(G,n);
60
> print AbelianQuotientInvariants(n);
[ 0, 0, 0, 0, 0, 0 ]
> r:=sub<n|t^2*u,a*t^2*u*a,t^-1*a*t^2*u*a*t,t*a*t^2*u*a*t^-1,t^-2*a*t^2*u*a*t^2,
t^2*a*t^-3*u*a*t^-2>;
> print Index(n,r);
1
```

We now deal with case of $\Gamma(3)$, which is a normal subgroup of $\mathrm{PSL}(2,\mathrm{O}_1)$ of index 360. In this case P_∞ maps to a group of order 18 so that $\mathbf{H}^3/\Gamma(3)$ has 20 cusps, and the group P in this case is $<t^3, u^3>$. Magma again checks that $<P> = \Gamma(3)$ (see below).

In this case we find it helpful to work with an intermediate subgroup $\Gamma(3) < \Gamma < \mathrm{PSL}(2,\mathrm{O}_1)$, where Γ is defined to be the group $<\Gamma(3), \delta> = \Gamma(3).\delta$ where $\delta = atu^{-1}$. Magma shows that $[\Gamma : \Gamma(3)] = 5$, and so we may deduce that the cover $\mathbf{H}^3/\Gamma(3) \to \mathbf{H}^3/\Gamma$ is a regular 5-fold cyclic cover with $\mathbf{H}^3/\Gamma$ having four cusps, and each cusp of $\mathbf{H}^3/\Gamma(3)$ projecting one-to-one to a cusp of $\mathbf{H}^3/\Gamma$.

As in the previous case we need to determine appropriate parabolic elements. We briefly discuss how this tedious but straightforward computation is done, and spare the reader details.

First, the four parabolic fixed points ∞, ± 1 and $1-i$ (the set of which we again denote by S) can be shown all to be mutually inequivalent under the action of Γ. For since $\gamma \in \Gamma$, then $\gamma = \gamma_0\delta^n$ for some $\gamma_0 \in \Gamma(3)$ and $n \in \mathbf{Z}$. Furthermore, from the previous paragraph, $\delta^5 \in \Gamma(3)$ and so it suffices to check, using Theorem 2.6, that for $n \in \{0, \pm 1, \pm 2\}$, the δ^n-orbits of the four elements of S are all $\Gamma(3)$-inequivalent.

Now the following parabolic elements in Γ fix these four points:

$$S' = \{t^3u^3, tat^3u^{-3}at^{-1}, t^{-1}au^3at, u^{-1}tau^3at^{-1}u\}.$$

As can be readily checked, these are primitive parabolic elements in Γ.

Magma now shows that the normal closure of S' in Γ is $\Gamma(3)$. Since the parabolic elements listed above represent inequivalent cusps of $\mathbf{H}^3/\Gamma$, if we now perform Dehn filling on $\mathbf{H}^3/\Gamma$ along the curves corresponding to these parbolic elements, the normal closure computation shows that we obtain a group of order 5. Since these are primitive parabolic elements, this group is the fundamental group of a closed 3-manifold, namely some Lens Space L (by Geometrization). Hence we deduce that $\mathbf{H}^3/\Gamma$ is a 4 component link in L with fundamental group of order 5. From above we can compatibly fill the cusps of $\mathbf{H}^3/\Gamma(3) \to \mathbf{H}^3/\Gamma$ resulting in a 5-fold cover $N \to L$, and so $N \cong S^3$ as required.

