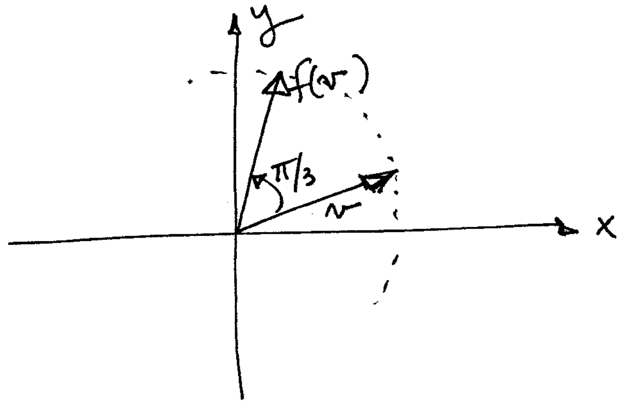
M341. Overview of Linear Transformations

Example: rotation of angle $\frac{\pi}{3}$ in $\mathbb{R}^2$.



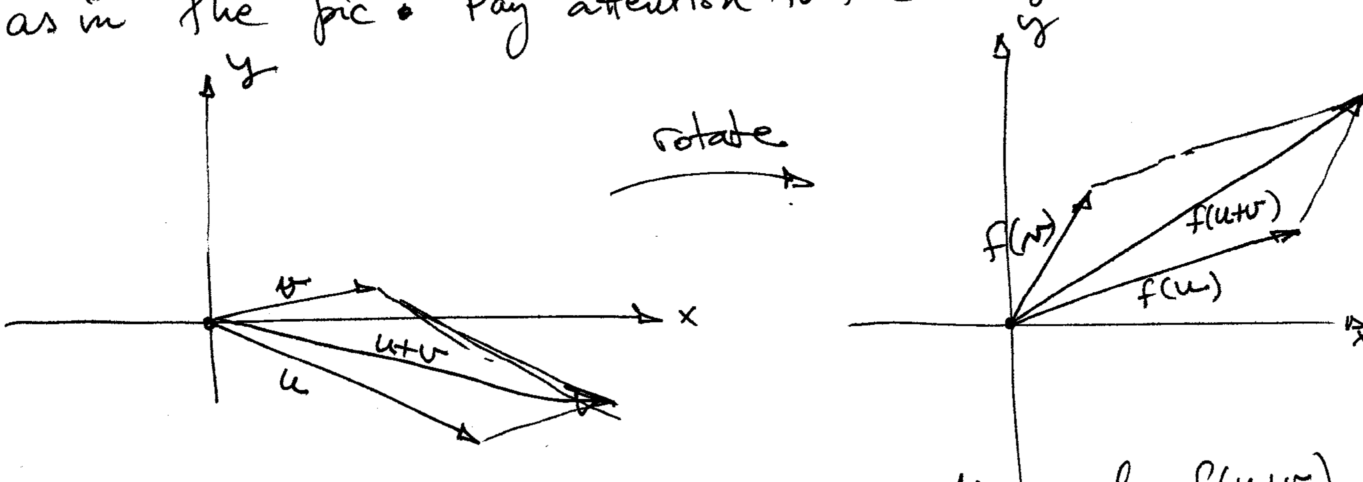
$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$v \longmapsto f(v)$ as in pic.

1) it's the same to rotate a scaled vector or to first rotate and then scale

i.e. (1) $\boxed{f(av) = a f(v)}$ for all $v \in \mathbb{R}^2$, for all $a \in \mathbb{R}^2$

2) consider the rotation of a parallelogram of sides u & v as in the pic. Pay attention to the diagonal u+v:



diagonal : u+v

diagonal : f(u+v) is <u>also</u> the sum of the sides f(u) & f(v)

i.e. (2) $\boxed{f(u+v) = f(u) + f(v)}$ for all $u, v \in \mathbb{R}^2$

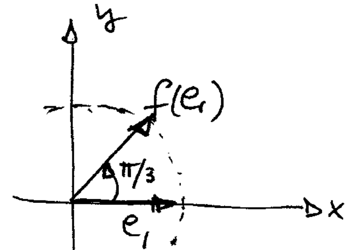
With a little induction, you can see that f behaves well with respect to linear combinations, i.e.

$$f(a_1u_1 + \cdots + a_ku_k) = a_1f(u_1) + \cdots + a_kf(u_k)$$

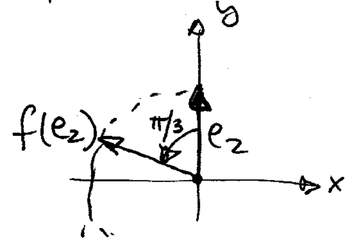
for all $u_i \in \mathbb{R}^2$, $a_i \in \mathbb{R}$.

