p-adic interpolation of half-integral weight modular forms

Adriana Sofer
asofer@math.princeton.edu

1 Introduction

The p-adic interpolation of modular forms on congruence subgroups of $SL_2(\mathbb{Z})$ has been succesfully used in the past to interpolate values of L-series.

In [12], Serre interpolated the values at negative integers of the ζ-series of a totally real number field (in fact of L-series of powers of the Teichmuller character) by interpolating Eisenstein series, which are holomorphic modular forms, and looking afterwards at the resulting constant terms of the q-expansions. This can be interpreted as evaluating at $q = 0$, the degenerate elliptic curve.

In [8], Katz interpolated the values of the Hecke L-series $L(\psi_k, r)$ by interpolating first "derivatives" of Eisenstein series, which are in general non-holomorphic, and then evaluating at elliptic curves.

In [18], we interpolated square roots of values of Hecke L-series by interpolating "derivatives" of theta functions, which are non-holomorphic half-integral weight modular forms, and evaluating at elliptic curves. This last step required appealing to the theory of integral weight p-adic modular forms, since there is not a complete theory for half-integral weight yet. There is ongoing research towards this (see the recent work of Jochnowitz [2], and Stevens [19]).

There is renewed interest in half-integral weight forms, after the work of Hida, Kohnen, Rodriguez-Villegas, Waldspurger, Zagier.

We here present a p-adic interpolation of the sequence $\left\{\left(Nq\frac{d}{dq}\right)^k f\right\}_{k\geq 0}$ where f is a given p-adic modular form of half-integral weight and of level $N \geq 1$, $4|N$, $p \not| N$. The integral weight version of this appears in [8, 1].

For integral weight, we use Serre's and Katz's theories of p-adic modular forms ([12, 8, 6, 5, 7, 1]). We use Koblitz's ([10]) and Jochnowitz's ([2, 3, 4] definitions for half-integral weight p-adic modular forms ([10]).

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2 Definitions and properties

2.1 p-adic modular forms

Let $p \geq 3$ be a rational prime. Fix $\mathbb{C}_p$ the completion of an algebraic closure of $\mathbb{Q}_p$. Let $B \subset \mathbb{C}_p$ be the ring of integers of a finite extension of $\mathbb{Q}_p$. We fix an isomorphism $\mathbb{C} \simeq \mathbb{C}_p$ and use it to consider B as included in $\mathbb{C}$.

The reader is referred to [16] or [9, Ch.4] for the following definitions.

Let j be the multiplier system on $\Gamma_0(N)$, $4|N$, corresponding to the standard theta series $\theta = 1 + 2\sum_{n\geq 1} e^{2\pi i n^2 z}$, and $\chi : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}$ be a character acting on $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ via $\chi(d)$. $\mathcal{H}$ is the upper-half plane in $\mathbb{C}$.

For $k \in \mathbb{Z}$, we denote by $\mathbf{M}_k(\Gamma_0(N), \chi)$ the set of holomorphic $f : \mathcal{H} \to \mathbb{C}$ such that $f(\gamma z) = \chi(d)(cz+d)^k f(z)$ for every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ and f is holomorphic at all cusps.

For $k \in \mathbb{Z} + \frac{1}{2}$, we denote by $\mathbf{M}_k(\Gamma_0(N), \chi)$ the set of holomorphic $f : \mathcal{H} \to \mathbb{C}$ such that $f(\gamma z) = \chi(d) j(\gamma)^{2k+1} f(z)$ for every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ and f is holomorphic at all cusps.

In both cases, and for every cusp t for $\Gamma_0(N)$, we associate to $f \in \mathbf{M}_k(\Gamma_0(N), \chi)$ a q-expansion $f_t \in \mathbb{C}[[q_N]]$, ($q_N = e^{2\pi i z/N}$) in a consistent manner. We choose $f_\infty(q_N) = f(z)$. Given a ring $A \subset \mathbb{C}$, we denote $\mathbf{M}_k(\Gamma_0(N), \chi, A)$ the set of $f \in \mathbf{M}_k(\Gamma_0(N), \chi)$ satisfying $f_t \in A[[q_N]]$ for every t. We extend these definitions to $\mathbf{M}_k(\Gamma_1(N)) \simeq \bigoplus \mathbf{M}_k(\Gamma_0(N), \chi)$.