Magma routine for $\Gamma(3)$

```
G<a,l,t,u>:=Group<a,l,t,u|l^2,a^2,(t*l)^2,(u*l)^2,(a*l)^2,(t*a)^3,(u*a*l)^3,
(t,u)>;
h:=sub<g|t^3,u^3>;
> n:=NormalClosure(G,h);
> print Index(G,n);
360
print AbelianQuotientInvariants(n);
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
> d:=sub<G|n,a*t*u^-1>;
> print Index(G,d);
72
> print AbelianQuotientInvariants(d);
[ 5, 0, 0, 0, 0 ]
> d1:=sub<d|t^3*u^3,t*a*t^3*u^-3*a*t^-1,t^-1*a*u^3*a*t,u^-1*t*a*u^3*a*t^-1*u>;
> d2:=NormalClosure(d,d1);
> print Index(d,d2);
5
> d2 eq n;
true
```

We now sketch some of the details for the case of $(1+i)^3$. In this case $\Gamma(<(1+i)^3>)$ has index 192, and $\mathbf{H}^3/\Gamma(<(1+i)^3>)$ has 12 cusps. The subgroup P to be used here is $< t^2u^2, t^4 >$, and Magma confirms that $< P >= \Gamma(< (1+i)^3 >)$. Using Theorem 2.6 it can be shown that the following twelve parabolic fixed points are inequivalent under $\Gamma(< (1+i)^3 >)$:

$$\{\infty, 0, 1/2, \pm 1, 2, \pm i, 1 \pm i, (1 \pm i)/2\}$$

Parabolic elements were then constructed and the Dehn filling argument as in the previous cases applied. The Magma routine is shown below:

Magma routine for $\Gamma(< (1+i)^3 >)$

```
G<a,l,t,u>:=Group<a,l,t,u|l^2,a^2,(t*l)^2,(u*l)^2,(a*l)^2,(t*a)^3,(u*a*l)^3,
(t,u)>;
> h:=sub<G|t^2*u^2,t^4>;
> n:=NormalClosure(G,h);
> print Index(G,n);
192
> print AbelianQuotientInvariants(n);
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
s:=sub<G|t^2*u^2,a*t^-2*u^2*a,t^2*a*t^2*u^2*a*t^-2,t*a*t^2*u^2*a*t^-1,
```

```
t^-1*a*t^2*u^2*a*t, u*a*t^2*u^2*a*u^-1,u^-1*a*t^2*u^2*a*u,
t*u*a*u^4*a*u^-1*t^-1,t*u^-1*a*t^4*a*u*t^-1,a*u*t^-1*a*t^2*u^2*a*t*u^-1*a,
a*u^-1*t^-1*a*t^2*u^2*a*t*u*a,a*t^-2*a*t^2*u^2*a*t^2*a>;
> print Index(s,n);
1
> s eq n;
true
```

This completes the proof of Proposition 3.1. □

d=2:

Proposition 3.3. $\Gamma(< 2+\sqrt{-2} >)$ *is a twelve component link group.*

Proof: The proof is similar to that of Case 3 of Proposition 3.1 and so we only include relevant information. Since $< 2+\sqrt{-2} >$ has norm 6, the principal congruence subgroup $\Gamma(< 2+\sqrt{-2} >)$ is a normal subgroup of index 72. In this this case the image of P_∞ has order 6 and so $\mathbf{H}^3/\Gamma(< 2+\sqrt{-2} >)$ has 12 cusps. Taking $P =< t^2u, t^6 >$, and as before Magma confirms that $< P >= \Gamma(< 2+\sqrt{-2} >)$ (see below).

