First Mid-Term for M365C

- This is an exam. You may consult your notes and textbooks but MUST NOT consult other people.
- Please write neatly on only 1 side of the sheet of paper.
- Please use a separate sheet for each question.
- All problems are of equal value though not necessarily equal difficulty.

Due Monday October 11th at 9 am

(1) Let $C_0 = [0,1]$. We get C_1 by deleting the open interval $(1/3, 2/3)$ to obtain

$$C_1 = [0, 1/3] \cup [2/3, 1].$$

To get C_2 we delete the open intervals $(1/9, 2/9)$ and $(7/9, 8/9)$ from C_1 to obtain

$$C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1].$$

We obtain C_n by deleting the middle third from each interval in C_{n-1}.

We define

$$C = \bigcap_{n=0}^{\infty} C_n.$$

This is called the middle third Cantor set. It is a closed subset of the unit interval and hence is compact.

(a) Find an expression for the area of C_n. Hence determine the area of C.

(b) Show that $x \in C$ if and only if x has a ternary expansion

$$x = \sum_{n=1}^{\infty} \frac{x_n}{3^n}$$

with $x_n \neq 1$ for all n.

(c) Prove that $|C| = |[0,1]|$. Hint: Consider binary expansions for points in $[0,1]$.

(2) Let X be a set endowed with two metrics d_1 and d_2. Suppose that there exists $C_1, C_2 > 0$ such that

$$C_1 d_1(x,y) \leq d_2(x,y) \leq C_2 d_1(x,y)$$

for all $x, y \in X$. Such metrics are called uniformly equivalent.

(a) Prove that a seqence (x_n) converges to x in the d_1 metric if and only if (x_n) converges to x in the d_2 metric.

(b) Consider the two metrics

$$d_1\big((x_1, y_1), (x_2, y_2)\big) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$
$$d_2\big((x_1, y_1), (x_2, y_2)\big) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$$

on $\mathbb{R}^2$. Describe the geometry of the open balls in each metric and then prove they are uniformly equivalent.

(3) Let (X, d) be a metric space and $A \subset (X, d)$.

(a) Prove

(i) $(A^\circ)^c = \overline{A^c}$.

(ii) $(\overline{A})^c = (A^c)^\circ$.

(b) Show that for every $x \in X$ the single point set $\{x\}$ is closed in (X, d).

(c) Show that a sequence (x_n) in (X, d) can have at most one limit i.e. if $x_n \to x$ and $x_n \to y$ then $x = y$.

(4) Consider the metric space (X, d) where X is the set of sequences which are eventually zero, i.e.

$$X = \{(x_n) \subset \mathbb{R} : \text{there exists } N > 0 \text{ such that, for all } n > N,\ x_n = 0\}$$

and

$$d\big((x_n), (y_n)\big) = \max_n |x_n - y_n|.$$

(a) Show that for each N the subset

$$X_N = \{(x_n) : \text{for all } n > N\ x_n = 0\}$$

is nowhere dense. You may use the lemma we used to prove the Baire theorem.

(b) Show that (X, d) is not a complete metric space. You may use the Baire theorem.

(5) Let V be a vector space with norm $\|\cdot\|$.

(a) Prove that if $u, v \in V$ then $\big|\|v\| - \|w\|\big| \leq \|v - w\|$.

(b) Prove that if $v_n \to v$ in the metric on V induced by $\|\cdot\|$ then $\|v_n\| \to \|v\|$ in the usual metric on $\mathbb{R}$.