Recall that θ does not vanish on $\mathcal{H}$, and that its q-expansions are:

$$\left\{\begin{array}{lcl}\theta_\infty & = & 1+2\sum_{n\geq 1} q^{n^2} \\ \theta_0 & = & \frac{1}{\sqrt{2}}\left(1+2\sum_{n\geq 1} q_4^{n^2}\right) \\ \theta_{\frac{1}{2}} & = & \theta(\frac{z}{4})-\theta(z)\end{array}\right. \tag{1}$$

[Serre-Koblitz] We call a p-adic modular form over B for $\Gamma_1(N)$ a collection of q-expansions $\{f_t(q_N)\}_{\{t \text{ cusp}\}} \subset B[[q_N]]$ such that there is a sequence of complex modular forms $f_j \in \mathbf{M}_{k_j}(\Gamma_1(N), B)$, $k_j \in \mathbb{Z}$ (or $k_j \in \mathbb{Z}+\frac{1}{2}$) with

$$(f_j)_t(q_N) \xrightarrow{p} f_t(q_N)$$

uniformly on the coefficients, for every cusp t.

[Serre-Jochnowitz] We call a p-adic modular form over B for $\Gamma_1(N)$ a power series $f(q_N) \in B[[q_N]]$ such that there is a sequence of complex modular forms $f_j \in \mathbf{M}_{k_j}(\Gamma_1(N), \bar{\mathbb{Q}})$ with p-adically integer coefficients , $k_j \in \mathbb{Z}$ (or $k_j \in \mathbb{Z}+\frac{1}{2}$) with

$$f_j(q_N) \xrightarrow{p} f(q_N)$$

uniformly on the coefficients, for every cusp t.

The Corollary in the next section shows that these two definitions are equivalent. Until that is established, we will work with definition 2.1.

Any p-adic modular form has a weight $k \in X$ (or $k+\frac{1}{2} \in X+\frac{1}{2}$) (see [2, Theorem 1.2], [12, 10]), where $X = \mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z}$. Moreover, if $p \not| N$, k is the p-adic limit of $(k_j, k_j \bmod p-1)$. We denote $\mathcal{M}_k^p(\Gamma_1(N), B)$ (or $\mathcal{M}^p_{k+\frac{1}{2}}(\Gamma_1(N), B)$) the set of such f's.

We immediately obtain that any classical holomorphic modular form with coefficients in B can be interpreted as a p-adic form.

Proposition 2.1 *The Ramanujan function* $P(z) = 1-24\sum_{n\geq 1}\sigma(n)e^{2\pi i n z}$ *is not modular in the classical sense, but* $P \in \mathcal{M}_2^p(SL_2(\mathbb{Z}), \mathbb{Z}_p)$.

Proof: [12, Example p.210].

Let "$q\frac{d}{dq}$" be the operator $D+\frac{1}{12}PH$, where D is the Halphen-Fricke operator, P is the Ramanujan function, and $H(g) = kg$ if $g \in \mathbf{M}_k(\Gamma_0(N), \chi)$ (see [8, 18]). We have $\left(Nq\frac{d}{dq}f\right)_\infty = \sum n a_n q_N^n$ if $f_\infty = \sum a_n q_N^n$.

Proposition 2.2 *Let $p \geq 5$, $k \in X \cup X + \frac{1}{2}$, and $f \in \mathcal{M}_k^p(\Gamma_1(N), B)$. Then $q\frac{d}{dq}f \in \mathcal{M}_{k+2}^p(\Gamma_1(N), B)$, where the weight $k+2$ means $k + (2,2)$.*

Proof: $Nq\frac{d}{dq} = ND + \frac{N}{12}PH$. For $k \in \mathbb{Z} \cup \mathbb{Z} + \frac{1}{2}$, ND is a derivation $\mathbf{M}_k(\Gamma_0(N), \chi, A) \to \mathbf{M}_{k+2}(\Gamma_0(N), \chi, A[\frac{1}{12}])$ that acts continuously. Hence, for $k \in X \cup X + \frac{1}{2}$,

$$\begin{array}{lcl} D : \mathcal{M}_k^p(\Gamma_1(N), B) & \longrightarrow & \mathcal{M}_{k+2}^p(\Gamma_1(N), B) \\ H : \mathcal{M}_k^p(\Gamma_1(N), B) & \longrightarrow & \mathcal{M}_k^p(\Gamma_1(N), B) \\ P \in \mathcal{M}_2^p(SL_2(\mathbb{Z}), \mathbb{Z}_p) & & \end{array}$$

and we deduce the result.