As in Case 3 of Proposition 3.1, we find it convenient to work with a group Γ where $\Gamma(< 2+\sqrt{-2} >) < \Gamma < \mathrm{PSL}(2, O_2)$ and $[\Gamma : \Gamma(< 2+\sqrt{-2} >)] = 2$; namely $\Gamma =< \Gamma(< 2+\sqrt{-2} >), a >$. As before we find a collection of parabolic elements in Γ whose normal closure in Γ is $\Gamma(< 2+\sqrt{-2} >)$. This is illustrated in the Magma routine shown below:

Magma routine for $\Gamma(< 2+\sqrt{-2} >)$

```
> G<a,t,u>:=Group<a,t,u|a^2,(t*a)^3,(a*u^-1*a*u)^2,(t,u)>;
> h:=sub<G|t^2*u,t^6>;
> n:=NormalClosure(G,h);
> print Index(G,n);
72
> print AbelianQuotientInvariants(n);
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
> d:=sub<G|t^2*u,t*a*t^2*u*a*t^-1,t^2*a*t^2*u*a*t^-2,t^-2*a*t^2*u*a*t^2,
t^3*a*t^-4*u*a*t^-3,t*a*t^-2*a*t^-4*u*a*t^2*a*t^-1>;
> m:=sub<G|n,a>;
> print Index(G,m);
36
> d2:=NormalClosure(m,d);
> print Index(m,d2);
2
> d2 eq n;
true
```

Arguing as before we see that the 2-fold cover $\mathbf{H}^3/\Gamma(< 2+\sqrt{-2} >) \to \mathbf{H}^3/\Gamma$ can be Dehn filled compatibly to extend to a cover $S^3 \to \mathbf{RP}^3$, and the proposition is proved. □

d=7:

Proposition 3.4. *1.* $\Gamma(< (3+\sqrt{-7})/2 >)$ *is a six component link group.*

2. $\Gamma(< 1+\sqrt{-7} >)$ *is an eighteen component link group.*

Proof: The first case is dealt with exactly as some of the previous cases. Here $\Gamma(<(3+\sqrt{-7})/2>)$ is a normal subgroup of index 24, and $\mathbf{H}^3/\Gamma(<(3+\sqrt{-7})/2>)$ has 6 cusps. The Magma routine is included below.

Magma routine for $\Gamma(<(3+\sqrt{-7})/2)>)$

```
> G<a,t,u>:=Group<a,t,u|a^2,(t*a)^3,(a*t*u^-1*a*u)^2,(t,u)>;
> h:=sub<G|t*u,t^4>;
> n:=NormalClosure(G,h);
> print Index(G,n);
24
> s:=sub<G|t*u,a*t*u*a,t*a*t*u*a*t^-1,t^-1*a*t*u*a*t,t^2*a*t*u*a*t^-2,a*t^2*a*\
t^-3*u*a*t^-2*a>;
> print Index(G,s);
24
> s eq n;
true
```

The second case of Proposition 3.4 involves some additional work as we now describe. The ideal $<1+\sqrt{-7}>$ has norm 8 and factorizes as $<\omega_7>^2<\overline{\omega_7}>$. Thus $[\mathrm{PSL}(2,\mathrm{O_d}):\Gamma(<1+\sqrt{-7}>)]=144$ and the image of P_∞ has order 8. Hence $\mathbf{H}^3/\Gamma(<1+\sqrt{-7}>)$ has 18 cusps. From the Magma routine shown below we see that $\Gamma(<1+\sqrt{-7}>)=<u^2,t^4>$

Now consider the group $\Delta=<\Gamma(<1+\sqrt{-7}>),x>$ where $x=at^{-2}a$. Magma shows that $[\Delta:\Gamma(<1+\sqrt{-7}>)]=2$, and so we have the following sequence of 2-fold covers.

$$\mathbf{H}^3/\Gamma(<1+\sqrt{-7}>)\to\mathbf{H}^3/\Delta\to\mathbf{H}^3/\Gamma(2).$$

To show that $\mathbf{H}^3/\Gamma(<1+\sqrt{-7}>)$ is homeomorphic to a link complement in S^3 we show that $\mathbf{H}^3/\Delta$ homeomorphic to a link complement in $\mathbf{RP}^3$, and the double cover is compatible with $S^3\to\mathbf{RP}^3$.

As before this is done by exhibiting a collection of parabolic elements that normally generate, and we refer the reader to the Magma routine below for this list. Now $\mathbf{H}^3/\Delta$ has twelve cusps and in the double cover $\mathbf{H}^3/\Gamma(<1+\sqrt{-7}>)\to\mathbf{H}^3/\Delta$ six of these cusps each lift to two cusps of $\mathbf{H}^3/\Gamma(<1+\sqrt{-7}>)$ and each of the remaining six is covered by a single a cusp of $\mathbf{H}^3/\Gamma(<1+\sqrt{-7}>)$. These cusps fill compatibly to give the required cover $S^3\to\mathbf{RP}^3$.

