Finding fibre faces in finite covers

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1 Introduction

A well-known conjecture about closed hyperbolic 3-manifolds asserts that the first Betti number can be increased without bound by passage to finite sheeted covers. If the manifold is fibred, a strengthening of this conjecture is that the number of fibred faces (see §2.1 for the definition of a fibred face) of the unit ball of the Thurston norm can be made arbitrarily large by passage to finite sheeted covers. The main result of this note is the following.

**Theorem 1.1.** Suppose that $M$ is a closed arithmetic hyperbolic 3-manifold which fibres over the circle.

Then given any $K \in \mathbb{N}$, there is a finite sheeted covering of $M$ for which the unit ball of the Thurston norm has $> K$ fibred faces.

Of course an immediate corollary of Theorem 1.1 is.

**Corollary 1.2.** Let $M$ be a closed arithmetic hyperbolic 3-manifold that fibres over the circle. Then the rank of its second homology can be increased without bound.

While this follows from a stronger result proved in [4] (see also [1] and [11]), namely that the conclusion of Corollary 1.2 holds for an arbitrary closed arithmetic hyperbolic 3-manifold with positive first Betti number, the proof given here is somewhat different.

The proof of Theorem 1.1 is purely geometric, using ideas of [3], and the density of the commensurator of an arithmetic Kleinian group. The first example of this phenomenon was recently given by Dunfield and Ramakrishnan [6], also using arithmetic hyperbolic 3-manifolds, but appealing to quite sophisticated number theoretic aspects of these manifolds.

Explicit small examples of arithmetic hyperbolic 3-manifolds which fibre over the circle are known, see e.g. the first example of [3] as analysed in [9] and described briefly in §3. Many other examples are provided by the tables of [2].

2 Proof of the main result

2.1

We begin with a preliminary discussion of some facts about bundles that will be needed. All manifolds are oriented.

Let $M$ be a closed hyperbolic 3-manifold that fibres of the circle with pseudo-Anosov monodromy $\phi$ and fibre $F$. Associated to $\phi$ is the suspension flow on $M$, denoted by $\mathcal{F}_\phi$, and is constructed as the image in $M$ of the foliation of the product $F \times I$ by lines.

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Recall that Thurston shows in [10] that the set $C$ of cohomology classes in $H^1(M; \mathbb{R})$ represented by non-singular closed 1-forms is some union of the cones on open faces of the unit ball of the Thurston norm minus the origin. Furthermore, the set of elements in $H^1(M; \mathbb{Z})$ whose Poincare duals are represented by fibres consists of the set of all lattice points in $C$. Thus if $M$ is fibred with monodromy $\phi$ we can associate the (open) face $T(\phi)$ to $\phi$. We call $T(\phi)$ a fibre face. The following result is proved by Fried (see Theorem 7 of [7]) and provides the connection from flows to fibre faces of the unit ball of the Thurston norm.

**Theorem 2.1.** Let $M$ be a closed hyperbolic 3-manifold that fibres over $S^1$ and for which $H^1(M; \mathbb{Z}) \neq \mathbb{Z}$. The face $T(\phi)$ determines $F_\phi$ up to strict conjugacy (that is to say, the flows are conjugate via a diffeomorphism isotopic to the identity map).

2.2

Before giving the proof of Theorem 1.1 we make a simple observation that will be helpful.

Let $X = \mathbb{H}^3/\Gamma$ be a closed hyperbolic 3-manifold, and let $S ↪ X$ be a $\pi_1$-injective oriented immersion of a quasi-Fuchsian surface whose lift to the universal covering is an oriented embedding $\tilde{S} ↪ \tilde{X}$. We consider the components of $\tilde{S}$ being oriented with a positive side and negative side given by such a designation on $S$.

We claim the following.

**Lemma 2.2.** Fix disjoint open sets $U$ and $V$ in the 2-sphere at infinity, whose complement contains an open set.

Then there are disjoint translates $\tilde{S}_1$ and $\tilde{S}_2$ in $\tilde{S}$ with the properties that

- $\partial \tilde{S}_1 \subset U$
- $\partial \tilde{S}_2 \subset V$
- The positive sides of $\tilde{S}_1$ and $\tilde{S}_2$ face each other.

**Proof:** Let $W$ be an open set which lies in the complement of $U \cup V$. We can find a hyperbolic element $\gamma \in \Gamma$ such that its attracting fix point lies in $W$, and thus arrange that there is a component of $S$, denoted $S'$, with $\partial S' \subset W$. Now standard dynamical properties of hyperbolic elements in $\Gamma$ allow us to find two hyperbolic elements $\gamma_1$ and $\gamma_2$ with the property that each has one fix point inside the disc spanned by $\partial S' \subset W$ and further so that the other fix point of $\gamma_1$ lies in $U$ and the other fix point of $\gamma_2$ lies in $V$.