Note: Also see an alternate proof of proposition 2.2 in [2].

2.2 The q-expansion principle

Recall the q-expansion principle for (integral weight) p-adic modular forms ([8, 5.2], [1, Theorem I.3.1, Prop. I.3.2]). We will prove a similar result for half-integral weight.

When $k \in \mathbb{Z}$, the weight k means (k,k), and $k + \frac{1}{2}$, $(k,k) + \frac{1}{2}$. Let us denote $\mathcal{M}_{\bullet+\frac{1}{2}}^p(\Gamma_1(N), B)$ (and $\mathcal{M}_\bullet^p(\Gamma_1(N), B)$) the set of $\mathbb{Z}$-linear combinations of forms in $\bigcup_{k \in \mathbb{Z} k \geq 0}(\mathcal{M}_{k+\frac{1}{2}}^p(\Gamma_1(N), B) \otimes \mathbb{Q}_p)$ (and, for integral weight, $\bigcup_{k \in \mathbb{Z} k \geq 0}(\mathcal{M}_k^p(\Gamma_1(N), B) \otimes \mathbb{Q}_p)$) such that the q-expansions coefficients (of the $\mathbb{Z}$-linear combinations) at all cusps belong to B (the q-expansions are defined by linearity). Notice that any sequence approximating an element of $\mathcal{M}_{\bullet+\frac{1}{2}}^p(\Gamma_1(N), B)$ (or $\mathcal{M}_\bullet^p(\Gamma_1(N), B)$) will eventually have coefficients in B, rather than in $B \otimes \mathbb{Q}_p$, due to the uniform convergence. Also, for $k \in \mathbb{Z}$, $k \geq 0$,

$$(\mathcal{M}_{k+\frac{1}{2}}^p(\Gamma_1(N), B) \otimes \mathbb{Q}_p) \cap \mathcal{M}_{\bullet+\frac{1}{2}}^p(\Gamma_1(N), B) = \mathcal{M}_{k+\frac{1}{2}}^p(\Gamma_1(N), B),$$

and similarly for weight k. We define $q\frac{d}{dq}$ on $\mathcal{M}_{\bullet+\frac{1}{2}}^p(\Gamma_1(N), B)$ and $\mathcal{M}_\bullet^p(\Gamma_1(N), B)$ by linearity.

Jochnowitz independently proved results similar to the next proposition.

Proposition 2.3 (**q-expansion principle**) *Let $\mu_N \cup \{\sqrt{2}\} \subset B' \subset B \subset \mathbb{C}_p$, B and B' the ring of integers of $K, K' \subset \mathbb{C}_p$, finite extensions of $\mathbb{Q}_p$. Let $p \geq 3$, $p \not| N$, and assume $f \in \mathcal{M}^p_{\bullet+\frac{1}{2}}(\Gamma_1(N), B)$.*

1. *If $f_\infty \in B'[[q_N]]$, then $f \in \mathcal{M}^p_{\bullet+\frac{1}{2}}(\Gamma_1(N), B')$.*

2. *Let π be a prime element in B. The map*

$$\mathcal{M}^p_{\bullet+\frac{1}{2}}(\Gamma_1(N), B) \overset{q\text{-}exp\ at\ \infty}{\longrightarrow} B[[q_N]]$$

is injective, and the cokernel has trivial π-torsion.