Magma routine for $\Gamma(<1+\sqrt{-7}>)$

```
> G<a,t,u>:=Group<a,t,u|a^2,(t*a)^3,(a*t*u^-1*a*u)^2,(t,u)>;
> h:=sub<G|u^2,t^4>;
> n:=NormalClosure(G,h);
> print Index(G,n);
144
> print AbelianQuotientInvariants(n);
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
> m1:=sub<G|n,a*t^2*a>;
> print Index(G,m1);
72
> m2:=sub<m1|u^2,a*(t^-4*u^2)*a,t*a*u^2*a*t^-1,t^2*a*u^2*a*t^-2,u*a*u^2*a*u^-1\
,t^-1*u*a*u^2*a*u^-1*t,t^-2*u*a*u^2*a*u^-1*t^2,a*t^-1*u*a*u^2*a*u^-1*t*a,t^-1*\
a*t^-1*u*a*u^2*a*u^-1*t*a*t,t*a*t^-1*u*a*u^2*a*u^-1*t*a*t^-1,a*u*a*u^2*a*u^-1*\
a,a*t^-1*a*t^-1*u*a*u^2*a*u^-1*t*a*t*a>;
```

```
> d:=NormalClosure(m1,m2);
> print Index(m1,d);
2
> d eq n;
true
```

d=11:

Proposition 3.5. $\Gamma(< (3+\sqrt{-11})/2 >)$ *is a twelve component link group.*

Proof: In this case $\Gamma(< (3+\sqrt{-11})/2 >)$ is a normal subgroup of $\mathrm{PSL}(2,\mathrm{O}_{11})$ of index 60, and $\mathbf{H}^3/\Gamma(< (3+\sqrt{-11})/2 >)$ has 12 cusps. Setting $P =< tu, t^5 >$, Magma shows that $< P >= \Gamma(< (3+\sqrt{-11})/2 >)$. As in some of the other cases we will exploit an intermediate group $\Gamma =< \Gamma(< (3+\sqrt{-11})/2 >), a >$, so that $[\Gamma : \Gamma(< (3+\sqrt{-11})/2 >)] = 2$. We then argue as in Proposition 3.3. The Magma routine illustrates this.

Magma routine for $\Gamma(< (3+\sqrt{-11})/2) >)$

```
> G<a,t,u>:=Group<a,t,u|a^2,(t*a)^3,(a*t*u^-1*a*u)^3,(t,u)>;
> h:=sub<G|t*u,t^5>;
> n:=NormalClosure(G,h);
> print Index(G,n);
60
> print AbelianQuotientInvariants(n);
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 ]
> d:=sub<G|n,a>;
> print Index(G,d);
30
> m:=sub<d|t*u,t*a*t*u*a*t^-1,t^2*a*t^3*u^-2*a*t^-2,a*t^-2*a*t*u*a*t^2*a,t*a*t\
^-2*a*t*u*a*t^2*a*t^-1,t^2*a*t^2*a*t^6*u*a*t^-2*a*t^-2>;
> M:=NormalClosure(d,m);
> print Index(d,M);
2
> M eq n;
true
```

This completes the proof of Proposition 3.5 and the proof of Theorem 1.1. $\square$

4 Proof of Theorem 1.2

In this section we prove Theorem 1.2. Since it will be useful in what follows in this section and in §5, we begin with some comments on the argument used in [16] to establish finiteness.

4.1

Suppose that $d \in \{1,2,3,7,11,19\}$ so that Q_d has one cusp. Assume that $S^3 \setminus L = \mathbf{H}^3/\Gamma(I)$ is a principal congruence link complement. Then, following our earlier notation, $P_\infty = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in I\}$.

