373K Algebra I, Homework 14

From Artin

Chapter 12 (pages 379–381) 3.2, 4.3, 4.4, 4.7.

Others:

1. Construct a finite field of order 7^3.

2. Let R be an integral domain. Elements $a, b \in R$ are called *associates* if $a = bu$ for some unit $u \in R$.

(a) Show that $(2+i)$ and $1+2i$ are associates in $Z[i]$.

(b) Are $3+2i$ and $1+i$ associates?

3. Let $f(x) \in \mathbf{Z}[x]$ be a monic polynomial. Suppose that for some prime $p \in \mathbf{Z}$, $f(x)$ viewed as a polynomial in $\mathbf{F}_p[x]$ (ie reduce coefficients modulo p) is irreducible, then $f(x)$ is irreducible in $\mathbf{Q}[x]$ **5.** Determine whether the following polynomials are irreducible in $\mathbf{Q}[x]$:

(i) $3x^2 - 7x - 5$, (ii) $2x^3 - x - 6$, (iii) $x^3 + 6x^2 + 5x + 25$

(iv) $x^5 - 4x + 2$, (v) $x^4 + x^2 + x + 1$.

4. Let k be a field, and let $f(x) = a_0 + a_1 x + \ldots + a_n x^n \in k[x]$ have degree n. Suppose that f is irreducible, show that the polynomial $a_n + a_{n-1}x + \ldots a_0 x^n$ is irreducible.

5. Let p be a prime congruent to 3 mod 4. Prove that $\mathbf{Z}[i]/ < p >$ is a field with p^2 elements.

6. Prove that group $\mathrm{SL}(2, \mathbf{Z}[\mathrm{i}])$ is residually finite (recall HW 4, Qn 5).

7. Prove that $\mathbf{Z}[\sqrt{-2}]$ is a Euclidean domain.

8. Go back over the questions in Midterm 2.

373K Midterm 2, Spring 2017

DO BOTH QUESTIONS IN PART A AND TWO FROM PART B

All questions are worth **25 points**. Indicate which four you want graded. State clearly Theorems you are using.

Notation: S_n denotes the symmetric group on n letters.

PART A

1.(a) Describe up to isomorphism the Sylow 2-subgroups and 3-subgroups of the group $S_4 \times S_3$.

(b) Prove that if $|G| = 351 = 3^3 \cdot 13$, then G has a normal Sylow p-subgroup for some prime p dividing its order.

(c) Is there a non-abelian group of order 351?

2. Answer the following *true* or *false*. You must explain your answers to get full credit.

(a) There is a unique isomorphism class of abelian groups of order 210?

(b) The following is a possible class equation for a group of 30: $1+1+1+1+6+10+10$?

(c) The following permutations are conjugate in S_{11}?

$$(1,3,5,7)(2,4,11)(9,10) \text{ and } (1,11)(2,3,5,6)(7,9,10)$$

(d) There is a simple group of order $3 \cdot 11^2$?

PART B

3. (a) Let R be a commutative ring with 1, and let

$$\mathcal{A}_1 \subset \mathcal{A}_2 \subset \ldots \subset \mathcal{A}_n \subset \ldots$$

be an ascending chain (not necessarily finite) of ideals of R. Prove that $\mathcal{A} = \bigcup \mathcal{A}_n$ is an ideal of R.

(b) Assume now that R has the property that all ideal are principal. Prove that any sequence of ideals as in (a) above terminates at some N. That is, there exists an N such that for all $m \geq N$, $\mathcal{A}_m = \mathcal{A}_N$.

4.(a) Let R be a commutative ring with identity and $\mathcal{M}$ an ideal. Prove that if $R/\mathcal{M}$ is a field, then $\mathcal{M}$ is maximal.

(b) Let $\mathbf{Z}[i] = \{a + bi : a, b \in \mathbf{Z}\}$. Prove that $< 1 + i >$ is a maximal ideal of $\mathbf{Z}[i]$. Is $< 13 >$ maximal?

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