Proof:

1. Since $p \geq 3$, $\theta \in \mathbf{M}_{\frac{1}{2}}(\Gamma_0(4), \mathbb{Z}[\frac{1}{\sqrt{2}}]) \subset \mathcal{M}^p(\Gamma_1(N), B')$. We know that $f\theta \in \mathcal{M}^p_\bullet(\Gamma_1(N), B)$, and $(f\theta)_\infty \in B'[[q_N]]$. By the integral-weight q-expansion principle , $f\theta \in \mathcal{M}^p_\bullet(\Gamma_1(N), B')$, i.e. $(f\theta)_t \in B'[[q_N]]$ for every cusp t. Since $\theta_t | (f\theta)_t$, and $\theta_t \in B'[[q_N]]$ with invertible first coefficient (see equations (1)), $(f)_t = \frac{(f\theta)_t}{(\theta)_t} \in B'[[q_N]]$ for every cusp t. Take a sequence of classical modular forms, with coefficients in B approaching f in the following way: if $f = \sum_i h_i$, $h_i \in \mathcal{M}^p_{k_i+\frac{1}{2}}(\Gamma_1(N), B) \otimes \mathbb{Q}_p$, take sequences $f_{ij} \in \mathbf{M}_{k_{i_j}}(\Gamma_1(N), B) \otimes \mathbb{Q}_p$ with

$$(f_{ji})_t \longrightarrow (h_i)_t.$$

We have

$$(f_j)_t = \sum_i (f_{ji})_t \longrightarrow f_t.$$

Since f has integral coefficients, the uniform convergence guarantees that f_j has coefficients in B, for every $j >> 0$.
Let $\bar{K}$ be the Galois closure of K in $\mathbb{C}_p$. Let $d = [\bar{K}, K']$. Define

$$g_j = \frac{1}{d} tr_{\bar{K},K'} f_j = \sum_i \frac{1}{d} tr_{\bar{K},K'} f_{ji},$$

a sequence such that dg_j has coefficients in $B' \cap \mathbb{C}$ for $j >> 0$. The forms $g_{ji} = \frac{1}{d} tr_{\bar{K},K'} f_{ji}$ belong to $\mathbf{M}_{k_{i_j}+\frac{1}{2}}(\Gamma_1(N), B') \otimes \mathbb{Q}_p$. Call $s_i \in \mathcal{M}^p_{k_i+\frac{1}{2}}(\Gamma_1(N), B') \otimes \mathbb{Q}_p$ the limit of g_{ji}. So

$$(g_j)_t = \sum_i (g_{ji})_t \longrightarrow (\sum_i s_i)_t.$$

Also

$$(g_j)_t \longrightarrow f_t.$$

We deduce that $f = \sum_i s_i \in \mathcal{M}^p_{\bullet+\frac{1}{2}}(\Gamma_1(N), B')$.

2. Assume $f, g \in \mathcal{M}^p_{\bullet+\frac{1}{2}}(\Gamma_1(N), B)$, $f_\infty = g_\infty$. Then $(f\theta)_\infty = (g\theta)_\infty$, and $f\theta, g\theta \in \mathcal{M}^p_\bullet(\Gamma_1(N), B)$. Again by the integral-weight q-expansion principle, $(f\theta)_t = (g\theta)_t$ for every t, hence $f_t = g_t$ for every t, and $f = g$.
Assume $f_\infty \in B[[q_N]]$ and $\pi^r f \in \mathcal{M}^p_{\bullet+\frac{1}{2}}(\Gamma_1(N), B)$ for some $r \geq 0$. Equivalently, $f \in \mathcal{M}^p_{\bullet+\frac{1}{2}}(\Gamma_1(N), B) \otimes \mathbb{Q}_p$. Again we multiply by θ and use the integral-weight q-expansion principle for $\pi^r f\theta \in \mathcal{M}^p_\bullet(\Gamma_1(N), B)$, and deduce that $f\theta \in \mathcal{M}^p_\bullet(\Gamma_1(N), B)$. Therefore, since θ_t is π-integral (see equations (1)), $\pi^r f_t$ is divisible by π^r for every t. Hence f_t has coefficients in B for every t. Take a sequence $(f_j)_j$ with coefficients in B, approaching $\pi^r f$. Then the sequence $(\pi^{-r} f_j)_j$ will approach f. We deduce that $f \in \mathcal{M}^p_{\bullet+\frac{1}{2}}(\Gamma_1(N), B)$. This ends the proof.

Corollary (Jochnowitz). If $p \geq 3$, $p \not| N$, $\mu_N \cup \{\sqrt{2}\} \subset B$, definitions 2.1 and 2.1 are equivalent.