For every cusp C of $\mathbf{H}^3/\Gamma(I)$, we have a cusp torus T_C equipt with a Euclidean metric. Now, since $\mathbf{H}^3/\Gamma(I)$ is a link complement in S^3, the 6-Theorem (see [2] Theorem 6.2, [22] Theorem 3.1

and [16] Lemma 1.6.1) implies that there is some cusp C for which there is a geodesic on T_C whose length in the Euclidean metric is < 6. Moreover, since $S^3 \setminus L$ is a regular cover of Q_d (with one cusp) it follows that for *all* cusps C, the tori T_C have a geodesic with length < 6.

Referring now to the cusp C_∞ associated with P_∞, the length of the geodesic η on T_{C_∞} can be computed as follows. There is a horopshere $\mathcal{H}_\infty$ at height h say so that P_∞ acts by translations on $\mathcal{H}_\infty$ and the length of η is $|z|/h$ for some $z \in I$. From above we have $|z|/h < 6$.

If we now take h to be the height of a maximal horosphere $\mathcal{H}_\infty$, then $h \le 1$ (see [7] Lemma 2.5 for example), and so $|z| < 6$. There are only finitely many such z.

Remark: The 6-theorem of [2] and [22] usually gives $|z| \le 6$. However as is pointed out in [16], in the case of S^3 (or more generally finite fundamental group) the proof can be improved to show that $|z| < 6$ (since the cores of the Dehn fillings still have infinite order).

4.2

For those (d, n) stated in Theorem 1.2, it follows from Theorem 1.1 that these do provide principal congruence link complements in S^3. It remains to show that these are the only ones. To that end we recall the following result from [4] that places further restrictions on the list of possible d's stated in Theorem 2.1.

Theorem 4.1. *If $h_d > 1$, and $\Gamma(n) < \mathrm{PSL}(2, \mathrm{O_d})$, then $\mathbf{H}^3/\Gamma(n)$ is not homeomorphic to a link complement in S^3.*

We deduce the following corollary of this result together with discussion in §4.1.

Corollary 4.2. *Suppose that $\Gamma(n) < \mathrm{PSL}(2, \mathrm{O_d})$ and $\mathbf{H}^3/\Gamma(n)$ is homeomorphic to a link complement in S^3. Then $d \in \{1, 2, 3, 7, 11, 19\}$ and $n \in \{2, 3, 4, 5\}$.*

The proof will be completed by a combination of methods as detailed in the following subsections.

4.3

First, as in the proof of Theorem 1.1, we can quickly eliminate some pairs (d, n) by using Magma to show (in the notation of §2.5) that $< P > \neq \Gamma(n)$. The table below shows the cases for which this works. In this table, N is a normal subgroup of $\mathrm{PSL}(2, \mathrm{O_d})$ that contains the group $< P >$.

d	n	N	**Order of** $\mathrm{PSL}(2, \mathrm{O_d})/\mathrm{N}$	**Order of** $\mathrm{PSL}(2, \mathrm{O_d}/\mathrm{I})$
2	3	$< t^3, u^3, (u^2at^2)^6 >$	2304	288
3	4	$< t^4, u^4 >$	3840	1920
7	3	$< t^3, u^3 >$	1080	360
11	2	$< t^2, u^2 >$	120	60
11	4	$< t^4, u^4, (t^2au^2)^2 >$	7680	1920

Remark: The cases of $(7, 3)$ and $(11, 2)$ can also be handled by the arguments in §4.4.

4.4

We now deal with the case when the level is divisible by an inert rational prime or certain split primes. Recall that if $p \in \mathbf{Z}$ then p is called *inert* if the O_d-ideal $< p >$ remains prime, and p is said to *split* if the O_d-ideal $< p >= \mathcal{P}_1\mathcal{P}_2$.

