The Bianchi groups

Let $d$ be a square-free positive integer, and $O_d$ the ring of algebraic integers in the field $\mathbb{Q}(\sqrt{-d})$. The collection of groups

$$\text{PSL}_2(\mathbb{Z}) \subset \begin{array}{c} \text{PSL}(2, O_d) \end{array}$$

discrete subgps of $\text{PSL}_2(\mathbb{C})$.

is called the **Bianchi groups**.

The quotients (**Bianchi orbifolds**):

$$Q_d = \mathbb{H}^3 / \text{PSL}(2, O_d)$$

are finite volume hyperbolic orbifolds.

**Definition:** Let $M = \mathbb{H}^3 / \Gamma$ be a non-compact finite volume hyperbolic 3-manifold (or orbifold). Then $\Gamma$ is **arithmetic** if some conjugate of $\Gamma$ in $\text{PSL}(2, \mathbb{C})$ is commensurable with $\text{PSL}(2, O_d)$. 
Hyperbolic Manifolds

Let $H^3$ denote hyperbolic 3-space.

The full group of orientation-preserving isometries can be identified with $\text{PSL}(2, \mathbb{C})$.

**Linear fractional action**

$$z \rightarrow \frac{az + b}{cz + d}$$

extended to $H^3$ by Poincaré Extn.
We will only be interested in non-compact finite volume hyperbolic 3-manifolds (and orbifolds). These manifolds (and orbifolds) have the form:

These manifolds can be described as complements of links in closed orientable 3-manifolds.
Remarks: (1) Let $h_d$ denote the class number of $\mathbb{Q}(\sqrt{-d})$. Hurwitz showed:

$Q_d$ has $h_d$ cusps.

(2) When $d \neq 1, 3$, every cusp cross-section of $Q_d$ is a torus.

When $d = 1$ the cusp cross-section is

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \text{ fixes } \infty$$

When $d = 3$ the cusp cross-section is

$$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix} \text{ fixes } \infty$$

$\omega = -1 + \frac{5 - 3}{2}$

Mark: When $h_d > 1$ there can be orbifolds in the commensurability class with $\omega$.

There are only finitely many such comm classes.
Some Bianchi orbifolds

\[ d = 1 \]

\[ d = 2 \]

\[ d = 3 \]

\[ d = 5 \]

\[ d = 7 \]
Comparison with the $\text{PSL}(2, \mathbb{Z})$

**Cuspidal Cohomology**

Let $\Gamma$ be a non-cocompact Kleinian (resp. Fuchsian) group acting on $\mathbb{H}^3$ (resp. $\mathbb{H}^2$).

Let $\mathcal{U}(\Gamma)$ denote the subgroup of $\Gamma$ generated by parabolic elements of $\Gamma$ and define:

$$V(\Gamma) = (\Gamma/\mathcal{U}(\Gamma))^{ab} \otimes_{\mathbb{Z}} \mathbb{Q}$$

Then $r(\Gamma) = \dim_{\mathbb{Q}}(V(\Gamma))$ denotes the dimension of the space of non-peripheral homology or equivalently $r(\Gamma)$ is the dimension of the Cuspidal Cohomology of $\Gamma$. 
Examples:

1) If $\mathbb{H}^2/\Gamma$ is genus $g$ surface with $p$ punctures and finitely many orbifold points, then $r(\Gamma) = g$.

Thus $r(\text{PSL}(2, \mathbb{Z})) = 0$.

2) Let $n$ be a square-free positive integer and define:

$$\Gamma_0(n) = \text{P}\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : c = 0 \mod n \right\}$$

and $O_n = \mathbb{H}^2/\Gamma_0(n)$.

Now the Riemann-Hurwitz formula shows easily that $g(O_n) = 0$ if and only if $1 \leq n \leq 10$ and $n = 12, 13, 16, 18, 25$. 
(2) If $L \subset S^3$ is a link then $r(\Gamma) = 0$. The link group is generated by meridians.

$$\pi_1(S^3 \setminus L)$$

(3) If $L$ is a link in a rational homology 3-sphere, $r(\Gamma) = 0$. 
Grunewald’s work on Bianchi groups

with Schwermer:


with Elstrodt and Mennicke:

*Eisenstein series for imaginary quadratic number fields.* Contemp. Math., 53.


*On the group PSL₂(\(\mathbb{Z}[i]\)).* London Math. Soc. Lecture Note Ser., 56.

*PSL(2) over imaginary quadratic integers.* Astérisque, 94.
with Mennicke:


with Helling and Mennicke:

$\text{SL}_2$ over complex quadratic number fields. I. Algebra i Logika 17 (1978).


Preprint with Finis and Tirao

The cohomology of lattices in $\text{SL}_2(\mathbb{C})$. 