Proof: The reader is referred to [2, 3, 4].

Therefore, we identify f to f_∞ in what follows.

2.3 Measures

A measure on $\mathbb{Z}_p$ with values in a p-adic ring A (i.e. $A = \varprojlim A/p^n A$) is an additive function

$$\mu : \{\text{compact open subsets of } \mathbb{Z}_p\} \longrightarrow A.$$

We call $Meas(\mathbb{Z}_p, A)$ the ring of measures (with convolution as product). The Fourier transform

$$\hat{\mu}(X) = \int_{\mathbb{Z}_p} (X+1)^x d\mu(x) = \sum_{n \geq 0} \left(\int_{\mathbb{Z}_p} \binom{x}{n} d\mu \right) X^n$$

gives a ring isomorphism between $Meas(\mathbb{Z}_p, A)$ and $A[[X]]$. For example, the Dirac measure at $a \in \mathbb{Z}_p$ has Fourier transform $(X+1)^a = \sum_{n \geq 0} \binom{a}{n} X^n$.

By abuse of notation, we will often use $\hat{\mu}(T)$ instead of $\hat{\mu}(X)$, where $T = X+1$. Also, $Meas(\mathbb{Z}_p, A) = \varprojlim Meas(\mathbb{Z}/p^n\mathbb{Z}, A)$, and

$$Meas(\mathbb{Z}/p^n\mathbb{Z}, A) \simeq A[\mathbb{Z}/p^n\mathbb{Z}] = \{\sum_{a \bmod p^n} m_a T^a, m_a \in A\}.$$

If $\mu \in Meas(\mathbb{Z}_p, A)$ and C is a compact open subset of $\mathbb{Z}_p$, $\mu|_C$, the restriction of μ to C, extended by 0, is in $Meas(\mathbb{Z}_p, A)$ too.

The following properties can easily be checked by proving them first for the dense subring of finite linear combinations of Dirac measures.

- If $\hat{\mu}(T) = \sum_{n\geq 0} a_n T^n$, then $\hat{\mu}|_{p\mathbb{Z}_p}(T) = \sum_{n\geq 0} a_{pn} T^{pn}$.
- For any $a \in \mathbb{Z}_p$, and any measure $\mu \in Meas(\mathbb{Z}_p, A)$, define $\mu_a \in Meas(\mathbb{Z}_p, A)$ to be the unique measure with $\hat{\mu}_a(T) = \hat{\mu}(T^a)$. Then $\int h(x) d\mu_a = \int h(ax) d\mu$ for every continuous function $h : \mathbb{Z}_p \longrightarrow \mathbb{Z}_p$.

References for measures are [8, 17, 20].

3 Interpolation

3.1 The measure associated to a modular form

Let $p \geq 5$, $p \not| N$. Define $\mathcal{M} = \mathcal{M}^p(\Gamma_1(N), B)$ to be the smallest p-adic ring containing $\mathcal{M}^p_{\bullet+\frac{1}{2}}(\Gamma_1(N), B)$. We define the q-expansions on $\mathcal{M}$ by continuity. Given $f \in \mathcal{M}^p_{\bullet+\frac{1}{2}}(\Gamma_1(N), B)$, we define μ_f by

$$\hat{\mu}_f(X) = \sum_{m\geq 0} \left(\sum_{r=0}^{m} c_{m,r} \left(Nq\frac{d}{dq} \right)^r f \right) X^m$$

where $c_{m,r} \in \mathbb{Q}$ are such that $\left(\begin{array}{c} x \\ m \end{array} \right) = \sum_{r=0}^m c_{m,r} x^r$. Notice that, if $f = \sum_{n\geq 0} a_n q_N^n$,

$$\hat{\mu}_f(T) = \sum_{n\geq 0} a_n q_N^n T^n.$$

Proposition 3.1 *$\mu_f \in Meas(\mathbb{Z}_p, \mathcal{M})$.*

Proof: To see that μ_f is a measure, it is enough to prove that $\hat{\mu}_f \in \mathcal{M}[[X]]$ (rather that in $\mathcal{M} \bigotimes \mathbb{Q}_p$ as the definition seems to suggest). An easy computation shows that

$$\hat{\mu}_f(X) = \sum_{m \geq 0} \left(\sum_{n \geq 0} \left(\begin{array}{c} n \\ m \end{array} \right) a_n q_N^n \right) X^m.$$