Proposition 4.3. *Assume that* $d \neq 1, 3$, *then* $\Gamma(p) < \mathrm{PSL(2, O_d)}$ *is not a link group in the following two cases:*

- *p is an inert prime in O_d.*
- *$p \geq 5$ splits in O_d.*

Proof: We begin with some comments relevant for both. For $d \in \{2, 7, 11, 19\}$, $H_1(Q_d; \mathbf{Z}) \cong \mathbf{Z} \oplus T$ where T is finite. Thus for all integers $n > 1$ we have epimorphisms

$$\phi_n : \mathrm{PSL(2, O_d)} \to \mathbf{Z} \to \mathbf{Z}/\mathrm{n}\mathbf{Z}.$$

Let $\Delta_n = \mathrm{ker}\phi_\mathrm{n}$. Now for $d \in \{2, 7, 11, 19\}$ inspection of the presentations in §2.4 shows that on abelianizing, the image of u has infinite order and on projecting to $\mathbf{Z}$ the image of t is trivial. Thus $\alpha^n \in \Delta_n$ for all parabolic elements $\alpha \in \mathrm{PSL(2, O_d)}$.

If now p is a prime and $\Gamma(p) < \mathrm{PSL(2, O_d)}$ is a link group then $\Gamma(p)$ is generated by parabolic elements, and so we deduce from this and the previous paragraph that $\Gamma(p) < \Delta_p$.

When p is inert, we therefore have $\mathrm{PSL(2, O_d)}/\Delta_\mathrm{p}$ a normal subgroup of $\mathrm{PSL(2, O_d/ < p >)}$, and this latter group is always a finite simple group in this case, a contradiction.

When $p \geq 5$ splits, say $p = \pi\overline{\pi}$, then a composition series for $G = \mathrm{PSL(2, O_d/ < p >)} = \mathrm{PSL(2, O_d)/\Gamma(p)}$ is

$$G = G_0 > G_1 > G_2 > G_3 = 1,$$

where $G_0/G_1 \cong G_1/G_2 \cong \mathrm{PSL}(2, \mathbf{F}_\mathrm{p})$, and $G_2/G_3 \cong \mathbf{Z}/2\mathbf{Z}$. Since $p \geq 5$, then $\mathrm{PSL}(2, \mathbf{F}_\mathrm{p})$ is a non-abelian simple group and so we cannot also have a composition series with $\mathrm{PSL(2, O_d)}/\Delta_\mathrm{p} \cong \mathbf{Z}/\mathrm{p}\mathbf{Z}$. □

Using this we can eliminate the following cases.

Corollary 4.4. *For* $(d, n) \in \{(2, 5), (7, 3), (7, 5), (11, 2), (11, 5), (19, 2), (19, 3), (19, 5)\}$, *the groups* $\Gamma(n)$ *are not link groups.*

4.5

Let $\{1, \omega_d\}$ be the usual integral basis for O_d, and for $r \in \mathbf{N}$, let $O_d(r) \subset O_d$ be the order generated over $\mathbf{Z}$ by $\{1, r\omega_d\}$.

Lemma 4.5. *If* $r|n$ *then* $\Gamma(n) < \mathrm{PSL(2, O_d(r))}$.

Proof: The entries of an element of $\Gamma(n)$ have the form nx or $1 + nx$ for some $x \in O_d$. If $r|n$, then clearly, such integers belong to $O_d(r)$. □

Given that a link complement in S^3 has trivial degree 1 cuspidal cohomology, Lemma 4.5 and the following result completes the proof of Theorem 1.2.