13
Theorem: (Cuspidal Cohomology Problem) (Harder, Zimmert, Grunewald-Schwermer, Rohlfs, ...., Vogtmann)

The only Bianchi orbifolds \( Q_d \) that can admit a finite sheeted cover that is a link complement in \( S^3 \) arise when

\[ d \in \{1, 2, 3, 5, 6, 7, 11, 15, 19, 23, 31, 39, 47, 71\}. \]

(i.e. \( \left( \text{PSL}_2(\mathbb{O}_d) \right) = 0 \) \( \iff \) \( d \in \text{list above} \))

Theorem: (Baker) For all values \( d \) as above, there exists a link complement covering \( Q_d \).
Arithmetic Knots

What can one say about \( M \rightarrow Q_d \) with \( M \) having 1 cusp. (or more generally 1-cusped auth. 3-mflds)

Note: If \( M \rightarrow Q_d \) and \( M \) has 1 cusp, then \( Q_d \) has 1 cusp. Thus \( h_d = 1 \) (by Hurwitz's theorem).

Solution to the class number 1 problem: \( h_d = 1 \) if and only if

\[
d \in \{1, 2, 3, 7, 11, 19, 43, 67, 163\}.
\]

Examples:

\[
d = 1
\]

\[
d = 3
\]

[ cyclic covers ]

\[
((-11), (-10))
\]

Remark: \( d = 2, 7, 11, 19 \) there are also 1 cusped covers.

(Brunner, Frame, Lee Wieland)
Dehn Surgery

$S^3 \setminus N(K)$

$\mu \to m^p \times q$

$\langle p, q \rangle = 1$

V solid torus

Glue by homeo of boundaries
Understanding 1-cusped \_\_is the same as understanding arithmetic knots.

**Definition:** A knot $K$ (or link $L$) in a closed orientable 3-manifold is called arithmetic if $M \setminus K$ (resp. $M \setminus L$) is arithmetic.

**Example—$S^3$**

**Theorem:** (R) *The figure eight knot is the only arithmetic knot.*

Links are different. *There are infinitely many arithmetic links (even of two components).*

Take $m$-fold cyclic covers $(3, m) = 1$.

$d = 2$. 
A question that naturally arise from this:

**Question:** Does every closed orientable 3-manifold contain an arithmetic knot?

**Why Care?**

A positive answer implies the Poincare Conjecture.

The proof that the figure eight knot is the only arithmetic knot in $S^3$ shows that the figure eight knot in $S^3$ is the only arithmetic knot in a homotopy 3-sphere.
Remark: Once again links are different.

*Every closed orientable 3-manifold contains an arithmetic link.*

The reason is:

The figure eight knot is *universal* (every closed orientable 3-manifold arises as a branched cover of $S^3$ with branch set $K$).
Theorem 1: (Baker-R) Suppose $L$ is a Lens space with $\pi_1(L)$ of odd order $\neq 5$. Then $L$ does not contain an arithmetic knot.

Some ideas in Proof:

1. \[ L \rightarrow \mathbb{H}^3 / \Gamma \] 
   \[ \Rightarrow d \in \{1, 2, 3, 7, 11, 19\} \]

2. $P_0 \subset \Gamma \subset \text{PSL}_2(\mathbb{O}_d)$ be peripheral subgroup fixing $\infty$. 
   \[ P_0 = \langle (1, x) (0, 1) \rangle, \quad x, y \in \mathbb{O}_d \]
   \[ \cong \mathbb{Z} \oplus \mathbb{Z} \]
   \[ \mu = \text{"meridian" of } K. \]

Gromov-Thurston $2$-Thu. $\implies 1 \times 1 \leq 6$ "small" 
(Improvement, Agol, Lackenby 
6 Theorem)

$O_d \subset \mathbb{C}$ discrete $\implies$ only finitely many $x.$
D) Two Cases

\( x \neq \text{a unit} \)

(i) \( x \text{ a unit } \implies L \setminus K \cong S^3 \times \mathbb{R} \)

Impossible as \( S^3 \times \mathbb{R} \)

has no Lens Space Dehn Surgery.

(ii) \( x \neq \text{a unit} \).

\( \langle x \rangle \) non-trivial ideal, so \( \exists \mathfrak{p} | \langle x \rangle \)

\( \mathfrak{p} \) a prime ideal.

\[
\begin{align*}
\operatorname{PSL}_2(\mathbb{Q}_d) & \overset{\langle \mathfrak{p} \rangle}{\longrightarrow} \operatorname{PSL}_2(\mathbb{Q}_\mathfrak{a}/\mathfrak{p}) \\
\Gamma & \longrightarrow \Gamma/\langle \langle \mathfrak{p} \rangle \rangle = \pi_1 \mathbb{T}
\end{align*}
\]

\( \mathfrak{p}(\Gamma) \) cyclic \( (\neq 1, \mathbb{H}^3/\ker \epsilon_\mathfrak{p} > 1 \text{ ungr}) \)

\( \mathfrak{p}(\Gamma) \) is cyclic of large prime order

But \( |\mathfrak{F} = \mathbb{Q}_d/\langle \mathfrak{p} \rangle| \) is bounded on \( 1 \times 1 \leq 6 \)

Contradiction.
Unlike the case of $S^3$, there are examples of closed orientable hyperbolic 3-manifolds that contain more than one arithmetic knot.