So the m^{th}-coefficient is in $\mathcal{M}^p_{\bullet+\frac{1}{2}}(\Gamma_1(N), B) \otimes \mathbb{Q}_p$, and its q-expansion at ∞ is $\sum_{n \geq 0} \left(\begin{array}{c} n \\ m \end{array} \right) a_n q_N^n \in B[[q_N]]$.
Therefore, the m^{th}-coefficient belongs to the p-torsion of $B[[q_N]]/\mathcal{M}^p_{\bullet+\frac{1}{2}}(\Gamma_1(N), B)$, which is trivial by the q-expansion principle. We get that all coefficients are in $\mathcal{M}^p_{\bullet+\frac{1}{2}}(\Gamma_1(N), B)$.

By density, we extend the definition to $f \in \mathcal{M}$.

The expression $\int_{\mathbb{Z}_p} x^r d\mu_f$ is called the r^{th}-moment of the measure ($r \in \mathbb{Z}$). The r^{th}-moment of μ_f satisfies

$$\int_{\mathbb{Z}_p} x^r d\mu_f = \sum_{n \geq 0} a_n n^r q_N^n = \left(Nq\frac{d}{dq} \right)^r f.$$

Therefore, the r^{th}-moment is $\left(Nq\frac{d}{dq} \right)^r f$.

Any measure $\mu \in Meas(\mathbb{Z}_p, A)$ "interpolates" its moments, via the Gamma transform

$$\begin{array}{rcl} \Gamma_\mu : \mathbb{Z}_p & \longrightarrow & A \\ s & \mapsto & \int_{\mathbb{Z}_p} < x >^s d\mu|_{\mathbb{Z}_p^*} \end{array}$$

This can be seen as follows: if $r \xrightarrow{p} s$, $|r| \to \infty$, $r \equiv 0 \bmod p-1$, then $\int_{\mathbb{Z}_p} x^r d\mu \to \Gamma_\mu(s)$. For $r \equiv 0 \bmod p-1$,

$$\Gamma_\mu(r) = \int_{\mathbb{Z}_p} x^r d\mu|_{\mathbb{Z}_p^*} = \int_{\mathbb{Z}_p} x^r d\mu - \int_{\mathbb{Z}_p} x^r d\mu|_{p\mathbb{Z}_p}.$$

This is the (r^{th}-moment) – (something related to it). We next pay a closer look at $\Gamma_\mu(r)$.

3.2 Restriction of the measure to $p\mathbb{Z}_p$

If $\hat{\mu}_f(T) = \sum_{n\geq 0} a_n q_N^n T^n$, then

$$\hat{\mu}_f|_{p\mathbb{Z}_p}(T) = \sum_{n\geq 0} a_{pn} q_N^{pn} T^{pn} = \hat{\mu}_{Rf}(T)$$

where $Rf = \int_{\mathbb{Z}_p} d\mu|_{p\mathbb{Z}_p} \in \mathcal{M}$, with q-expansion at ∞: $\sum_{m\geq 0} a_{pm} q_N^{pm}$. The operator R equals VU, where $Vf = f(q^p)$, and $Uf = \sum a_{np} q_N^n$ if $f = \sum a_n q_N^n$, operators that respect the level. (See [4]).

Let $p \geq 5$, $p \not| N$, and $f \in \mathcal{M}^p_{\bullet+\frac{1}{2}}(\Gamma_1(N), B)$, and let μ_f be the measure on $\mathbb{Z}_p$ with values in $\mathcal{M}$ associated to it. Then its Gamma transform $\Gamma_f = \Gamma_{\mu_f}$ "interpolates" the sequence $\left(Nq\frac{d}{dq}\right)^r f$. More precisely, for $r \equiv 0 \bmod p-1$,

$$\begin{aligned}
\Gamma_f(r) &= \left(Nq\frac{d}{dq}\right)^r f - R\left(\left(Nq\frac{d}{dq}\right)^r f\right) \\
&= \left(Nq\frac{d}{dq}\right)^r f - \left(\left(Nq\frac{d}{dq}\right)^r Rf\right) \\
&= \left(Nq\frac{d}{dq}\right)^r f - p^r V\left(\left(Nq\frac{d}{dq}\right)^r Uf\right)
\end{aligned}$$