Proposition 4.6. *For those pairs* (d, r) *listed below, the groups* $\mathrm{PSL(2, O_d(r))}$ *have non-trivial degree 1 cuspidal cohomology.*

$$\{(1, 4), (1, 5), (2, 4), (3, 5), (7, 4), (11, 3), (19, 4)\}.$$

Proof: This follows from [19] as we now describe. In §2 of [19] the authors describe a construction of homology classes for $\mathbf{H}^3/\mathrm{PSL(2, O_d(r))}$ which can be used to generate a subspace of $\mathrm{PSL(2, O_d(r))^{ab}} \otimes \mathbf{Q}$. At the heart of this construction is a set of positive integers (the so-called Zimmert set) $W(d, m)$, which we will not define here, but suffice it to say that if $w(d, m) = |W(d, m)|$ satisfies $w(d, m) \geq 2$, then the degree 1 cuspidal cohomology of $\mathrm{PSL(2, O_d(r))}$ is non-trivial. In the cases stated, the Zimmert sets all have at least 2 elements. This proves the proposition. □

5 Handling the remaining cases

In this section we summarize for $d \in \{1,2,3,7,11,19\}$ what remains to be done to complete the enumeration of all principal congruence link complements covering Q_d. Of course there still remains the cases of $d \in \{5,6,15,23,31,39,47,71\}$ to deal with.

We first discuss the case of $d = 3$. In this case, it is proved in Goerner's thesis [16] (Theorem 1.7.1) that beyond the levels indicated in Theorem 1.1, the only possible levels that could provide congruence link groups are

$$< (7 \pm \sqrt{-3})/2 >, < 4 \pm \sqrt{-3} >, < (11 \pm \sqrt{-3})/2 > .$$

We remark that Goerner's list excludes the possibility $< (9 \pm \sqrt{-3})/2 >$, however our computations indicate that the first homology group is torsion-free and seems to still be a possibility.

Using §2.2 and §4.1, to enumerate the possible levels of principal link complements, we can use both the systole bound from Theorem 2.4, and the norm of an element $z \in I$ as described in §4.1. Using Magma as described earlier some levels arise as candiates for link groups, whilst some can be eliminated using the methods above. On the other hand on many occasions, Magma cannot determine whether the index is finite. In what follows we tabulate levels that provide candidates for link groups, and those that we can eliminate. All other levels (subject to the bounds above) are such that Magma provides no useful information.

In the second, third, and fourth columns, we list x a generator of the ideal being considered, its norm N and the order O of $\mathrm{PSL}(2, \mathrm{O_d})/ < \mathrm{P} >$ where P denotes, as before, the stabilizer of infinity in the principal congruence subgroup.

d	x	N	O	**Comments**
1	$1+i$	2	6	Torsion in $\Gamma(< 1+i >)$
1	$3+i$	10	360	Possible 18 component link group
1	$3+2i$	13	1092	Possible 42 component link group
1	$4+i$	17	2448	Possible 72 component link group
1	$4+2i$	20	1474560	Not a link group, $\lvert\mathrm{PSL}(2, \mathrm{O_d}/ < 4+2\mathrm{i} >)\rvert = 2880$
1	$5+i$	26	$--$	Not a link group, see (i) below
2	$\sqrt{-2}$	2	6	Torsion in $\Gamma(< \sqrt{-2} >)$
2	$1+2\sqrt{-2}$	9	324	Possible 36 component link group
2	$3+\sqrt{-2}$	11	660	Possible 60 component link group
2	$2+2\sqrt{-2}$	12	8432	Not a link group, $\lvert\mathrm{PSL}(2, \mathrm{O_d}/ < 2+2\sqrt{-2} >)\rvert = 576$
2	$4+\sqrt{-2}$	18	$--$	Not a link group, see (ii) below
2	$2+3\sqrt{-2}$	22	$--$	Not a link group, see (iii) below
7	$\sqrt{-7}$	7	168	Possible 24 component link group
7	$(5+\sqrt{-7})/2$	8	192	Possible 24 component link group
7	$2+\sqrt{-7}$	11	660	Possible 60 component link group
7	$(7+\sqrt{-7})/2$	14	1008	Possible 72 component link group
7	$3+\sqrt{-7}$	16	$--$	Not a link group, see (iv) below
11	$(5+\sqrt{-11})/2$	9	324	Possible 36 component link group

(i) $\lvert\mathrm{PSL}(2, \mathrm{O_1}/ < 5+\mathrm{i} >)\rvert = 6552$, and $< P >$ is contained in a normal subgroup of index 46800.