Examples:

1. $S^2 \times S^1$ contains at least 2 arithmetic knots (the complements being commensurable with $Q_3$ and $Q_7$).

2. $\mathbb{RP}^3 \# \mathbb{RP}^3$ contains at least 2 arithmetic knots (the complements being commensurable with $Q_1$ and $Q_3$).

3. $L(4,1) \# L(4,1)$ contains at least 2 arithmetic knots (both the complements being commensurable with $Q_7$).

4. $\mathbb{RP}^3 \# (S^2 \times S^1)$ contains at least 2 arithmetic knots (the complements being commensurable with $Q_1$ and $Q_3$).
There are hyperbolic examples:

The manifold obtained by 5/1-Dehn surgery on the figure eight knot contains at least 2 arithmetic knots.

One obvious one, and the other is shown:

\[ \text{Brunner - Frenkel - Lee}\]
\[\text{Wiclenberg.}\]

**Question:** *Is the number of arithmetic knots in a closed orientable 3-manifold finite?*
One can generalize the question about the uniqueness of the figure eight knot in $S^3$ in two obvious ways.

$$ S^3 \xrightarrow{\text{spherical}} \text{integral homology 3-sphere.} $$

Question: What can one say about arithmetic knots in spherical 3-manifolds or integral homology 3-spheres?

**Theorem 2:** (Baker-R) Suppose $M$ be a spherical 3-manifold or an integral homology 3-sphere. Suppose that

$$ M \setminus K \rightarrow \mathbb{Q}_d. $$

Then,

(1) If $M$ is spherical then $d = 3$.

(2) If $M$ is an integral homology 3-sphere, $d = 1, 3$. 

23
**Remain:** If $M$ is an integral homology 3-sphere and $K \subset M$ an arithmetic knot then one can show that $M \setminus K \to Q_d$ for some $d$.

[True for knots in mod 2 homology spheres]

**Conjecture:** Let $M$ be an integral homology 3-sphere. If $M$ contains an arithmetic knot $K$, then $M$ is obtained by $1/n$-Dehn surgery on the figure eight knot complement and $K$ is "the core of the surgery solid torus".

i.e. $M \setminus K \cong S^3 \setminus \text{fig 8 knot}$.
**Final Comments**

**1-cusped congruence subgroups**

In the case of the modular group $H$, Petersson showed that there are only finitely many 1-cusped congruence subgroups of $\text{PSL}(2, \mathbb{Z})$.

In her (2005) thesis, K. Petersen (my former student) showed that there are only finitely many **maximal 1-cusped congruence subgroups**.

Indeed for $d = 11, 19, 43, 67, 163$ there are only finitely many 1-cusped congruence subgroups.

For $d = 19, 43, 67, 163$ there are **no torsion-free 1-cusped congruence subgroups**.
Arithmetic number of a closed orientable 3-manifold

Let $M$ be a closed orientable 3-manifold. The arithmetic number of $M$, denoted $\mathcal{A}(M)$, is the minimal number of components of a non-empty arithmetic link in $M$.

As remarked, $M$ contains an arithmetic link, so $\mathcal{A}(M)$ is well defined positive integer.

Examples: (1) A Lens Space $L$ is a Dehn surgery on the Whitehead link, so that $\mathcal{A}(L) \leq 2$. Theorem 1 therefore shows $\mathcal{A}(L) = 2$ for $L$ with $\pi_1(L)$ odd order $\neq 5$. 

\begin{center}
\includegraphics[width=0.2\textwidth]{diagram.png}
\end{center}
(2) The Poincare homology sphere $\Sigma$ contains a 2-component arithmetic link so $A(\Sigma) \leq 2$. 

This prompts:

**Question:** Does the Poincare homology sphere contain an arithmetic knot?

(3) Methods of proof of Theorem 1 show "many" non-hyperbolic 3-manifolds have arithmetic number $\geq 2$. 

Brunner - Frence - Lee - Wienberg.
Challenges:

(1) Prove that there exists closed orientable 3-manifolds for which $\mathcal{A}(M)$ is arbitrarily large.

(2) Prove that there exists a closed orientable hyperbolic 3-manifold that does not contain an arithmetic knot; ie $\mathcal{A}(M) \geq 2$.

(1) Looks like it is related to Heegaard genus.

(2) There are candidate integral homology 3-spheres.