Proof: We prove the first equality. The other two are left to the reader.

$$\begin{aligned}
\Gamma_f(r) &= \int_{\mathbb{Z}_p} x^r d\mu_f - \int_{\mathbb{Z}_p} x^r d\mu_{Rf} \\
&= \left(Nq\frac{d}{dq}\right)^r f - \int_{\mathbb{Z}_p} x^r d(\sum_{n\geq 0} a_{pn} q_N^{pn} T^{pn}) \\
&= \left(Nq\frac{d}{dq}\right)^r f - \sum_{n\geq 0} a_{pn} (pn)^r q_N^{pn} \\
&= \left(Nq\frac{d}{dq}\right)^r f - R\left(\left(Nq\frac{d}{dq}\right)^r f\right)
\end{aligned}$$

We include the following particular case.

Corollary. If $f \in \mathcal{M}^p_{\bullet+\frac{1}{2}}(\Gamma_1(N), B)$, $f_\infty(q_N) = \sum_{n\geq 0} a_n q_N^n$, satisfies $v_p(n) \neq 1$ when $a_n \neq 0$ (the standard theta function θ, and $\theta_{\psi,t}$, the generators of all

forms of weight $\frac{1}{2}$ given in [13] are examples), then the interpolation can be further refined:
For $r \equiv 0 \bmod p-1$,

$$\Gamma_f(r) = \left(Nq\frac{d}{dq}\right)^r f - p^{2r}\left[\left(Nq\frac{d}{dq}\right)^r (U^2 f)\right](q^{p^2})$$

Proof:

$$\begin{aligned}
\Gamma_f(r) &= \left(Nq\frac{d}{dq}\right)^r f - \int_{\mathbb{Z}_p} x^r d\left(\sum_{n\geq 0} a_{np^2} q_N^{np^2} T^{np^2}\right) \\
&= \left(Nq\frac{d}{dq}\right)^r f - p^{2r}\int_{\mathbb{Z}_p} x^r d\left(\sum_{n\geq 0} a_{np^2} q_N^{np^2} T^{n}\right) \\
&= \left(Nq\frac{d}{dq}\right)^r f - p^{2r}\sum_{n\geq 0} a_{np^2} n^r q_N^{np^2} \\
&= \left(Nq\frac{d}{dq}\right)^r f - p^{2r}\left[\left(Nq\frac{d}{dq}\right)^r (U^2 f)\right](q^{p^2})
\end{aligned}$$

4 Applications

As mentioned in the introduction, this kind of interpolation was used in several instances to obtain p-adic L-functions. For that, we start with a classical modular form f, we interpret it as a p-adic form, and we compose the interpolating function $\Gamma_f : \mathbb{Z}_p \longrightarrow \mathcal{M}$ from Proposition 3.2 with evaluation at an appropriate modular point. For half-integral weight, this can not be done directly, since we do not know how to evaluate half-integral weight p-adic forms yet. In [18], we evaluated, instead, the function Γ_f/θ, that takes values on Katz's generalized p-adic modular forms. This provides a function $\mathbb{Z}_p \longrightarrow B$, that interpolates the values obtained by evaluating the sequence $\left\{\frac{1}{\theta}\left(\left(Nq\frac{d}{dq}\right)^r f - R\left(\left(Nq\frac{d}{dq}\right)^r f\right)\right)\right\}$ at the modular point.

Via Shimura Reciprocity, these evaluations become an Euler Factor times the evaluation of $\left\{\frac{1}{\theta}\left(Nq\frac{d}{dq}\right)^r f\right\}$.

Finally, the evaluation of $\frac{1}{\theta}\left(Nq\frac{d}{dq}\right)^r f$ is proved to equal the evaluation of $\frac{1}{\theta}W^r f$, where W is the Maass-Weil operator. This puts us in the realm of the classical theory, where the connection between $W^r f$ and special values of L-series is established. The extra θ in the denominator is finally absorbed by the power of the complex period that naturally appears in the resulting formula.

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