This indicates that we can get control of all the "action" if we know what f does to the vectors in a basis.

Here: (use $\mathcal{E}$) $f(e_1) = f([1,0]) = [1/2, \sqrt{3}/2]$ (use trig.)



& $f(e_2) = f([0,1]) = [-\sqrt{3}/2, 1/2]$



Let $v \in \mathbb{R}^2$: $v = [x,y] = [v]_{\mathcal{E}}$

i.e. $v = xe_1 + ye_2$

So:
$$\begin{aligned} f(v) &= f(xe_1 + ye_2) \\ &= f(xe_1) + f(ye_2) && \text{by (2)} \\ &= x f(e_1) + y f(e_2) && \text{by (1)} \\ &= x[1/2, \sqrt{3}/2] + y[-\sqrt{3}/2, 1/2] \\ &= [1/2\, x - \sqrt{3}/2\, y,\ \sqrt{3}/2\, x + 1/2\, y] \end{aligned}$$

So:
$$\boxed{f(v) = \begin{bmatrix} \underbrace{\begin{matrix} 1/2 \\ \sqrt{3}/2 \end{matrix}}_{f(e_1)} & \underbrace{\begin{matrix} -\sqrt{3}/2 \\ 1/2 \end{matrix}}_{f(e_2)} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}}$$

<u>**Defn:**</u> Let $V, +, \cdot$ & $W, +, \cdot$ be v.sps. A function $f: V \to W$ is called a <u>linear transformation</u> if it satisfies:

① $f(u+v) = f(u) + f(v)$ for all $u, v \in V$

② $f(au) = af(u)$ for all $u \in V$, $a \in \mathbb{R}$

<u>OBS</u>: $f(\mathbf{0}) = f(0 \cdot \mathbf{0}) = 0 \cdot f(\mathbf{0}) = \mathbf{0}$

Example : (check)

1) $f : \mathcal{M}_{2,3} \longrightarrow \mathcal{M}_{2,2}$

$f\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right) = \begin{bmatrix} f & b \\ 0 & a+e \end{bmatrix}$ is a lin. comb.

2) $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$f([x, y, z]) = [x-z, 2y+x]$ is a lin. comb.

3) Let $u \in \mathbb{R}^n$ fixed.

$f_u : \mathbb{R}^n \longrightarrow \mathbb{R}$

$f_u(x) = x \cdot u$ is a lin. transf.

4) Let $A \in \mathcal{M}_{m,n}$ fixed.

$f_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$f_A(v) = \underbrace{\underset{m\times n}{A} \cdot \underset{n\times 1}{v}}_{m\times 1}$ is a lin. transf.

5) $V, +, \cdot$ v. sp. $\mathcal{B}$ an ord. basis for V. $\dim V = n$

$f : V \longrightarrow \mathbb{R}^n$

$f(v) = [v]_{\mathcal{B}}$ is a lin. transf.

Thm : $V, +, \cdot$ & $W, +, \cdot$ v. sps with ord. bases $\mathcal{B}_V$ & $\mathcal{B}_W$ resp., $\mathcal{B}_V = (u_1, u_2, \dots, u_n)$.

Let $f : V \longrightarrow W$ be a lin. transf.

Then there is a matrix A such that :

$$\boxed{[f(v)]_{\mathcal{B}_W} = A \cdot [v]_{\mathcal{B}_V}}$$

where $$A = \left[[f(u_1)]_{\mathcal{B}_W} \middle| [f(u_2)]_{\mathcal{B}_W} \middle| \cdots \middle| [f(u_n)]_{\mathcal{B}_W} \right]$$