(ii) $\lvert\mathrm{PSL}(2, \mathrm{O_2}/ < 4+\sqrt{-2} >)\rvert = 1944$, and $< P >$ is contained in a normal subgroup of index 2654208.

(iii) $\lvert\mathrm{PSL}(2, \mathrm{O_2}/ < 2+3\sqrt{-2} >)\rvert = 3960$, and $< P >$ is contained in a normal subgroup of index 36432.

(iv) $|\mathrm{PSL}(2, \mathrm{O}_7/ < 3 + \sqrt{-7} >)| = 1152$, and $< P >$ is contained in a normal subgroup of index 4608.

Remark: Using a computer M. H. Sengun checked for us that for $d = 1, 2, 7, 11$ and $\mathcal{P}$ a prime of norm p having a generator x satisfying $|x| < 6$, that $\Gamma(\mathcal{P})$ has trivial cuspidal cohomology except for two cases. These are $d = 1$ and level $< 5 + 2i >$, and $d = 11$ and level $< \sqrt{-11} >$.

In addition, in the cases of $d = 2$ and level $< 3 + 2\sqrt{-2} >$, $d = 7$ and levels $< 4 + \sqrt{-7} >$, $< 1 + 2\sqrt{-7} >$ and $d = 11$ and levels $< (9 + \sqrt{-11})/2 >$, $< (5 + 3\sqrt{-11})/2 >$, Sengun checked that the principal congruence groups have torsion in first homology. The orders of the torsion subgroups are: 2^9, 2^{22}, 5^{15}, 2^{112} and $2^{96}5^{30}29^{32}31^{30}$ respectively.

Hence the corresponding principal congruence subgroups in all of the above cases are not link groups.

6 Final Remarks

As mentioned in the Introduction, the following question remains open:

Question: *Are there finitely many congruence link complements in S^3?*

It is worth comparing the 3-dimensional setting with what is known in dimension 2. In this setting a *link complement in S^3* is replaced by a *punctured S^2*, and it was conjectured by Rademacher that there are only finitely many congruence subgroups of $\mathrm{PSL}(2, \mathbf{Z})$ of genus 0. The proof of this was completed in a sequence of papers [11], [12] and [13]. Different proofs of this (actually of a slightly stronger version of this result) were also given by Thompson [30] and Zograf [34]. Indeed, in these two papers it is proved that there are only finitely congruence subgroups of $\mathrm{PSL}(2, \mathbf{Z})$ of any fixed genus.

The list of torsion-free congruence subgroups of genus 0 was completed in 2001 and given in [27] (there are 33 and the levels are all of the form $2^a 3^b 5^c 7$ with $a \leq 5$, $b \leq 3$, and $c \leq 2$ with 2^5 being the largest level). Of those only 4 are principal congruence subgroups (of levels 2, 3, 4 and 5).

Now congruence manifolds admit a spectral gap; i.e. there exists a number $C > 0$ (conjectured to be 1) so that if $M = \mathbf{H}^3/\Gamma$ (or $\mathbf{H}^2/\Gamma$) is any congruence manifold, then $\lambda_1(M) > C$. The argument of [34] to prove the finiteness result in dimension 2 for congruence surfaces of genus 0 mentioned above is done by playing off the spectral gap for congruence manifolds in dimension 2, together with a result proved in [34] that says that for a sequence of genus 0 manifolds with increasing numbers of punctures we must have $\lambda_1 \to 0$.

Thus a natural question is whether there exists a "Zograf type result" in dimension 3. The answer to this in general is no since Lackenby and Souto (in preparation) have shown that there exists a family of hyperbolic link complements in S^3 (say M_n) with $\mathrm{Vol}(\mathrm{M_n}) \to \infty$ and a constant $C_1 > 0$ such that $\lambda_1(M_n) > C_1$.

On the other hand there are classes of links known for which sequences as above do not arise (see [15] and [23]). In particular, it follows from [23] that: *There are only finitely many alternating links whose complements are congruence link complements.*

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