Now by applying sufficiently high powers of the hyperbolic elements $\gamma_i$’s we see that we achieve the situation of the lemma. \(\square\)

**Proof of Theorem 1.1:**

We begin by showing how to obtain two fibre faces for the Thurston norm ball, since this illustrates the main idea. By assumption $M = \mathbb{H}^3/\Gamma$ fibres over the circle, and we let the associated suspension flow be denoted by $F_1$.

It follows from [8], Theorem 1.1 (see also Lemma 5.8 therein), that one can find an immersion $S ↪ M$ transverse to the flow $F_1$ (and therefore necessarily $\pi_1$-injective) which is quasi-Fuchsian. It follows from [3] (see the discussion on p. 264 following the proof of Lemma 3.3) that the lift of this immersion to the universal covering is embedded. Hence, by Lemma 2.2, there is a pair of disjoint lifts, $\tilde{S}_1$ and $\tilde{S}_2$ whose positive sides face each other. Thus the negative sides of these lifts define a pair of disjoint open discs in the 2-sphere at infinity, which we will refer to informally as *caps*. 

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Choose any closed flowline of $F_1$ and let $\gamma$ be a lift of this closed flowline to $H^3$. By standard arguments, there is an element $g \in \text{PSL}(2, \mathbb{C})$ which maps one of the endpoints of $\gamma$ into one cap, and the other endpoint of $\gamma$ into the other. Thus $g(\gamma)$ projects to a closed flowline of a flow on $H^3/g\Gamma g^{-1}$, which has one of its endpoints in one cap and one in the other.

Since $M$ is arithmetic, the commensurator of $\Gamma$ is dense in $\text{PSL}(2, \mathbb{C})$. Hence, by adjusting $g$ slightly, we can assume in addition that $g\Gamma g^{-1}$ is commensurable with $\Gamma$. Let $\Gamma_{12}$ be the intersection of these two subgroups, $M_{12}$ the cover of $M$ determined by $\Gamma_{12}$ and let $F_2$ be the conjugated flow on $H^3/g\Gamma g^{-1}$ lifted to $M_{12}$. Lift the original flow $F_1$ to $M_{12}$, where to avoid clumsy notation, we continue to denote it by $F_1$.

We claim that one cannot isotope $F_2$ to $F_1$ in $M_{12}$. The reason is this. By construction, $g(\gamma)$ covers some closed flowline of $F_2$. However the positioning of the endpoints of $g(\gamma)$ ensures that any loop isotopic to the closed flowline must meet the two chosen lifts of the $F_1$-transverse surface $S$ in opposite orientations. This implies that the loop cannot be isotoped into the flow $F_1$. By Theorem 2.1, these flows represent different faces of the unit ball of the Thurston norm as required.

The general case is similar: We now work on $M_{12}$. Fix some quasi-Fuchsian immersion $G$ transverse to the flow $F_2$. Taking $U$ and $V$ to be the cap regions defined by $\tilde{S}_1$ and $\tilde{S}_2$, we apply Lemma 2.2 to find disjoint lifts of $G, \tilde{G}_1$ and $\tilde{G}_2$ with boundaries inside the $S$-caps and with positive sides facing each other. This gives new, smaller, $G$-caps into which we may place the endpoints of a flowline using an element of the commensurator. An identical argument now shows that the new flow this defines cannot be strictly conjugate to either of the first two flows. \(\Box\)

3 Example 1 of [3]

Example 1 of [3] is described explicitly using a branched flat structure, however from the discussion in [3] this example can easily be seen to be described as follows.

Let $T$ be the 1-punctured torus bundle over the circle with monodromy $R^2L^2$. Fix a framing for the boundary torus so that a longitude is the boundary of a fiber, and a meridian taken as the suspension of a point on the boundary of a fiber. Then $(0, 2)$ orbifold filling on $T$ provides a 2-orbifold bundle $Q$ over the circle. The genus 2 bundle $M$ described in [3] is a finite cover of this. As shown in [3] this genus 2 bundle admits a cut and cross join surface that is quasi-Fuchsian. Thus we need to check that this example is arithmetic.

This can be verified directly by using Snap [5] to check that $Q$ is arithmetic. The invariant trace-field is $\mathbb{Q}(\sqrt{-3})$ and the invariant algebra is ramified at the places above 2 and 3.

Remark: In addition the bundle $Q$ is also easily seen to be commensurable with the 2-orbifold bundle described in §4.2 of [9].

References


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