Notation : $A = \|f\|_{\mathcal{B}_V, \mathcal{B}_W}$

(proof based on computations like this one:

$$u = a_1 u_1 + a_2 u_2 \overset{f \text{ lin.tr.}}{\Longrightarrow} f(u) = a_1 f(u_1) + a_2 f(u_2)$$

$$\Longrightarrow [f(u)]_{\mathcal{B}_W} = [a_1 f(u_1) + a_2 f(u_2)]_{\mathcal{B}_W}$$

$$= a_1 [f(u_1)]_{\mathcal{B}_W} + a_2 [f(u_2)]_{\mathcal{B}_W}$$

$$= \left[[f(u_1)]_{\mathcal{B}_W} \,\middle|\, [f(u_2)]_{\mathcal{B}_W} \right] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$
)

<u>Thm</u>: $V, W, \mathcal{U}$ v.sps ; f & g lin transfs:

$$\underset{\mathcal{B}_V}{V} \xrightarrow[\text{matrix } A]{f} \underset{\mathcal{B}_W}{W} \xrightarrow[\text{matrix } B]{g} \underset{\mathcal{B}_\mathcal{U}}{\mathcal{U}}$$

then $g \circ f : \underset{\mathcal{B}_V}{V} \longrightarrow \underset{\mathcal{B}_W}{W}$ has assoc. matrix $B \cdot A$

<u>Example</u> (easy): $f : \mathcal{M}_{2,3} \longrightarrow \mathbb{R}^2$

$f\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right) = [f, a+e]$. Check lin. transf.

$\mathcal{M}_{2,3}$: $\mathcal{E}_1 = (E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23})$ ord. basis

$\mathbb{R}^2$: $\mathcal{E}' = (e_1, e_2)$ ord. basis.

Build matrix associated to f:

$f(E_{11}) = [0,1] = e_2$ so $[f(E_{11})]_{\mathcal{E}'} = [0,1]$

$f(E_{12}) = [0,0]$ so $[f(E_{12})]_{\mathcal{E}'} = [0,0]$

$f(E_{13}) = [0,0]$ so $[f(E_{13})]_{\mathcal{E}'} = [0,0]$

$f(E_{21}) = [0,0]$ so $[f(E_{21})]_{\mathcal{E}'} = [0,0]$

$f(E_{22}) = [0,1] = e_2$ so $[f(E_{22})]_{\mathcal{E}'} = [0,1]$

$f(E_{23}) = [1,0] = e_1$ so $[f(E_{23})]_{\mathcal{E}'} = [1,0]$

matrix $A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

Compute $f\left(\underbrace{\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \end{bmatrix}}_{\text{call } B}\right)$ using coords of matrix A:

$$[f(B)]_{\mathcal{E}'} = A \cdot [B]_{\mathcal{E}}$$

Need $[B]_{\mathcal{E}} = [1, 2, -1, 0, 1, 3]$

So: $[f(B)]_{\mathcal{E}'} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \\ 1 \\ 3 \end{bmatrix} = [3, 2]$

<u>Ex</u>: (more complicated) use $\mathcal{B} = ([1,1], [1,-1])$ for $\mathbb{R}^2$.

<u>Thm</u>: $f : \underset{\substack{\mathcal{B}, \mathcal{D} \\ \text{two bases}}}{V} \longrightarrow \underset{\substack{\mathcal{C}, \mathcal{F} \\ \text{two bases}}}{W}$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
{[v]_{\mathcal{B}}} \arrow[r, "\|f\|_{\mathcal{B},\mathcal{C}}"] \arrow[d, "P_{\mathcal{B}\to\mathcal{D}}"', "\text{(transition)}"] \arrow[dr, bend left=60, "\text{PATH 2}"] \arrow[dr, bend right=60, "\text{PATH 1}"'] & {[f(v)]_{\mathcal{C}}} \arrow[d, "P_{\mathcal{C}\to\mathcal{F}}", "\text{(transition)}"'] \\
{[v]_{\mathcal{D}}} \arrow[r, "\|f\|_{\mathcal{D},\mathcal{F}}"] & {[f(v)]_{\mathcal{F}}}
\end{tikzcd}$$

Paths agree: $[f(v)]_{\mathcal{F}} = \|f\|_{\mathcal{D},\mathcal{F}} \cdot P_{\mathcal{B}\to\mathcal{D}} \cdot [v]_{\mathcal{B}} = P_{\mathcal{C}\to\mathcal{F}} \cdot \|f\|_{\mathcal{B},\mathcal{C}} \cdot [v]_{\mathcal{B}}$