373K Algebra I, Homework 5

From Artin
Chapter 2 (p. 74): 11.3, 11.5, 11.6, 11.9.

Others:
1. Let $G$ be a group and $S \subset G$ (only a subset, possibly empty). Let $\langle S \rangle = \cap \{H < G : S \subset H\}$.
   (i) Prove that $\langle S \rangle$ is a subgroup of $G$, called the subgroup generated by $S$.
   (ii) Let $G$ denote the dihedral group $D_n$, $\rho$ a rotation of order $n$, and $\sigma$ any reflection. Prove that $\langle \{\rho, \sigma\} \rangle = G$.

2. Referring to Question 1 above. Let $G$ be a group and $S = \{aba^{-1}b^{-1}\}$. The group $\langle S \rangle$ is called the commutator subgroup of $G$ and denote $G'$.
   (i) Prove that $G'$ is a normal subgroup of $G$.
   (ii) Prove that $G/G'$ is abelian.
   (iii) Prove that if $N$ is a normal subgroup of $G$ and $G/N$ is abelian then $G' < N$.

3. Referring to Question 2 above. What is $D_4/D_4'$?

4. (i) Compute the orders of $\text{PSL}(2, \mathbb{Z}/3\mathbb{Z})$ and $\text{PSL}(2, \mathbb{Z}/5\mathbb{Z})$. Prove that $\text{PSL}(2, \mathbb{Z}/3\mathbb{Z})$ has no subgroup of order 6.
   (ii) Prove that $\text{PSL}(2, \mathbb{Z}/3\mathbb{Z})$ is not simple but $\text{PSL}(2, \mathbb{Z}/5\mathbb{Z})$ is simple.

5. Let $G$ be a finite group, let $p$ be a prime, and let $H < G$ be a normal subgroup. Prove that if $H$ and $G/H$ have orders which are a power of $p$, then $|G|$ is a power of $p$.

6. Let $G = S_4$ and $H = \langle 1, (1 \ 2)(3 \ 4), (1 \ 3)(2 \ 4), (2 \ 3)(1 \ 4) \rangle$. Prove that $H < G$ is normal and identify $G/H$.

Sample Midterm 1 Questions

1. Answer the following true or false. You must explain your answers to get full credit.
   (a) There is a non-abelian group of order 17.
   (b) There is a non-abelian group of order 14.
   (c) There is an abelian non cyclic group of order 49.
   (d) $S_5$ contains an element of order 6.

2. Suppose $A$ is a normal subgroup of $G$ and $H < G$ and these satisfy:
   (i) $A$ is abelian,
   (ii) $A.H = G$.
   Show that $A \cap H$ is a normal subgroup of $G$.
   \textbf{(Hint:} Show $A \subset N_G(A \cap H)$ and $H \subset N_G(A \cap H)$.)
3. (a) Let $G$ be a group containing normal subgroups $H$, $K$ such that $H \cap K = 1$ and $G = HK$. Show that $G$ is isomorphic to $H \times K$ (Hint: Consider $G/K$ and $G/H$).
(b) Use (a) to deduce that an Abelian group of order 9 is cyclic or isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

4. Prove that if $\alpha \in S_n$ is an $m$-cycle (i.e., has the form $(a_1 \ a_2 \ldots a_m)$ for distinct integers $a_1, \ldots, a_m \in \{1, \ldots, n\}$) then $\alpha$ is a product of transpositions (i.e., permutations of the form $(i \ j)$ for some $1 \leq i \neq j \leq n$).

5. Suppose that $G$ is a group and $N < G$ a normal subgroup. Assume that there is no normal subgroup $M < G$ with $N < M$. Prove that $G/N$ is simple.